Nonlinear Differential Systems: A Canonic, Multivariable Theory

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Abstract—An attached (multivariable) nonassociative algebra is shown to be a useful tool for the study of general polynomic differential systems. The main result is that the theory allows the reduction of all such systems to the canonic quadratic form $\dot{x} = x \cdot x$ within the algebra. This canonic square law differential equation is obtained through multivariable introductions to make all terms quadratic; nonstationary systems are included as are nonpolynomic differential systems through approximation. In the algebra a canonical power series solution is found which is useful for computer aided analysis. Examples are given to illustrate the concepts.

Prologue:

"When they heard the command, the articifers split a rock in two and they made a record in writing." [1, p. 12]

I. Introduction

"The dry land and all that dwells thereon are known to me, but the sea and all that dwells therein—what might they be like?" [1, p. 11]

RECENTLY polynomic systems have come under study because of their generality and wide range of applicability to engineering systems (see especially [2]). Within this class of systems lies the stationary polynomic differential systems, the main class to be studied here, these being described by $\dot{x} = p(x)$ where $p(x)$ is polynomial in the components of the vector x.

Certainly the theory of ordinary differential equations is a well studied and established one [3]–[6]; for these classical theories we will refer to the excellent treatment of H. Cartan [7] in the following. But for nonlinear differential equations these theories essentially consider special cases, for example, concentrating on second order ones [8], or tabulating results on individual equations [9], or placing emphasis upon stability [10], or the existence of solutions in an abstract setting [11]. If one turns from a functional to an algebraic approach, then more general results can be found [12], though these are on a very abstract footing (using solutions in Galois type extensions of differential fields [13]). To be sure this type of algebraic theory is very useful for certain classes of linear systems problems, especially time-variable network synthesis on a scattering basis [14], but for nonlinear systems concrete results of engineering interest are in the same situation as their mathematical base. Nevertheless, the merger of algebraic and functional concepts points out the fruitfulness of the kind of treatment we will discuss here.

Our treatment takes as its starting point the work of L. Markus [15], where it is shown that for quadratic polynomic differential systems there is a related, possibly nonassociative, algebra. By looking at nilpotents and idempotents in the related algebra, Markus was able to deduce certain properties of the differential system; in the case of two variables this led to a complete classification of the solutions in the phase-plane. Here we make a further association with the related algebra by transferring the system dynamics to the algebra. This allows the reduction of the multivariable functional differential equation to a single variable canonical differential equation in the algebra, as discussed in Section III. This canonical differential equation is readily solved by a power series within the algebra, Section IV, thus yielding a solution to the original system. The algebra itself is reviewed in Section III after the discussion, and introduction, of quadratic differential systems in Section II. Section VI shows how all other polynomic systems come within the theory through the adjoinment of new variables; this method also allows the treatment of nonstationary systems, while nonpolynomic ones are covered by using the Weierstrass approximation. Simple, but illustrative, examples are given throughout with the theory developed for real systems, though extendable to more general ones.

Manuscript received May 5, 1976; revised September 9, 1976. This work was supported in part by the US National Science Foundation under Grant NSF ENG 75-03227 and in part by a Fulbright-Hays Grant.

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II. Canonic (Quadratic) Systems

"Into this lake the king released fish of every sort, and in the middle of it stood an island of great height, over which vapour constantly hovered as though the summit was wrapped in dewy mist." [1, p. 9]

In this section we define and discuss the quadratic differential systems which form the backbone of the theory. Later, in Section VI, we will show how all polynomic, and many other, systems can be brought to this quadratic form.

Let $x = [x^i]$ be a (column) n-vector of real-valued functions of time t; physically x can be interpreted as a state-vector. Then, given a set of n real symmetric $n \times n$ matrices, $A^i = [a^i_{jk}]$, a quadratic differential system $\mathcal{S}$ is defined by

$$\dot{x}^i = \tilde{x} A^i x \quad i = 1, \cdots, n. \tag{1}$$

Where the denotes time differentiation and ~ the transpose. Written in component form these equations are

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} \sum_{k=1}^{n} a^i_{jk} x^j x^k. \tag{1a}$$

There is of course no loss in generality in normalizing the A^i such that they are symmetric, since their off-diagonal entries are always paired in a sum in (1a) anyway.

