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PROPERTIES OF POSITIVE-REAL MATRICES

by

Robert W. Newcomb

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Systems Theory Laboratory
Stanford Electronics Laboratories
Stanford University Stanford, California

ABSTRACT

In most known generality
The positive-real property
Here reports, compiles, and deeds,
Qualities, as synthesis needs.

I. INTRODUCTION

Positive-real scalars are familiar to most graduate electrical engineers because of their use in the synthesis of passive one-ports, [1]. It is also recognized by most such engineers that some sort of extension of this concept is necessary, but except for special cases, [2, p. 263], there appears to be only one place where positive-real matrices are defined in their full generality, [3, p. 122], and nowhere can one locate their properties.

Because of the need of many of these properties in a nonreciprocal Brune synthesis, it became necessary to assemble many of these properties, [4]. Although this requires a theory for rational matrices only, it is just as easy to give results incorporating more general matrices. Consequently, it seems worthwhile assembling the important properties which are presently available in a report separate from that containing the synthesis method.

We will adhere to the following notation and conventions: $p = \sigma + j\omega$, $j = \sqrt{-1}$, denotes the complex frequency variable; a superscript asterisk, *, denotes complex conjugation; a subscript asterisk, $_*$, denotes Hurwitz conjugation (replacement of p by -p); a superscript tilde, $\tilde{}$, denotes matrix transposition; a subscript $_H$ denotes the Hermitian part, i.e., $A_H = \frac{1}{2}[A+\tilde{A}^*]$; $\dotplus$ denotes the direct sum of two matrices; 1_n and 0_n denote, respectively, the n x n identity and zero matrices;

$| \; |$ denotes the absolute value; $\measuredangle$ denotes "the angle of."

II. DEFINITION

We consider a passive n-port network, N, which satisfies suitable restrictions to have a Laplace transform description. N need not be reciprocal or contain a finite number of lumped elements, but, for our purposes we can assume it possesses an impedance matrix Z(p). By exciting with square integrable currents we see that Z(p) is analytic in the open right half plane $\sigma > 0$, [5, Chapters 5, 10, 13]. By exciting with exponential currents whose frequencies are on the positive σ axis and then using Schwarz's reflection principle we see that $Z^*(p) = Z(p^*)$ in $\sigma > 0$, [5, Chapter 13]. By exciting with currents of the form $i(t) = \text{Re}\, I \exp[p_o t]$, $\sigma_o > 0$, $-\infty < t < \infty$, we see that $Q_-(Z, I, p) \geq 0$ for every complex n-vector I and every p in $\sigma > 0$ where, [5, Chapter 13],

$$Q_-(Z,I,p) = \begin{cases} \tilde{I}^* Z_H(p) I - \frac{\sigma}{|p|} \left| \tilde{I}\, Z(p) I \right| & \text{if } \omega \neq 0 \\ \tilde{I}^* Z_H(p) I & \text{if } \omega = 0 \end{cases} \tag{1}$$

Since a necessary condition for $Q_- \geq 0$ is that the Hermitian part of Z be positive semi-definite in $\sigma > 0$, the above considerations lead us naturally to the following definition, [3, p. 122].

Definition:

An n x n matrix A(p) is called positive-real if it satisfies all of the following conditions:

a) A(p) is analytic in $\sigma > 0$

b) $A^*(p) = A(p^*)$ in $\sigma > 0$

c) $A_H(p)$ is positive semi-definite in $\sigma > 0$.

In contrast to what one finds in the literature, [6, p.2], or even in recent textbooks, [2, p. 244], $\sigma = 0$ must be excluded from 3), since the Hermitian part of a matrix is not defined at singularities, [7, p. 2]. That singularities can actually occur on $\sigma = 0$ is shown by the driving-point impedance $Z(p) = 1/p$, which one would of course wish to call positive-real. That a) is not a consequence of b) and c) is seen by $A(p) = \sigma - j\omega$. In the case where A(p) is rational and positive-real, we will call it <u>PR</u>. If further A(p) is PR and symmetric, that is $A = \tilde{A}$, we will call it <u>SPR</u>. Finally, if A(p) is SPR and a scalar, that is, n = 1, then we will call it <u>pr</u>. In the PR case, a) is a consequence of b) and c) and can be rightfully omitted from the definition. Further, in the PR case, b) merely states that the coefficients are real.

Before continuing we consider the following examples.

Example 1:

a) $Z(p) = \begin{bmatrix} p+1 & 1 \\ 1 & 1+1/p \end{bmatrix}$ is SPR since it is symmetric and the impedance matrix of a tee network of unit inductance, resistance and capacitance.

b) $Z(p) = \begin{bmatrix} p+1 & 2 \\ 0 & 1+1/p \end{bmatrix}$ is PR since it is the impedance matrix of the network of a) in series with a unit gyrator. Z is not SPR.

c) $Z(p) = \begin{bmatrix} \text{ctnh } p & \text{csch } p \\ \text{csch } p & \text{ctnh } p \end{bmatrix}$ is positive-real since it is the impedance matrix of a lossless transmission line. Z(p) is meromorphic but not PR.

d) $Z(p) = \begin{bmatrix} 1+\sqrt{p} & 2 \\ 0 & p+1 \end{bmatrix}$ is positive-real since it comes from a tee consisting of an infinite R-L transmission line, resistor, inductor put in series with a gyrator. Z is not meromorphic nor PR. Z has a branch point at p=0 for which the positive branch for $p=\sigma$ is taken.

e) None of a)-d) are pr while A(p)=p is pr.

f) None of A(p)= sinh p, $A(p)= j1_n$, $A(p) = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}$ are positive-real.

III. PR TEST AND σ=0 POLES

Before turning to the properties of general positive-real matrices we give an alternate characterization of PR matrices. This serves as a useful PR test, since the definition of section II requires searching in the entire (open) right half plane. The following theorem also allows us to explicitly exhibit the factors for poles on σ=0 and consequently lends some insight into the structure of PR matrices. An alternate, but in many cases less useful, test is given in T-15 of section IV.

