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DIFFERENTIAL SYSTEMS ON ALTERNATIVE ALGEBRAS*

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Abstract

Continuing a previous study, which gave a universal square - law canonical form for differential systems by embedding the description in an attached commutative nonassociative algebra, we show here that when the algebra is alternative the description can further be reduced to zero.

"It is my wish that the Treasury shall make a chronicle setting forth the geneology."
[1, p.1]

1. Introduction

Previously [2] we have shown that any system $\mathcal{S}$ described in the state variable form $x = f(x,t)$, with f reasonably behaved, can be reduced to the canonical form

$$\dot{\underline{x}} = \underline{x}\cdot\underline{x} \qquad \underline{x}(0) = \underline{x}_0 \text{ given} \tag{1}$$

where $\underline{x}(t)$ is a vector in an attached algebra $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ and $\dot{\underline{x}} = d\underline{x}/dt$. This algebra is commutative but not necessarily associative. Among the more extensively studied nonassociative algebras are the alternative algebras [3, Chap. 3]. Here we show that if $\mathcal{A}(\mathcal{S})$ is an alternative algebra the canonical form can be further reduced to the ultimate form of zero.

2. Main Result

We first recall that an algebra is a vector space in which multiplication of any two vectors is properly defined [4, p. 144]. An alternative algebra $\mathcal{A}$ is an algebra in which for all $\underline{x}, \underline{y} \in \mathcal{A}$

$$(\underline{x}\cdot\underline{x})\cdot\underline{y} = \underline{x}\cdot(\underline{x}\cdot\underline{y}) \tag{2a}$$

and

$$\underline{y}\cdot(\underline{x}\cdot\underline{x}) = (\underline{y}\cdot\underline{x})\cdot\underline{x} \tag{2b}$$

Since the attached algebras for (1) are assumed commutative, there is no difference between (2a) and (2b) in the considerations here. Too, as is known [5, p.319], we may assume that $\mathcal{A}$ has a unity element $\underline{e}$, since otherwise one may be adjoined while preserving the alternative nature of the algebra.

Now, given an alternative algebra $\mathcal{A}(\mathcal{S})$ associated with a system $\mathcal{S}$ to yield its representation $x = x\cdot x$, that is, representation by (1) within the algebra, we can proceed somewhat as with Riccati differential equations [6, p.12] by introducing a new vector variable y through

$$\dot{\underline{y}} = -\underline{x}\cdot\underline{y} \tag{3}$$

Then differentiation of (3), using (1), yields $\ddot{\underline{y}} = -\dot{\underline{x}}\cdot\underline{y} - \underline{x}\cdot\dot{\underline{y}} = -(\underline{x}\cdot\underline{x})\cdot\underline{y} + \underline{x}\cdot(\underline{x}\cdot\underline{y})$. On using (2a) this gives our main result

$$\ddot{\underline{y}} = \underline{0} \tag{4}$$

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In other words, any system whose attached algebra is alternative can be reduced further within the algebra to $\ddot{\underline{y}} = \underline{0}$.

3. Solution

Equation (4) can be integrated to yield a solution:

$$\dot{\underline{y}} = \underline{a} \tag{5a}$$

$$\underline{y}(t) = \underline{a}t + \underline{b} \qquad \underline{a} \;\&\; \underline{b} \text{ constant} \tag{5b}$$

Substitution in (3) gives, if $\underline{y}^{-1}$ and then $\underline{b}^{-1}$ exist,

$$\underline{x} = -\dot{\underline{y}}\cdot\underline{y}^{-1} = -\underline{a}[\underline{a}t + \underline{b}]^{-1} \tag{6a}$$

$$= -\underline{a}\underline{b}^{-1}[\underline{a}\underline{b}^{-1}t + \underline{e}]^{-1} \tag{6b}$$

As the initial conditions are

$$\underline{x}(0) = \underline{x}_o = -\underline{a}\underline{b}^{-1} \tag{6c}$$

we obtain

$$\underline{x}(t) = \underline{x}_o[\underline{e} - t\underline{x}_o]^{-1} \tag{7}$$

which agrees with [1, Eq.(14)] previously obtained for division algebras. Since only the ratio of $\underline{a}$ to $\underline{b}$ is important, in (5b) we may always take $\underline{a} = \underline{x}_o$, $\underline{b} = \underline{e}$. In this case $\underline{b}$ is clearly nonsingular, and, hence (as $\underline{y}(0) = \underline{b} = \underline{e}$), by continuity $\underline{y}(t)$ for sufficiently small t. By analytic continuation (7) is then seen to be the solution within the attached alternative algebra.

4. Discussion

We have seen here that if the attached algebra $\mathcal{A}(\circ)$ of a system $\mathcal{S}$ is alternative, the canonical equations $\dot{\underline{x}} = \underline{x}\cdot\underline{x}$ take the very simple linearized form $\ddot{\underline{y}} = \underline{0}$ within the same algebra. This further leads to a simple solution $\underline{x}(t) = \underline{x}_o[\underline{e} - t\underline{x}_o]^{-1}$ given the initial conditions $\underline{x}_o = \underline{x}(0)$, irrespective of whether or not the alternative algebra is a division algebra. It is of interest to note that alternative division algebras themselves are well-studied and are either associative or isomorphic to an eight - dimensional Cayley - Dickson division algebra over their center [7, p.141].

To H. S. H. Jamalullail with remembrance of Malaysian developments.

"When he heard the word of his Highness, he took the command upon his head and his limbs were bowed beneath the weight of it."
[1, pp.1-2]

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