As a side point, from (1a) one notes that the superscripts on the x's of (1a) are used not to denote powers but the components of the vector x, in order to distinguish x as a contravariant vector [16, p. 270]. Indeed, by making a change of variable, $x = By$ with B a real, constant, nonsingular matrix, equivalent systems are found; a study of these equivalent systems yields the same information as a study of the original system. In the $n = 2$ case Markus has catalogued, with the use of these variable changes, all such systems giving the nature of their phase-plane portraits [15, p. 208]. Here we merely point out that, if one defines a critical point of a differential system $\dot{x} = f(x)$ as an x_c such that $f(x_c) = 0$ [5, p. 38], then the zero vector $x = 0$ is a critical point of (1).

The notation is illustrated by

$$\dot{x}^1 = kx^2 \cdot x^2 \qquad k \text{ real} \tag{2a}$$

$$\dot{x}^2 = 2x^1 \cdot x^2 + x^2 \cdot x^2 \tag{2b}$$

which has $n = 2$ and

$$A^1 = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \tag{2c}$$

If we denote by $\mathcal{S}_{i \cdot j}$ the system classified as the jth system in the ith theorem of Markus [15], (2) is found to define $\mathcal{S}_{6.10}$. For $\mathcal{S}_{6.10}$, since $\dot{x}^1 = \dot{x}^2 = 0$ if $x^2 = 0$, the x^1 axis is a line of critical points.

III. Canonic (Nonassociative) Algebra

"And on this island he planted trees of all kinds and every sort of flower and fruit-tree that exists in this world was to be found there." [1, p. 9]

Although it was by nature of determining the critical points and phase-plane behavior that Markus was successful in his introduction of a related algebra, we shall turn, in Section IV, to such an algebra for obtaining explicit solutions for the quadratic differential system equations, (1). At this point we set up this related algebra and show that within it the system equations take a canonical square law form as in (7).

We recall that by an algebra $\mathcal{A}$ is meant a vector space in which any two vectors can be multiplied to give another vector in the space [17, p. 225]. Since any vector can be represented in terms of its basis vectors, an algebra is defined once the multiplication table of basis vectors is specified. Consequently, to relate an algebra to our quadratic differential systems (1) we take the underlying vector space to have n basis vectors, $u_1, \cdots, u_n$, all assumed time independent. Thus any vector x in the algebra $x = x(t) \in \mathcal{A}$, can be written as

$$x = \sum_{i=1}^{n} x^i u_i \tag{3}$$

where, as before, we take the x^i to be real-valued functions of time. This latter being the case, it is worth the parenthetical comment that we are really working in an algebra defined on a module [18, p. 106] rather than on a vector space, but since the concepts differ little, for physical insights we shall continue to refer to the elements of $\mathcal{A}$ as vectors. Too, as will become clear, it is not necessary to calculate or spell out the basis vectors more than by just labeling them with indices.

Again following Markus [15, p. 186], we set up the multiplication table for basis vectors by

$$u_j \cdot u_k = \sum_{i=1}^{n} a^i_{jk} u_i. \tag{4}$$

This defines $\mathcal{A}(\mathcal{S})$, the algebra related to $\mathcal{S}$ of (1). Conversely, given a multiplication table (4) with real $a^i_{jk} = a^i_{kj}$, a quadratic differential system is related to it by (1). It is worth noting that the multiplication table of basis vectors is simply $\Sigma A^i u_i$ when expressed as an array.