We first recall that the residue at infinity is defined in network theory as the coefficient of p in a Laurent expansion about infinity (this differs from the mathematical usage, [8, p. 227]).

T-1:

(PR Test) An nxn matrix A is PR if and only if it has all of the following properties

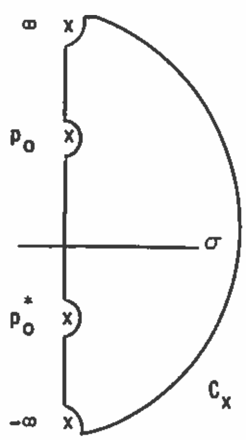
1) A is rational with real coefficients
2) A has no poles for $\sigma > 0$
3) Poles of A for $\sigma = 0$ are simple
4) The residue matrix of A for each pole on $\sigma = 0$ (including infinity) is Hermitian with every principal minor non-negative
5) All principal minors of A_H are non-negative for each p on $\sigma = 0$ for which they are defined.

Proof:

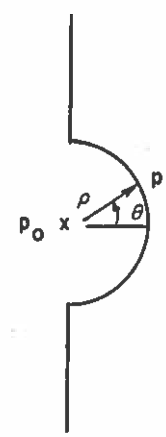
Clearly 1) must hold for any PR matrix as it is part of the definition. We will first show that if A satisfies the remaining properties it is PR. Define, for any complex constant vector x

$$f_x(p) = \tilde{x}^* A x \tag{2}$$

We must show that Re $f_x \geq 0$ for all p in $\sigma > 0$. By 2) f_x is analytic in $\sigma > 0$. Further by 1), f_x has only a finite number of poles on $\sigma = 0$, which by 3) are simple. Now consider Fig. 1 where the contour C_x is indented to avoid every pole of f_x on Re p = 0 (note that C_x may change with x).



a. p Plane Contour C_x for f_x



b. Detail near a Re p = 0 Pole at p_o

FIG. 1. CONTOUR FOR f_x.

For p on C_x but near a finite pole p_o of f_x, $p_o = j\omega_o$, we can write, by 3) and 4) and Fig. 1b

$$f_x \approx k_x/(p-p_o) = k_x \rho^{-1} e^{-j\theta} \text{ with } \begin{cases} k_x = \text{residue} > 0 \\ -(\pi/2) \leq \theta \leq (\pi/2) \end{cases} \quad (3)$$

Because of the limits on θ, we see that $\text{Re } f_x \geq 0$ on the indentions of C_x for finite poles. An exactly similar argument holds for p near infinity, except $1/(p-p_o)$ is replaced by p. By a known theorem of matrix theory, [9,p.337],5) implies that $\text{Re } f_x \geq 0$ on the remainder of C_x. Consequently, we have

$$\text{Re } f_x(p) \geq 0 \quad \text{for } p \text{ on } C_x \quad (4)$$

We now apply the principle of the maximum modulus, [8, p. 133], to $\exp[-f_x]$ for p inside and on C_x. Thus

$$\exp[-\text{Re } f_x] = |\exp[-f_x]| \quad (5)$$

obtains its maximum on C_x, $\exp[-f_x]$ being analytic inside and on C_x. Consequently, $\text{Re } f_x$ obtains its minimum on C_x, for which the indentations can be chosen arbitrarily small. We then have

$$\text{Re } f_x = \tilde{x}^* A_H x \geq 0 \text{ for } p \text{ in } \sigma > 0 \quad (6)$$

Since these arguments hold for every x, A is PR.

We begin to show the only if statement by assuming that a PR A has an m^{th} order finite pole at p_o with $\text{Re } p_o \geq 0$. We wish to consider the coefficient matrix, K, of $(p-p_o)^{-m}$

in a Laurent expansion of A about p_o, since, for p arbitrarily close to p_o, there exist x's such that

$$f_x \approx k_x/(p-p_o)^m \text{ with } k_x = \tilde{x}^* Kx \neq 0 \tag{7}$$

We will then consider p on a circle of radius ρ about p_o (analogous to Fig. 1b) and look at

$$A_H \Rightarrow \rho^{-m}[Ke^{jm\theta}]_H = \rho^{-m} L \tag{8}$$

which serves to define the Hermitian matrix L. We have

$$L_{ii} = [\text{Re } K_{ii}] \cos (m\theta) + [\text{Im } K_{ii}] \sin (m\theta); \; i=1,\ldots,n \tag{9}$$

The assumption of A being PR, coupled with (8), shows that $L_{ii} \geq 0$ for all i and each p on the circle but in $\sigma \geq 0$. Further, there must be at least one L_{ii} which is not identically zero, because we require $L_{ii}L_{jj}-L_{ij}L_{ij}^* \geq 0$, for each i and j, and each p on the circle. If for all p on the circle all L_{ii} are zero, this shows that all L_{ij} are zero, giving K = 0, i.e., no pole would exist. Thus let $L_{kk} \neq 0$ for some p on the circle; then we require

$$-(\pi/2) \leq \theta \leq (\pi/2), \; m=1, \; \text{Re } K_{kk} > 0, \; \text{Im } K_{kk} = 0 \tag{10}$$

The condition on θ shows that only half of the circle about p_o lies in $\sigma \geq 0$, that is, there are no poles in $\sigma > 0$. The condition on m shows that the finite poles on Re p = 0 are simple.

We must show that K is Hermitian and has every principal minor non-negative. We have from (7) through (10), for p on the circle near p_o,

$$\tilde{x}^* A_H x \approx \rho^{-1} ([\mathrm{Re}\ k_x] \cos\theta + [\mathrm{Im}\ k_x] \sin\theta) \qquad (11)$$

Because of the requirement on θ and the non-negativeness of (11), we require

$$\mathrm{Im}\ k_x = 0 \qquad \text{for all } x \qquad (12a)$$

$$\tilde{x}^* K x = \mathrm{Re}\ k_x \geq 0 \text{ for all } x \qquad (12b)$$

This shows that K is Hermitian and applying the previously used result from matrix theory, [9, p. 337], we know it has every principal minor non-negative.