As an example, from (2c) we have the multiplication table for basis vectors in $\mathcal{A}(\mathcal{S}_{6.10})$ given by

$$\begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 \\ u_2 \cdot u_1 & u_2 \cdot u_2 \end{bmatrix} = A^1 u_1 + A^2 u_2 = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix} u_1 + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} u_2 = \begin{bmatrix} 0 & u_2 \\ u_2 & ku_1 + u_2 \end{bmatrix}.$$

We note that $u_1 \cdot u_1 = 0$ (that is, u_1 is a nilpotent element of the algebra) in line with the x^1-axis being a line of critical points of (2). Also we note that if $k \neq 0$, this example has

$$u_2 = u_1 \cdot (u_2 \cdot u_2) \neq (u_1 \cdot u_2) \cdot u_2 = ku_1 + u_2. \tag{5a}$$

Consequently this particular algebra is not associative [associativity meaning $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathcal{A}$]. All of the algebras are commutative, however, by virtue of $\tilde{A}^i = A^i$ [commutativity meaning $x \cdot y = y \cdot x$ for all $x, y \in \mathcal{A}$]. From these comments, and what follows below, it will be appreciated that the general theory of nonassociative algebras is important to further development of the subject. Although, when compared to the associative case [19], general works are few [20] [21], still there are available a number of research papers [22]–[31] and a good survey paper on structure theory [32].

Going now beyond Markus, we observe on using (1) in (3) that

$$\dot{x} = \sum_{i=1}^{n} \dot{x}^i u_i = \sum_{i=1}^{n} \tilde{x} A^i x u_i \tag{6a}$$

$$x \cdot x = \left(\sum_{j=1}^{n} x^j u_j\right) \cdot \left(\sum_{k=1}^{n} x^k u_k\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} x^j x^k u_j \cdot u_k$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} x^j x^k a^i_{jk} u_i = \sum_{i=1}^{n} \tilde{x} A^i x u_i. \tag{6b}$$

Noting that (6a) and (6b) are identical, we have in $\mathcal{A}(\mathcal{S})$ the key equation

$$\dot{x} = x \cdot x \tag{7}$$

irrespective of $\mathcal{S}$. That is, when interpreted in the related algebra $\mathcal{A}(\mathcal{S})$, the differential equation describing the (quadratic differential) system $\mathcal{S}$ is always the same. Or again, $\dot{x} = x \cdot x$ is a universal canonical differential equation describing all quadratic differential systems. Once we solve (7) we should have a universal solution to all quadratic differential systems. Of course to solve (7) we need to use the initial conditions $x_0 = x_0(t_0)$ for (1), these being represented in the related algebra by

$$x_0 = \sum_{i=1}^{n} x^i(t_0) u_i. \tag{7a}$$

We next turn to the solution of (7).

IV. Canonic Solutions

"Can that be the light of fire that glows yonder? Whisht! It may be the gleam of the gem on some great dragon's head!" [1, p. 13]

Classically, the equation $\dot{x} = x \cdot x$ is used to illustrate various properties of ordinary differential equations [5, p. 10], [7, p. 111] with its solutions being well known. These solutions are

$x(t) = x_0/[1 - (t - t_0)x_0]$; if t is considered a complex variable this solution is valid for all $t \neq t_0 + (1/x_0)$, $x(t)$ being an analytic function everywhere but at this singularity [3, p. 213]. Our case is somewhat similar, but differs in that our equation $\dot{x} = x \cdot x$ has as its domain of definition the possibly nonassociative algebra $\mathfrak{A}(\mathcal{S})$ rather than a real-valued function space. Using, then, similar ideas we can proceed to a solution in which it should be clear that it is through the domain of definition that the nature of the original system $\mathcal{S}$ will appear.

Considering that "analyticity" of the solution is still expected in the variable t, with coefficients in $\mathfrak{A}(\mathcal{S})$, we assume a power series expansion for the solution and let (and here the superscript i denotes the ith power)

$$x(t) = \sum_{i=0}^{\infty} (t - t_0)^i a_i \tag{8}$$

where the a_i are constant vectors in $\mathfrak{A}(\mathcal{S})$. Then

$$\dot{x} = \sum_{i=0}^{\infty} i(t - t_0)^{i-1} a_i \tag{9a}$$

$$x \cdot x = \left[\sum_{j=0}^{\infty} (t - t_0)^j a_j\right] \cdot \left[\sum_{k=0}^{\infty} (t - t_0)^k a_k\right] = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (t - t_0)^{j+k} a_j \cdot a_k. \tag{9b}$$

The solution to $\dot{x} = x \cdot x$ is formed by equating coefficients of $(t - t_0)^{i-1}$ in (9); this gives, since $j + k = i - 1$

$$i a_i = \sum_{j=0}^{i-1} a_j a_{i-1-j} \qquad i = 1, 2, \cdots \tag{10}$$

subject to the initial condition

$$a_0 = x_0 = x(t_0). \tag{10a}$$

Equations (10) can be solved iteratively, and, upon using commutativity of the algebra to simplify, we obtain

$$a_1 = a_0 \cdot a_0 \tag{11a}$$

$$a_2 = \tfrac{1}{2}[a_0 \cdot a_1 + a_1 \cdot a_0] = \tfrac{1}{2}[a_0 \cdot (a_0 \cdot a_0) + (a_0 \cdot a_0) \cdot a_0] = a_0 \cdot (a_0 \cdot a_0) \tag{11b}$$

$$a_3 = \tfrac{1}{3}[a_0 \cdot a_2 + a_1 \cdot a_1 + a_2 \cdot a_0] = \tfrac{1}{3}[2a_0 \cdot (a_0 \cdot \{a_0 \cdot a_0\}) + (a_0 \cdot a_0) \cdot (a_0 \cdot a_0)] \tag{11c}$$

etc.

Further simplification can not generally occur in (11) due to the nonassociativity of the algebra.

An expansion of the coefficient vectors a_i of the solution in terms of the basis vectors u_i, with an identification of the resulting expansion of (8) with (3), yields the desired $x^i(t)$ which solve the original quadratic differential system (1).

To illustrate let us consider $\mathcal{S}_{6,9}$ [15, p. 194] which is defined by

$$\dot{x}^1 = -x^2 \cdot x^2 \tag{12a}$$

$$\dot{x}^2 = 2x^1 \cdot x^2. \tag{12b}$$

This has as its bases product table

$$\begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 \\ u_2 \cdot u_1 & u_2 \cdot u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} u_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u_2. \tag{12c}$$

If the initial conditions are $t_0 = 0$, $x^1(0) = 0$, $x^2(0) = \beta$, then, using (10a) & (11)

$$a_0 = x^1(0)u_1 + x^2(0)u_2 = \beta u_2 \tag{12d}$$

$$a_1 = (\beta u_2) \cdot (\beta u_2) = -\beta^2 u_1 \tag{12e}$$

$$a_2 = (\beta u_2) \cdot a_1 = -\beta^3 u_2 \tag{12f}$$

$$a_3 = \tfrac{1}{3}[2(\beta u_2) \cdot (-\beta^3 u_2) + (-\beta^2 u_1) \cdot (-\beta^2 u_1)] = \tfrac{2}{3}\beta^4 u_1 \tag{12g}$$

etc.

Hence (8) becomes

$$x = \beta u_2 - t\beta^2 u_1 - t^2 \beta^3 u_2 + \tfrac{2}{3} t^3 \beta^4 u_1 + \cdots \tag{12h}$$

$$= x^1 u_1 + x^2 u_2 \tag{12i}$$

giving

$$x^1(t) = -\beta^2 t + \tfrac{2}{3}\beta^4 t^3 + \cdots \tag{12j}$$

$$x^2(t) = \beta - \beta^3 t^2 + \cdots. \tag{12k}$$

Some special cases of the theory are of interest. If the related algebra is power-associative (that is, all powers of a given element associate [27, p. 552]), then (11) shows that a_i is the $(i + 1)$st power of a_0

$$a_i = a_0^i \cdot a_0 = a_0^{i+1} \quad \text{(power-associative)} \tag{13a}$$

and (8) becomes

$$x = \sum_{i=0}^{\infty} [(t - t_0)a_0]^i \cdot a_0 \quad \text{(power-associative)}. \tag{13b}$$

In the further instance where division is possible [33] [34], e.g., if $\mathfrak{A}(\mathcal{S})$ is a division algebra, in which an identity e can always be assumed present [25. p. 704], then

$$x = a_0 \cdot [e - (t - t_0)a_0]^{-1} \quad \begin{pmatrix}\text{power-associative} \\ \text{division algebra}\end{pmatrix} \tag{14}$$

which reduces to the classical result in the scalar functional case.