We have shown that 2), 3) and 4) hold for finite poles, but they also hold for poles at infinity. To see this note that a pole of A(p) at infinity is a pole at zero of A(1/p). But A(1/p) is PR if A(p) is, since Re $p > 0$ implies $\mathrm{Re}(1/p) > 0$ and vice versa.

To show that 5) can't be violated we use the fact that $\tilde{x}^* A_H x$, for an arbitrary x, is a continuous function of the two real variables σ and ω. Since it is non-negative for p in $\sigma > 0$, it is non-negative for $\sigma = 0$, when it is defined, by continuity. Consequently every principal minor of A_H is non-negative on $\sigma = 0$, when it is defined. Q.E.D.

A similar test has been described for SPR matrices, essentially without proof, by Tellegen, [6, p. 2]. The proof given here is a modification of the test for scalars, [1, p. 182]. We can actually give a physical interpretation to each condition of T-1. Thus 1) states that PR matrices are concerned with lumped, finite networks whose elements are real valued. 3) and 2) state that the network is stable and perhaps asymptotically stable. 4) indicates that poles on $\sigma = 0$ result from lossless networks. 5) means that the average power input in the sinusoidal steady state is non-negative when any set of terminals is excited. Taken together, all four conditions tell us that PR matrices are linked to passive networks.

The following example illustrates the usefullness of T-1.

Example 2:

Let $A = (p^2 + 1)^{-1} \begin{bmatrix} p & 1 & 0 \\ -1 & p & 1 \\ 0 & -1 & p \end{bmatrix}$

Here A satisfies conditions 1), 2), 3) and 5), but the residue matrix for $p = j1$ has a determinant of $-1/8$. Hence A is not PR.

Using T-1 we can exhibit the required form for the $\sigma = 0$ pole terms in a partial fraction expansion of a PR matrix A. Let $p_o = j\omega_o$ be a finite, non-zero, pole of A; then p_o^* is also a pole. Let K be the residue matrix at p_o;

then K^* is the residue matrix at p_o^*. Combining these conjugate poles, the partial fraction term of A is then [10, p. 155],

$$\frac{K}{(p-j\omega_o)} + \frac{K^*}{(p+j\omega_o)} = \frac{[pA+B]}{p^2+\omega_o^2} \tag{13a}$$

where

$$A = K + \tilde{K} = K + K^* \tag{13b}$$

$$B = j\omega_o[K - \tilde{K}] = j\omega_o[K - K^*] \tag{13c}$$

Here, if a diagonal member A_{jj} of A is zero, the corresponding row and column of A and B are zero, since every 2 x 2 principal minor of K is non-negative. A is real, symmetric and positive semi-definite, (since $\tilde{x}Ax = 2\tilde{x}Kx$) while B is real and skew-symmetric. Since the residue matrices are real for poles at zero and infinity, the Hermitian property shows that they are symmetric. Thus, these poles contribute terms $pA^{(\infty)}$ and $p^{-1}A^{(o)}$ with $A^{(\infty)}$ and $A^{(o)}$ real, symmetric, positive semi-definite matrices.

Using these facts we see that, for a PR A, $A + \tilde{A}_*$ has no $\sigma = 0$ poles. Since $A + \tilde{A}_* = A + \tilde{A}^*$ on $\sigma = 0$, we also see that if properties 1) through 4) of T-1 are satisfied, then by forming $A + \tilde{A}_*$ and setting $p = j\omega$, A_H can be defined for all ω. This is actually implicit in the work of many engineers but should be done with caution.

Finally, we remark that if any positive-real matrix has poles on the imaginary axis these poles satisfy conditions 3) and 4) of T-1 and are thus of the form of (13a) if $\omega_0 \neq 0, \infty$. A test, such as T-1, should hold for meromorphic positive-real matrices, however, some trouble seems to arise because of the presence of essential singularities at infinity in meromorphic functions. Further, one should be able to say something about algebraic singularities on the imaginary axis, by using the appropriate expansion about such a singularity.

IV. FUNDAMENTAL PROPERTIES

We will list here the most important properties of positive-real matrices which are needed for synthesis. Several of the more specialized properties are important for specific synthesis methods, while some are needed for converting to the scattering matrix.

T-2:

If A is positive-real, then $\tilde{A}$ is also positive-real. This is easily seen by transposing $\tilde{x}^* A_H x$. Physically it means that the result of turning the gyrators around in a passive network is still a passive network.

T-3:

If A and B are positive-real and of the same order, then A + B is positive-real.

This is seen from $\tilde{x}^*(A+B)_H x = \tilde{x}^* A_H x + \tilde{x}^* B_H x$ and physically states that the series or parallel connections of two passive networks is again passive.

T-4:

> If T is a real constant n x r matrix and A is an n x n positive-real matrix then $\tilde{T}AT$ is an r x r positive-real matrix.

We have $\tilde{x}^*(\tilde{T}AT)_H x = (\widetilde{Tx})^* A_H (Tx)$. But $\tilde{x}^* A_H x$ assumes all the values, if not more, that $(\widetilde{Tx})^* A_H (Tx)$ does, which proves T-4. Physically this theorem states that the termination of a transformer or gyrator (n+r)-port by a passive n-port results in a passive r-port. By choosing T to isolate any desired submatrix we obtain the following corollary. Physically this states that every subnetwork of a passive network, which is obtained by ignoring terminal pairs, is again passive.

C-4:

> Every principal submatrix of a positive-real matrix is positive-real.