Among commutative nonassociative algebras, the division algebras are relatively scarce. For $n \leqslant 4$, though, a number of interesting division algebras have been investigated [33] [34], and it is shown that for $n = 2$ there is essentially only one division algebra. Further, it is known that in any real division algebra there is an element b satisfying $b^2 = -e$ [25, p. 705], and, thus, quantities which behave as complex numbers.

Examples of power-associative algebras are $\mathfrak{A}(\mathcal{S}_{6,10})$ of (2) when $k = 2$, as well as $\mathfrak{A}(\mathcal{S}_{7,1})$ defined by $u_1 \cdot u_1 = u_1$, $u_1 \cdot u_2 = 0$, $u_2 \cdot u_2 = u_2$. In fact any real commutative algebra $\mathfrak{A}$ is power-associative if and only if $((x \cdot x) \cdot x) \cdot x = (x \cdot x) \cdot (x \cdot x)$ for all $x \in \mathfrak{A}$ [27, p. 554]. The division algebra for $n = 2$ will be covered in the next section.

Before that, however, it is worth a comment on the validity of the power series solution (8) for x. Because the original quadratic differential system (1) when written as $\dot{x} = f(x)$, is described by a continuous and locally Lipschitz $f(\cdot)$ (i.e. [7, p. 110], there is a $k > 0$ such that, with $\|\cdot\|$ denoting the norm in R^n, $\|f(x_1) - f(x_2)\| < k\|x_1 - x_2\|$ when x_1 and x_2 are finite), there is a unique solution valid in a neighborhood of t_0 [7, p. 110]; consequently, there exists an $\epsilon > 0$ such that for all $t \epsilon [t_0 - \epsilon, t_0 + \epsilon]$ the power series expansion (8), $x = \Sigma(t - t_0)^i a_i$, will yield the desired solution, given the initial conditions $x_0 = x(t_0) = a_0$. Where possible this solution can be continued [7, p. 111], even as in the scalar case for complex t. Thus although in specific instances other solution methods may prove more convenient, the power series given here with its recursion relation (10) for the coefficients is universally applicable.

V. Division Algebra Example

"Come let us go and see what it was that glowed like fire last night." [1, p. 13]

In order to tie the various ideas together this is a good point to discuss the canonical power-associative division algebra catalogued for $n = 2$ through $\mathcal{S}_{8,4}$ in [15, p. 201]. This has

$$\dot{x}^1 = x^1x^1 - x^2x^2 \tag{15a}$$

$$\dot{x}^2 = 2x^1x^2 \tag{15b}$$

with the basis vectors for $\mathcal{A}(\mathcal{S}_{8,4})$ consequently satisfying

$$\begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 \\ u_2 \cdot u_1 & u_2 \cdot u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u_2. \tag{15c}$$

We observe directly that the identity is $e = u_1$. Letting $z = [au_1 + bu_2] \in \mathcal{A}(\mathcal{S}_{8,4})$ we can solve for z^{-1} by setting $z^{-1} = [cu_1 + du_2]$ giving, on using (15c),

$$e = u_1 = [au_1 + bu_2] \cdot [cu_1 + du_2]$$
$$= [ac - bd]u_1 + [ad + bc]u_2$$

or

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

which always has a unique solution if $z \neq 0$ since its determinant is $\Delta = a^2 + b^2 > 0$. Thus $z^{-1} = [(a/\Delta)u_1 - (b/\Delta)u_2]$. Note that $u_2 \cdot u_2 = -e$.