The following theorem, T-5, is needed to prove many of the remaining theorems. Before considering T-5 it is worth defining reduced matrices. A positive-real matrix which is analytic on Re p = 0 (including infinity) will be called <u>imaginary-reduced</u>. A matrix A(p) for which det $A_H = 0$ at a fixed p_o on Re p = 0 will be called <u>real-reduced at p_o.</u> A matrix which is both imaginary and real reduced (at some p_o) will simply be called <u>reduced</u>, or sometimes <u>minimum</u>.

T-5:

If A is positive-real and if $\tilde{x}_o^* A_H x_o = 0$ for some fixed $x_o \neq 0$ and some fixed p_o in $\sigma > 0$, then $\tilde{x}_o^* A_H x_o = 0$ for all p in $\sigma > 0$. If A is imaginary-reduced the last $\sigma > 0$ can be replaced by $\sigma \geq 0$.

Proof:

As in (2) let $f_{x_o} = \tilde{x}_o^* A x_o$, then exp $[-f_{x_o}]$ is an analytic function for p in $\sigma > 0$, or $\sigma \geq 0$ in case A is imaginary-reduced. Considering (5) we see that Re f_{x_o} can not have a minimum in $\sigma > 0$, or $\sigma \geq 0$ if A is imaginary reduced, unless it is a constant. However, Re $f_{x_o} = 0$ is clearly a minimum. Q.E.D.

The following result shows that if each inductance or capacitance is replaced by a passive one-port, the resulting network is again passive.

T-6:

If $f(p) \neq 0$ is a positive-real scalar and $A(p)$ is positive-real, then $A(f(p))$ is also positive-real.

This results from the fact that Re $f(p) > 0$ for Re $p > 0$.

The following theorem, which has been stated for the rational case by Oono, [10, p. 154], shows that synthesis methods valid for the impedance matrix are equally valid for the admittance matrix. We first make the following convention. The inverse of an n x n matrix function of p is said to <u>exist</u> if it exists when p is interpreted as an indeterminate p, [11, p. 60]. Consequently, in the rational case the rank is n for all but a finite number of p, when p is treated as a complex number.

T-7:

If A is positive-real then so is its inverse $B = A^{-1}$, if it exists.

Proof:

At each point where B exists we have $\tilde{x}^* A_H x = (\widetilde{Ax})^* B_H (Ax)$. Thus the Hermitian forms for A_H and B_H are identical, Ax assuming all values with x. We must show that this holds for all p in $\sigma > 0$, i.e., B exists everywhere in $\sigma > 0$. Assume the contrary. Then there is a p_o and an x_o such that $\tilde{x}_o^* A x_o = 0 = \tilde{x}_o^* A_H x_o$. But by T-5, this holds for all p in $\sigma > 0$. In other words B wouldn't exist, contrary to the hypothesis. Q.E.D.

The following lemma is needed to prove T-8.

L-1:

If $A = \tilde{A}$ is positive-real and A^{-1} exists, then $\tilde{x}^* A_H x > 0$ for all $x \neq 0$ and all p in $\sigma > 0$.

Proof:

Assume the contrary. Thus assume that at p_o in $\sigma > 0$ there is an x_o such that $\tilde{x}_o^* A_H x_o = 0$. By T-5 this is true for all p in $\sigma > 0$ and in particular for $p = \sigma > 0$. But for $p = \sigma$, $A = A_H$, and hence A would be singular, [11, p. 270], counter to the hypothesis. Q.E.D.

It should be noted that L-1 would in general be false if we deleted the symmetric requirement. This is illustrated by $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which has $A_H = 0$, even though A^{-1} exists. However, we can extend L-1 to other than symmetric matrices in many cases of importance.

L-2:

If A has all of the following properties

1) A is positive-real

2) A is imagninary-reduced

3) A is not real-reduced at $p = 0$

then $\tilde{x}^* A_H x > 0$ for all $x \neq 0$ and all p in $\sigma > 0$.

This is easily seen by assuming that there is an x_o such that $\tilde{x}_o^* A_H x_o = 0$ for some p in $\sigma > 0$. T-5 then applies to show that this holds for all p in $\sigma \geq 0$. However, investigating $p = 0$ shows that assumption 3) is violated. Note that in L-1 we assumed the existence of A^{-1} while in L-2 we didn't. A^{-1} does exist for L-2 as is seen by a converse for L-1 which holds for other than symmetric A.

L-3:

If A is positive-real and if $\tilde{x}^* A_H x > 0$ for all $x \neq 0$ and all p in $\sigma > 0$, then A^{-1} exists.

To see this we note that if A were singular at even one p_o in $\sigma > 0$, then there would exist an x_o such that $A(p_o)x_o = 0$. Using this x_o and p_o contradicts $\tilde{x}_o^* A_H(p_o)x_o > 0$ and thus proves the assertion.

We are now in a position to prove some of the more useful results needed for synthesis. The first of these is analogous to a theorem of McMillan for SPR matrices, [12,p. 544]. Physically it states that the series connection of a passive reciprocal n-port with no shorted terminal pairs and any other passive n-port results in a passive n-port with no shorted terminal pairs.

T-8:

If B and $A = \tilde{A}$ are positive-real and if A^{-1} exists, then the inverse of A + B exists.

Proof:

Assume A + B to be singular. Then there is an $x_o \neq 0$ such that for p in $\sigma > 0$, $\tilde{x}_o^* A_H x_o + \tilde{x}_o^* B_H x_o = 0$. Since A and B are positive-real $\tilde{x}_o^* A_H x_o = 0$. However, this contradicts the fact that A^{-1} exists, by L-1. Q.E.D.

Here $A = \tilde{A}$ is essential, as is seen by the fact that $A = \tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has A + B = 0. However, replacing L-1 by L-2 in the proof of T-8 we immediately get the following result, which is sometimes useful in the synthesis of nonreciprocal networks.

T-9:

If B is positive-real and A satisfies the hypothesis of L-2, then the inverse of A + B exists.

By analogy with Hurwitz polynomials and following Belevitch, we will call a rational matrix <u>Hurwitzian</u> if all matrix elements are analytic in $\sigma \geq 0$, [13, p. 304]. Physically the following result states that the addition of unit resistances to a passive network can never result in a lossless network or in shorted terminal pairs.

T-10:

$(A + 1_n)^{-1}$ is Hurwitzian, if A is PR.

Proof:

Clearly $A + 1_n$ is PR, T-3, $(A + 1_n)^{-1}$ exists, T-9, and is PR, T-7. By properties 2) and 3) of T-1 we must show that $B = (A + 1_n)^{-1}$ is imaginary reduced. Assume then that there is a simple pole of B of residue K at $p_o = j\omega_o$, where at first ω_o is finite. Then form

$$2(\widetilde{Bx})^*(A + \tilde{A}^*)(Bx) = \tilde{x}^*[1_n - (1_n - 2\tilde{B}^*)(1_n - 2B)]x \qquad (14)$$

which, for p in $\sigma > 0$ but near p_o, must be non-negative since Bx assumes all values with x. The right side then gives

$$\tilde{x}^*[-\tilde{K}^*K/|p - p_o|^2]x \geq 0 \qquad (15)$$

which can never hold for all x. For a pole at infinity we replace $(p - p_o)^{-1}$ by p to prove the theorem. Q.E.D.

Note that if A is meromorphic a similar result holds, that is, $(A + 1_n)^{-1}$ is analytic in $\sigma \geq 0$, perhaps excluding infinity, in such a case. This isn't true in the general case as is seen by $A(p) = \sqrt{p}$ for which $(1 + \sqrt{p})^{-1}$ has a branch point at $p = 0$.

The following theorem is a generalization of the result for symmetric, rational matrices, [14, p. 152], and has been stated in the rational case without proof by Oono, [10, p. 155]. Physically the theorem states that essentially the only way of making the impedance matrix singular is to adjoin short circuits through transformers to a network which possesses an impedance and an admittance matrix.

T-11:

> If an n x n positive-real matrix has rank $r < n$ then there is a real, constant, n x n, matrix T, which may be chosen orthogonal, such that $A = \tilde{T}(A_o \dotplus 0_{n-r})T$, where A_o is a non-singular positive-real matrix of rank r.

Proof:

Consider a real p_o in $\sigma > 0$ where A has its normal rank. Since $A(p_o)$ is real, we can find a real constant non-singular T_o such that T_oA has its last n-r rows zero at p_o. Multiplying this on the right by $\tilde{T}_o$ gives at p_o,

$$T_oA\tilde{T}_o = \begin{bmatrix} A'_o & A'_{12} \\ 0 & 0 \end{bmatrix} ; \quad \begin{matrix} A'_o \text{ of order } r \times r \\ A'_{12} \text{ of order } r \times (n-r) \end{matrix} \tag{16}$$

By T-4, $T_o A\tilde{T}_o$ is positive-real and thus $(T_o A\tilde{T}_o)_H$ has every principal minor non-negative at p_o. Considering the 2 x 2 principal minors for the last n-r rows and columns shows that $A'_{12} = 0$. We can show that (16) with $A'_{12} = 0$ holds for all p. Consider the 1 x n vector $\tilde{x} = [\tilde{x}_1 \tilde{x}_2]$ where $\tilde{x}_1$ and $\tilde{x}_2$ are constant row vectors of order 1 x r and 1 x (n-r) respectively. First let $x_1 = 0$, x_2 = arbitrary and form $\tilde{x}^*(T_o A\tilde{T}_o)_H x$. At $p = p_o$ this is zero and thus by T-5 it is zero for all p in $\sigma > 0$. This shows that the (n-r) x (n-r) submatrix of $T_o A\tilde{T}_o$ satisfies $A'_{22} = -\tilde{A}'^*_{22}$ in $\sigma > 0$. Now A'_{22} is analytic in p, and, hence, A'^*_{22} can't be analytic unless it is a constant, by the Cauchy-Riemann equations. Since it is zero at p_o, it is then zero everywhere in $\sigma > 0$. Then, letting x_1 be arbitrary we obtain

$$\tilde{x}^*(T_o A\tilde{T}_o)_H x = \tilde{x}_1^*(A'_o)_H x_1 + \mathrm{Re}[\tilde{x}_1^*(A'_{12} + A'^*_{21})x_2] \qquad (17)$$

with A'_{21} defined in an obvious manner. By properly choosing x_1 and x_2 we can isolate any element in the right hand term and varying x_2 we can make the right hand term as negative as desired (if it is non-zero). Thus, we require $\mathrm{Re}[\tilde{x}_1^*(A'_{12} + \tilde{A}'^*_{21})x_2] = 0$ for all x_1 and x_2 in p in $\sigma > 0$. By choosing x_1 and x_2 real we require $\mathrm{Re}(A'_{12} + \tilde{A}'^*_{21}) = 0$ and choosing x_1 real and x_2 imaginary we require $\mathrm{Im}(A'_{12} + \tilde{A}'^*_{21}) = 0$. Thus $A'_{12} = -\tilde{A}'^*_{21}$ for all p in $\sigma > 0$ and as for A'_{22} we see that A'_{12} and A'_{21} are zero for all p. Consequently

$$T_o A\tilde{T}_o = A'_o \dotplus 0_{n-r} \qquad (18)$$

To show that T_o can be replaced by an orthogonal matrix we can multiply (18) on the left by a real, constant, non-singular matrix T_1 and on the right by $\tilde{T}_1$. T_1 is determined in the following manner. Consider the rows of T_o as basis vectors and orthogonalize these vectors by the Gram-Schmidt process, [15, p. 127], beginning with the last row and working toward the first. This process can be performed by the matrix T_1 which in turn can be thought of as a product of elementary matrices. The first elementary matrix (on the right) of T_1 multiplies the last row of T_o by a constant, the second adds the n^{th} row of T_o to the $(n-1)^{th}$, etc. Since we will always add higher numbered rows to lower numbered rows through T_1, the 0_{n-r} is preserved and we arrive at

$$A = \tilde{T}(A_o \dotplus 0_{n-r})T \tag{19a}$$

$$T = T_1 T_o = \text{orthogonal} \tag{19b}$$

$$A_o \dotplus 0_{n-r} = T_1(A_o' \dotplus 0_{n-r})\tilde{T}_1 \tag{19c}$$

A_o is positive-real since A_o' was, and it is of rank r by inspection of (19), T being of rank n. Q.E.D.