Now, let arbitrary initial conditions be given as

$$x_0^1 = \alpha, \quad x_0^2 = \beta \tag{16a}$$

for which

$$a_0 = \alpha u_1 + \beta u_2 = x_0 \tag{16b}$$

Choosing $t_0 = 0$, we have, using z^{-1} found above to calculate the indicated inverse,

$$[e - ta_0]^{-1} = [(1 - t\alpha)u_1 - t\beta u_2]^{-1}$$
$$= \frac{(1 - t\alpha)}{\Delta} u_1 + \frac{t\beta}{\Delta} u_2 \tag{16c}$$

where $\Delta = (1 - t\alpha)^2 + (t\beta)^2$. Consequently, (14), the solution in the algebra is, on using (15c),

$$x = [\alpha u_1 + \beta u_2] \cdot \left[\frac{(1 - t\alpha)}{\Delta} u_1 + \frac{t\beta}{\Delta} u_2\right]$$
$$= \left[\frac{\alpha(1 - t\alpha) - t\beta^2}{\Delta}\right] u_1 + \left[\frac{\beta}{\Delta}\right] u_2. \tag{16d}$$

Then (3) yields

$$x^1 = \frac{\alpha(1 - t\alpha) - t\beta^2}{(1 - t\alpha)^2 + (t\beta)^2} \tag{16e}$$

$$x^2 = \frac{\beta}{(1 - t\alpha)^2 + (t\beta)^2}. \tag{16f}$$

If both initial conditions are nonzero, this solution is seen to hold for all t.

VI. Reduction to Quadratic Form

"In truth this universe is manifold, and manifold are the kinds that dwell in it." [1, p. 11]

To this point our discussion has been primarily upon quadratic differential systems. Here we show that such systems are in a sense universal in that all other polynomic differential systems can be reduced to the quadratic ones. The polynomic differential systems are in turn somewhat universal in that others can be brought into their form, either exactly or approximately. Although these comments are true, it will be observed that the reductions to be discussed all come about through something extraneous. Nevertheless, the universality of the quadratic differential systems is illustrated.

A. Polynomic Systems

By a polynomic differential system is meant one which is described in terms of multivariable polynomials $p_i(\cdot)$ with real constant coefficients through

$$\dot{x}^i = p_i(x^1, x^2, \cdots, x^n) \quad i = 1, \cdots, n. \tag{17}$$

First we cover homogeneous polynomials, that is ones for which all terms have the same degree, and then reduce nonhomogeneous ones to homogeneous form.

1) Homogeneous p_i: Assuming that each $p_i(\cdot)$ is homogeneous of degree greater than two, we increase the dimension by introducing new, product, variables. On iterating the process we can reduce the system to an equivalent quadratic one.

Let (recall that the superscripts are indices rather than powers)

$$\dot{x}^i = \Sigma\, a^i_{klm\ldots} x^k x^l x^m \cdots \quad i = 1, \cdots, n \tag{18}$$

where the summation is on the indices $k, l, m, \cdots$, and introduce the new variables, $n \times n$ in number,

$$y^{i,j} = x^i x^j \quad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n. \tag{19}$$

Then we have

$$\dot{x}^i = \Sigma\, a^i_{klm\ldots} y^{k,l} x^m \cdots \tag{20a}$$

$$\begin{aligned} \dot{y}^{i,j} &= \dot{x}^i x^j + x^i \dot{x}^j \\ &= \Sigma\, a^i_{klm\ldots} y^{k,l} x^j x^m \cdots + \Sigma\, a^j_{klm\ldots} y^{k,l} x^i x^m \cdots \\ &= \Sigma\, a^i_{klm\ldots} y^{k,l} y^{j,m} \cdots + \Sigma\, a^j_{klm\ldots} y^{k,l} y^{i,m} \cdots \end{aligned} \tag{20b}$$

from which it is seen that these polynomials are all of one degree less than the originals. It will be seen, also, that in some cases it is only necessary to introduce some of the $y^{i,j}$, as shown by the following example.