The following result justifies the resistance extraction which is one of the crucial steps in the Brune synthesis. The result is a generalization of that of McMillan, [12, p. 556], and allows that quoted by Belevitch as an important special case.

T-12:

If $A(p)$ is an $n \times n$ PR matrix, which is not real-reduced at any $p_o = j\omega_o$, and if A_o is any real, constant, positive semi-definite matrix, then there exists some constant $r > 0$ such that $A(p) - rA_o$ is PR and real-reduced at some p_o.

Proof:

We form $A_H(j\omega) - \lambda A_o$ where λ is a parameter and we guarantee that the entries of $A_H(j\omega)$ and A_o are bounded by setting $p = j\omega$ in $A + \tilde{A}_*$. Now $A_H(j\omega)$ and A_o are both positive semi-definite and can be simultaneously diagonalized, [17]; that is, we can find, for each ω, a non-singular (but perhaps complex) $T(\omega)$ such that

$$\tilde{T}^*(\omega)[A_H(j\omega) - \lambda A_o]T(\omega) = \Big\{(a'_{11}(\omega) - \lambda) \dotplus \dots \dotplus (a'_{kk}(\omega) - \lambda) \dotplus a'_{k+1,k+1}(\omega) \dotplus \dots \dotplus a'_{nn}(\omega)\Big\} \qquad (20)$$

where k is the rank of A_o. We then choose r as

$$r = \min_{1 \le j \le r}, \min_{0 \le \omega \le \infty} a'_{jj}(\omega) \qquad (21)$$

Substituting this r for λ in (20) shows that $A(p)$ is PR by T-1 and ω_o is the ω at which the minimum occurs in (21). Q.E.D.

Of especial interst is the choice of $A_o = [1 \dotplus 0_{n-1}]$. This is most convenient for the Brune synthesis, [16, p. 285], and was first suggested by Oono, [18, p. 168], and used by

Tellegen, [6, p. 4]. By using the Gauss diagonalization on $A_H(j\omega)$ one sees that in this case, [9, p. 26], $r = \min_{0 \le \omega \le \infty} \Delta(\omega)/\Delta_{11}(\omega)$ where Δ is the determinant and Δ_{11} the (1,1) minor, both of $A_H(j\omega)$. By (20) we see that $A_H(j\omega_o) - r A_o$ has rank n - 1. Because of its importance we state this explicitely.

C-2:

If A(p) is an n x n PR matrix which is not real-reduced at any p_o then

$$A_m(p) = A(p) - [r \dotplus 0_{n-1}] \tag{22a}$$

with

$$r = \min_{0 \le \omega \le \infty} \Delta(\omega)/\Delta_{11}(\omega) \tag{22b}$$

is real-reduced, of rank n-1, at the ω_o for which r is formed.

Physically, then, a finite n-port described by a non-real-reduced impedance matrix can be described by a real-reduced impedance matrix by extracting a series resistor from any port. The above theorem, T-12, appears to remain valid in the meromorphic case, but some trouble seems to arise in the general case as one apparently isn't guaranteed the minimum needed for (21).

We now turn to angle constraints and related concepts, which are based upon Schwarz's lemma, [8, p. 136].

Schwarz's lemma:

If $f(z)$ is analytic in $|z|<1$ and satisfies the two conditions

1) $f(o) = 0$ in $0 < |z| < 1$

2) $|f(z)| \leq 1$ in $0 < |z| < 1$

then either

a) $f(z) = \exp[j\psi]$, ψ = real constant

or

b_1) $|f(z)| < |z|$ in $0 < |z| < 1$

and

b_2) $|f'(o)| < 1$ where $f' = df/dz$

First consider a scalar positive-real function, $A(p) = a(p)$. By looking at real $p_o = \sigma_o > 0$ and forming the reflection coefficient $f(z) = [a(p) - a(p_o)]/[a(p) + a(p_o)]$ with $z = [p - p_o]/[p + p_o]$ and applying part b_1) of Schwarz's lemma, one shows in the standard manner that, [1, pp. 114-121],

$$|\angle a(p)| \leq |\angle p| \text{ in } \sigma > 0 \tag{20a}$$

Here equality holds if and only if either $p = \sigma$ or $a = c/p$ or $a = cp$, $c \geq 0$. Further by expressing the angles in terms of cosines (20a) is easily rewritten as

$$\text{Re } a(p) \geq \frac{\sigma}{|p|} |a(p)| \text{ in } \sigma > 0 \tag{20b}$$

This clearly shows that (20a) is equivalent to b) and c), under the assumption of a), of the positive-real definition (use Schwarz's reflection principle, [19, p. 89], with a) to get b)). Now consider a positive-real, symmetric matrix $A(p)$

and an arbitrary n-vector $\tilde{x} = \tilde{x}_1 + j\tilde{x}_2$ with x_1, x_2 both real. Then

$$a_x(p) = \tilde{x}^* A x \tag{21}$$

is a positive-real scalar, since $a_x(p) = \tilde{x}_1 A x_1 + \tilde{x}_2 A x_2$. In this case (20b) is identical to $Q_-(A,x,p) \geq 0$ in $\sigma > 0$. We have then proven,

T-12:

Necessary and sufficient conditions for an n x n <u>symmetric</u> matrix to be positive-real are

1) $A(p)$ is analytic in $\sigma > 0$ and either

2a) $|\measuredangle \tilde{x}^* A(p) x| \leq |\measuredangle p|$ in $\sigma > 0$ for every x or equivalently

2b) $Q_-(A,x,p) \geq 0$ in $\sigma > 0$ for every x.