Let

$$\dot{x}^1 = x^1x^2$$
$$\dot{x}^2 = x^1x^2x^2.$$

Then

$$y^{1,2} = x^1x^2$$

giving

$$\dot{y}^{1,2} = \dot{x}^1x^2 + x^1\dot{x}^2 = x^1x^2x^2 + x^1x^1x^2x^2$$
$$= y^{1,2}x^2 + y^{1,2}y^{1,2}.$$

Here the $y^{1,1}$ and $y^{2,2}$ equations are unnecessary, and we can take for our equivalent quadratic system

$$\dot{x}^1 = x^1 x^2$$
$$\dot{x}^2 = x^2 y^{1,2}$$
$$\dot{y}^{1,2} = x^2 y^{1,2} + y^{1,2} y^{1,2} \quad \text{with } y_0^{1,2} = x_0^1 x_0^2.$$

2) Nonhomogeneous p_i: If the p_i are not homogeneous we can bring them into homogeneous form by increasing the dimension of the space through the introduction of a new variable w such that

$$\dot{w} = 0, \quad w(t_0) = 1 = w_0.$$

Terms not of the highest power in p_i then get multiplied by the necessary power of w to make the polynomials homogeneous.

For example

$$\dot{x}^1 = 1 + x^1 x^2 x^2$$
$$\dot{x}^2 = x^1 x^2$$

can be made homogeneous by setting, since $w(t) \equiv 1$

$$\dot{x}^1 = www + x^1 x^2 x^2$$
$$\dot{x}^2 = x^1 x^2$$
$$\dot{w} = 0 \quad \text{with } w_0 = 1.$$

3) Linear p_i: If the polynomials p_i are linear they can also be brought into homogeneous quadratic form by the introduction of an auxiliary variable [15, p. 192], in the same manner as just discussed. Although one might expect the related algebra to be associative, this is not generally the case, as seen by the following example, so (11) must be used to evaluate the power series solution (8), which, however, must turn out to be in the customary exponential form valid for all t.

As an example consider the scalar equation

$$\dot{z} = 2kz \quad k = \text{real constant}.$$

This can be rewritten, with $x^2 = z$, as

$$\dot{x}^1 = 0 \quad x_0^1 = k$$
$$\dot{x}^2 = 2x^1 x^2$$

which is $\mathcal{S}_{6,2}$, having $u_1 \cdot u_1 = u_2 \cdot u_2 = 0, u_1 \cdot u_2 = u_2$. Since

$$[u_2 \cdot (u_1 + u_2)] \cdot (u_1 + u_2) = u_2 \neq 0$$
$$= u_2 \cdot [(u_1 + u_2) \cdot (u_1 + u_2)]$$

we see that the algebra is not associative. If

$$a_0 = \kappa u_1 + \alpha u_2$$

then (11) gives

$$a_1 = 2\alpha k u_2, \quad a_2 = \frac{1}{2!}\alpha(2k)^2 u_2, \quad a_3 = \frac{1}{3!}\alpha(2k)^3 u_2, \cdots$$

Thus, from (8), using $t_0 = 0$

$$x(t) = ku_1 + \alpha[1 + 2kt + \frac{1}{2!}(2kt)^2 + \frac{1}{3!}(2kt)^3 + \cdots]u_2$$

or

$$x^1 = k, \quad x^2 = \alpha e^{2kt}$$

as expected.

B. Nonstationary Systems

If we have

$$\dot{x} = f(x, t) \tag{21}$$

we can again increase the dimension by setting [7, p. 141]

$$y = \begin{bmatrix} x \\ t \end{bmatrix}, \quad g(y) = \begin{bmatrix} f(x,t) \\ 1 \end{bmatrix}, \quad z = t \tag{22}$$

for which our system becomes stationary.

$$\frac{dy}{dz} = g(y). \tag{23}$$

Again, although a valid technique, some loss of insight seems to occur through this increase of dimension [35].

C. Nonpolynomic Systems

If in the n-vector equation

$$\dot{x} = f(x)$$

f consists of continuous real-valued functions, then, over any compact interval of the n-dimensional cartesian space R^n, these functions can be uniformly approximated by n-variable polynomials p_i using the Weierstrass approximation theory [36, p. 69]. The resulting equations (17) will thus be approximations, as well as their solutions found via (10). However, over any finite time interval the approximate solution can be made as close as desired to the actual solution [7, p. 102] by making the Weierstrass polynomial approximation as close as necessary.