Belevitch has given an ingenious extension of 2a) for non-symmetric PR matrices, [20, p. 4]. For any p in $\sigma > 0$ he writes

$$A(p) = U(\sigma,\omega) + jV(\sigma,\omega) \tag{22}$$

where U and V are Hermitian. Letting subscripts r and i denote real and imaginary parts, he concludes, by using arguments on the scattering matrix, that in $\sigma > 0$

$$\omega^2 U^*(\sigma,\omega) - \sigma^2[V_r(\sigma,\omega) + U_i(\sigma,\omega)]U^{-1}(\sigma,\omega)[V_r(\sigma,\omega) - U_i(\sigma,\omega)]$$

is positive semi-definite (A is assumed non-singular by using T-11 so that U^{-1} exists).

If $A(p)$ is analytic and symmetric in $\sigma > 0$, then a simple manipulation of (20b) shows that a necessary and sufficient condition for $A(p)$ to be positive-real, using the notation of (22), is that both of $\omega U(\sigma,\omega) \pm \sigma V(\sigma,\omega)$ be positive semi-definite in $\sigma > 0$, [21, p. 3]. If $A(p)$ is positive-real and analytic at $p_o = j\omega_o$, then the use of the maximum modulus theorem, as in T-1, allows us to conclude that $A_H(p_o) = U(0,\omega_o)$ is positive semi-definite. In case A is symmetric and $A_H(p_o) = 0_n$ we can conclude Takahasi's theorem, [22, p. 58], [18, p. 167].

T-13 (Takahasi's theorem):

If $A(p)$ is a symmetric, positive-real matrix which is analytic at $p_o = j\omega_o$, $0 < \omega_o \neq \infty$, with $A_H(j\omega_o) = 0_n$, then

$$\left[\frac{dA(p)}{dp} \pm \frac{A(p)}{p}\right]\Bigg|_{p=p_o}$$

is positive semi-definite (for both signs). These are positive definite if $A(p) - pA^{(\infty)} - A^{(o)}/p$ is non-singlar, where $A^{(\infty)}$ and $A^{(o)}$ are the residue matrices at any poles of A at infinity or zero.

Proof:

By the analyticity at p_o, we can write, using (22),

$$\frac{dA(p)}{dp}\Bigg|_{p_o} = \frac{\partial A(p)}{\partial j\omega}\Bigg|_{p_o} = \left[\frac{\partial V(o,\omega)}{\partial\omega} - j\,\frac{\partial U(o,\omega)}{\partial\omega}\right]\Bigg|_{\omega_o} \quad (23a)$$

Now a Taylor series for $U(o,\omega)$ is

$$U(o,\omega) = U(o,\omega_o) + \frac{\partial U(o,\omega_o)}{\partial\omega}(\omega - \omega_o) + \dots \tag{23b}$$

and since $U(o,\omega_o) = 0_n$, the non-negativeness of U shows that $\partial U(o,\omega_o)/\partial\omega = 0_n$. A Taylor series for $A(p)$ then becomes

$$A(p) = A(p_o) + \frac{dA(p_o)}{dp}(p - p_o) + \dots \tag{23c}$$

$$= jV(o,\omega_o) + \sigma\frac{\partial V(o,\omega_o)}{\partial\omega} + j\frac{\partial V(o,\omega_o)}{\partial\omega}(\omega - \omega_o) + \dots \tag{23d}$$

for p near p_o. Choosing $\omega = \omega_o$ we have only the first two terms, to first order. Thus

$$A(\sigma + j\omega_o) = U(\sigma,\omega_o) + jV(\sigma,\omega_o)$$

$$= \sigma\frac{\partial V(0,\omega_o)}{\partial\omega} + jV(0,\omega_o) + \text{order } (\sigma^2) \tag{23e}$$

But $\omega_o U \pm \sigma V$ is positive semi-definite and factoring out σ we see that

$$\frac{\partial V(0,\omega_o)}{\partial\omega} \pm \frac{V(0,\omega_o)}{\omega_o}$$

is positive semi-definite. By noting the conditions under which equality holds in (20b), we see that these are positive definite under the assumptions stated. Q.E.D.

By the use of L'Hospital's rule the same result is obtained by Belevitch for SPR matrices, [20, p. 4]. Takahasi's theorem is a generalization of the positive slope condition for lossless one-ports and is recognized as being useful in the one-port Brune synthesis, [1, p. 517]. Note that in T-13 A need not be rational, in which case one must check the analyticity at p_o, for instance $A(p) = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p/n!}{p^2+(m/n)^2}$ is analytic nowhere on the imaginary axis.

Using part b_2) of Schwarz's lemma another interesting relationship can be obtained, which, however, we know of no use for as yet. The result, for scalars, was first stated by Richards', [23, p. 779], and again later by Reza, [24, p. 40].

T-14:

If A(p) is positive-real, then, for every complex constant n-vector x,

$$\left|\tilde{x}^* \frac{dA(p)}{dp} x\right| \leq \frac{\tilde{x}^* A_H(p)x}{\sigma} \text{ in } \sigma > 0 \tag{24}$$

Proof:

Let p_o be fixed, but arbitrary, and with $\sigma_o > 0$ and let $A_o = A(p_o)$. Then form $z = (p - p_o)/(p + p_o^*)$, $f(z) = [\tilde{x}^*A(p)x - \tilde{x}^*A_ox]/[\tilde{x}^*A(p)x + \tilde{x}^*\tilde{A}_o^*x]$. For $\sigma > 0$ the conditions of Schwarz's lemma are satisfied; note that $f(z) = 1$ at $p = p_o$ if $\tilde{x}^*A_H(p_o)x = 0$, by L'Hospital's rule. But

$$f'(z) = \left\{\frac{\tilde{x}^*A'x}{[\tilde{x}^*Ax + \tilde{x}^*\tilde{A}_o^*x]} - \frac{[\tilde{x}^*Ax - \tilde{x}^*A_ox]}{[\tilde{x}^*Ax + \tilde{x}^*\tilde{A}_o^*x]^2} \cdot \tilde{x}^*A'x\right\}\frac{dp}{dz} \tag{25a}$$

$$= \frac{2(\tilde{x}^*A_{oH}x)(\tilde{x}^*A'x)}{[\tilde{x}^*Ax + \tilde{x}^*\tilde{A}_o^*x]^2} \cdot \frac{2\sigma_o}{(z-1)} \tag{25b}$$