Consequently, we see that our quadratic differential systems are essentially universal, especially for purposes of engineering analysis.

VII. Discussion

"The Raja then stayed for ten days, and on the eleventh day he departed with traditional ceremony taking with him the princess." [1, p. 4]

Given a nonlinear differential system it has been shown here how to bring it into quadratic form; in this sense the quadratic differential systems (1) $\dot{x}^i = \Sigma\Sigma\, a_{jk}^i x^j x^k$, with which we started, are universal. By embedding this in a related algebra all such systems are seen to satisfy the canonic differential (7), $\dot{x} = x \cdot x$, for which the solution can be given in terms of the power series (8), $x = \Sigma(t - t_0)^i a_i$, where the constants satisfy the recursion relations (10), $ia_i = \Sigma_{j=0}^{i-1} a_j a_{i-1-j}$; these in turn hold universally with numerical values being fixed by the initial conditions, $a_0 = x_0$, and the structure of the algebra. This latter is further fixed in its turn by the multiplication constants a_{jk}^i of the algebra basis vectors, these being given through (4), $u_j \cdot u_k = \Sigma\, a_{jk}^i u_i$, which is fixed by the original system (1). With this, it is seen how the theory is completed within itself.

It is to be emphasized that the main point is the universality of $\dot{x} = x \cdot x$, accomplished through the introduction of an algebra. Once this has been realized, the equation and the algebra become available for engineering design or analysis using the recursion relations for the power series solution.

Besides the real systems discussed here the theory directly applies to any differential system with coefficients lying in any time-independent ring, for example, those with complex coefficients. Indeed the formulation holds for general differential fields [13], while it also may be possible by incorporating variable bases, with $\dot{u}_i$ generally nonzero, to handle time-variable systems without using the technique of replacing t by a new variable discussed in Section VI-B. For driven systems one would, though, like to see a different technique for introducing the input into the canonical equation $\dot{x} = x \cdot x$, such that the ideas could be used more for engineering design by including the input in an identifiable manner throughout. With further regard to Section VI, since the reduction to quadratic systems always proved possible one is curious if a similar reduction to linear equations would be possible. This is indeed true for alternative algebras, as will be discussed elsewhere. It should, though, be commented that the theory is readily developed in the noncommutative case, the main difference being in the form of the a_i; commutativity is, therefore, only used as a convenience.

Since many physical systems are essentially polynomic, development of the ideas toward design would be natural. This would be particularly true in the area of MOS circuits where the basic transistor law is a square one [37, p. 59], or in neural feedback systems where the basic nonlinearity is cubic [38].

By way of further areas for development we mention that although Markus gave a complete classification of quadratic systems for $n = 2$ [15], this classification has not as yet been undertaken for higher n. Nor has the region of convergence of the solution (8), which is dependent upon the system, been fully investigated. Since the structures of algebras are rather developed within mathematics [32], the field opens up a number of opportunities for further developments within the algebraic treatment of differential systems. Too, because of the nature of the algebra and solutions within it, the theory is convenient for computer-aided design of nonlinear systems. An initial program has been made by F. J. Belaza of Madrid.

Finally it should be commented that historically equation (3) was formed already in 1893 by Scheffers [39]. But, although attempts were made to differentiate with respect to vectors, little seems to have been accomplished in this field prior to the work of Markus [15], which, as we see here, gives the first step toward the general theory.

Epilogue:

> "Brilliant as the light of the sun was the light of her countenance, and she was endowed moreover with great wisdom and understanding." [1, p. 2]

Acknowledgment

The author wishes to acknowledge discussions with R. Rassai-Gilani and F. El-Mokadem which assisted in the presentation of the ideas, the encouragement and assistance of N. Bose (and his reviewer) in preparing the paper, the enthusiasm of his Malaysian colleagues, and the faith of Z. Azziz to whom the ideas are dedicated.

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