Applying b_2) of Schwarz's lemma gives (24), when $\tilde{x}^* A_{oH} x \neq 0$, since $z = 0$ is $p = p_o$. If $\tilde{x}^* A_{oH} x = 0$, then $\tilde{x}^* A_H(p) x = 0$ in $\sigma > 0$, by T-5. By the Cauchy-Riemann equations $\tilde{x}^* A' x$ is then zero in $\sigma > 0$, and (24) is satisfied with both sides identically zero. Q.E.D.

In many cases it is sometimes convenient to consider the following transformation. Let

$$S_- = 1_n - 2(A + 1_n)^{-1} \tag{26a}$$

$$= (A - 1_n)(A + 1_n)^{-1} \tag{26b}$$

$$S_+ = - S_- \tag{26c}$$

If A is an impedance or an admittance matrix, then S_- or S_+ is the respective scattering matrix (physically S_+ results from the dual of the network for S_-). From this definition and the definition of a positive-real matrix, it is clear that S_-, as well as S_+, satisfy: I) S(p) is analytic in $\sigma > 0$ ($A + 1_n$ is positive-real and hence analytic in $\sigma > 0$), II) $S^*(p) = S(p^*)$ in $\sigma > 0$, III) $1_n - \tilde{S}_-^*(p)\, S_-(p)$ is positive semi-definite in $\sigma > 0$ (note that $1_n - \tilde{S}_-^* S_- = 4(\tilde{A}^* + 1_n)^{-1} A_H (A + 1_n)^{-1}$) . Matrices satisfying conditions I), II), III) have been called bounded-real in the literature, where an extensive treatment of their relation to passive networks exists, [3, p. 116]. We merely wish to point out that sometimes a simpler test than T-1 exists, if one wishes to convert to the scattering matrix. Consequently, we prove T-15, which is of most use when A(p) has poles on the imaginary axis, since no special consideration need be given to this axis when using S_-.

T-15:

An n x n matrix A is PR if and only if S_- has all of the following properties:

1) $S_-(p)$ is rational with real coefficients

2) $S_-(p)$ is Hurwitzian

3) $1_n - \tilde{S}_-^*(j\omega)S_-(j\omega)$ has every principal minor non-negative (including $\omega = \infty$).

The same results hold if S_- is replaced by S_+.

Proof:

Clearly 1) can't be relaxed. For the only if part, we see that a relaxation of 2) would contradict T-10, while a relaxation of 3) counters (14). To prove the if part, we note that S_-x is analytic in $\sigma \geq 0$ for x an arbitrary constant vector. Thus, by the maximum modulus theorem, $\tilde{x}^*(1_n - \tilde{S}_-^*S_-)x \geq 0$ in $\sigma \geq 0$. By (14) and (26) the validity of the theorem rests upon the fact that $S_- - 1_n$ is non-singular at each p in $\sigma > 0$. To see that this is indeed the case we assume the contrary, i.e., that there is an x_o such that $S_-x_o = x_o$ for some p_o in $\sigma > 0$. Again applying the maximum modulus theorem to (14) shows that $S_-x_o = x_o$ for all p in $\sigma > 0$ and hence $(S_- - 1_n)^{-1}$ wouldn't exist. Clearly the results are equally valid in terms of S_+. Q.E.D.

The second condition essentially results from the stability of a finite passive network, while the third states that the average power input in the sinusoidal steady state is non-negative. T-15 can be extended to cover any positive-real A, [3, p. 116]. The essential change is that S_- need not be analytic on $\sigma = 0$, but must have $\sup \left\{ (\exp[-|p|^{\alpha}]) ||S_-(p)|| \right\} < \infty$ in $\sigma > 0$ where $||S|| = \sum_{i=1}^{n} \sum_{j=1}^{n} |s_{ij}|$. Condition 1) is also replaced by $S^*(j\omega) = S(-j\omega)$ for almost all ω.

For completeness, we finally state the recent Richards' theorem for matrices, [25, p. 3]. If A is positive-real and $k > 0$, define

$$\overline{A}(p) = A(p) - \frac{1}{2}[A(k) - \tilde{A}(k)] \qquad (27)$$

then

$$A_r(p) = [p\overline{A}(k) - k\overline{A}(p)][p\overline{A}(p) - k\overline{A}(k)]^{-1}\overline{A}(k) \qquad (28)$$

is positive-real.

V. CONCLUSIONS

Here we have presented the properties of positive-real matrices which are of use in synthesis. In the cases where it is no harder to state the results in the general case of positive-real, in contrast to merely PR or SPR, this has been done. Almost all of the properties are extensions of those which are well-known for the scalar case, however, we have been unable to previously find these results compiled for matrices.

Several of the results can be perhaps modified. For instance Bayard, [26, p. 384], states that by varying A_o of T-12, any p_o can be obtained. (28) can yield an extension following the ideas of Fialkow and Gerst, [27, p. 161], [28, p. 197]. One would still like to generalize T-12 to show that if A is positive-real then $Q_- \geq 0$. This must be true in the PR case, as one sees indirectly by noting that any PR impedance matrix, Z, has a passive network realization, and hence by the arguments of section II, $Q_- \geq 0$. Of course Q_- is a generalization of q_- of Desoer and Kuh, [29, p. 425], valid for distributed parameter networks.

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