Livsic's Chain Synthesis

T. T. Ha and R. W. Newcomb
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

ABSTRACT

The factorization method of M. S. Livsic for J-lossless matrices is presented in concise form.

"on the other bank, the palms beckoned us". [1]

I. INTRODUCTION

In the book of M. S. Livsic [2] a factorization theory for J-lossless matrices is presented which is useful for the cascade synthesis of lossless networks. Unfortunately this is spread throughout a large segment of the book and in a language not too familiar to western engineers. Consequently we here attempt to concisely present the ideas and in a form more accessible to western engineers. Although we hope this paper will be self-contained we point out the companion paper [3] where the background ideas are presented.

II. PRELIMINARIES

We consider as given an $n \times n$ matrix transmission operator $S(p)$ of the complex frequency variable $p = \sigma + jw$. associated with $S(p)$ is a constant matrix J satisfying $J = J^a$, where superscript a means adjoint, and $J^2 = I$, with I the identity. Physically a transmission operator can be interpreted as a chain, scattering or transfer scattering matrix (or various combinations of these), the interpretation depending upon the associated J. Thus, for example, $J = I$ corresponds to the scattering matrix while $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is associated with the chain matrix [3].

We assume, as per [3], that the transmission operator can be written in the form

$$S(p) = I + J\Gamma^a[T-pI]^{-1}\Gamma \tag{1a}$$

where T, the interior operator, and Γ, the input to state space operator, are consequently taken as known. The interior operator is also taken to satisfy the relationship [3]

$$T + T^a = -\Gamma J\Gamma^a \tag{1b}$$

Knowing the transmission operator in this form we clearly know the input to interior operator

$$R = [T-pI]^{-1}\Gamma \tag{2a}$$

from which

$$S(p) = I + J\Gamma^a R \tag{2b}$$

As a final preliminary we comment aht if Q is an isometry, $Q^aQ = I$, [4, p.15], then the replacement

$$\hat{T} = QTQ^a \tag{3a}$$

$$\hat{\Gamma} = Q\Gamma \tag{3b}$$

leaves $S(p)$ invariant, that is

$$S(p) = I+J\Gamma^a[T-pI]^{-1}\Gamma = I+J\hat{\Gamma}^a[\hat{T}-pI]^{-1}\hat{\Gamma} \tag{4a}$$

though we find

$$\hat{R} = Q^aR \tag{4b}$$

III. FACTORIZATION - RATIONAL CASE

At this point we assume that $S(p)$ is rational in p in which case T in (1a) can be taken to be an $m \times m$ matrix operator with m finite. By putting T in lower triangular form, (5), we will show that a decomposition of the interior operator R, (10), results to yield a factorization, (12).

By standard techniques [5, p. 75] there exists a unitary transformation Q, which is a special case of an isometry, for which T can be put in lower triangular form through (3a). Thus we can assume that T is in lower triangular form and, hence, by partition, written as

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{12} & T_{22} \end{bmatrix} \tag{5}$$

where T_{11} and T_{22} are also lower triangular, with T_{11} of any desired number of rows (smaller than m, assuming $m \geq 1$).

Assuming now that T is in lower triangular form and to go along with the partition of (5) we define projection matrix operators

$$\underset{\sim}{p}_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underset{\sim}{p}_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix} \tag{6}$$

where I_1 and I_2 are identities of appropriate dimension; here $\underset{\sim}{p}_1 + \underset{\sim}{p}_2 = I$. By (5)

$$T = (\underset{\sim}{p}_1+\underset{\sim}{p}_2)T(\underset{\sim}{p}_1+\underset{\sim}{p}_2) = \underset{\sim}{p}_1 T\underset{\sim}{p}_1+\underset{\sim}{p}_2 T\underset{\sim}{p}_2+\underset{\sim}{p}_2 T\underset{\sim}{p}_1 \tag{7a}$$

$$= T_1 + T_2 + \underset{\sim}{p}_2 T\underset{\sim}{p}_1 \tag{7b}$$

$$\Gamma = (\underset{\sim}{p}_1+\underset{\sim}{p}_2)\Gamma = \underset{\sim}{p}_1\Gamma + \underset{\sim}{p}_2\Gamma \tag{7c}$$

$$= \Gamma_1 + \Gamma_2 \tag{7d}$$

from which our definitions for $T_1, T_2, \Gamma_1, \Gamma_2$ should be clear. Further, as $\underset{\sim}{p}_1 T\underset{\sim}{p}_2 = 0$,

$$\underset{\sim}{p}_2 T\underset{\sim}{p}_1=\underset{\sim}{p}_2 T\underset{\sim}{p}_1+0=\underset{\sim}{p}_2 T\underset{\sim}{p}_1+\underset{\sim}{p}_2 T^a\underset{\sim}{p}_1=\underset{\sim}{p}_2(T+T^a)\underset{\sim}{p}_1 \tag{7e}$$

$$= -\underset{\sim}{p}_2(\Gamma J\Gamma^a)\underset{\sim}{p}_1 \tag{7f}$$

$$= -\Gamma_2 J\Gamma_1^a \tag{7g}$$

With these points in mind we can invert $T-pI$ in a relatively straightforward manner to get

$$[T-pI]^{-1} = \begin{bmatrix} (T_{11}-pI_1)^{-1} & 0 \\ -(T_{22}-pI_2)^{-1}T_{21}(T_{11}-pI_1)^{-1} & (T_{22}-pI_2)^{-1} \end{bmatrix} \tag{8a}$$

$$= \underset{\sim}{p}_1(T_1-pI)^{-1}\underset{\sim}{p}_1+\underset{\sim}{p}_2(T_2-pI)^{-1}\underset{\sim}{p}_2$$

$$-\underset{\sim}{p}_2(T_2-pI)^{-1}\underset{\sim}{p}_2 T\underset{\sim}{p}_1(T_1-pI)^{-1}\underset{\sim}{p}_1 \tag{8b}$$

$$= \underset{\sim}{p}_1(T_1-pI)^{-1}\underset{\sim}{p}_1+\underset{\sim}{p}_2(T_2-pI)^{-1}\underset{\sim}{p}_2$$

$$+\underset{\sim}{p}_2(T_2-pI)\Gamma_2 J\Gamma_1^a(T_1-pI)^{-1}\underset{\sim}{p}_1 \tag{8c}$$

consequently we obtain, on directly using (8c) in (2),

$$R=[T-pI]^{-1}\Gamma \tag{9a}$$

$$=\underset{\sim}{p}_1(T_1-pI)^{-1}\Gamma_1+\underset{\sim}{p}_2(T_2-pI)^{-1}\Gamma_2$$

$$+\underset{\sim}{p}_2(T_2-pI)^{-1}\Gamma_2 J\Gamma_1^a(T_1-pI)^{-1}\Gamma_1 \tag{9b}$$

$$=\underset{\sim}{p}_1(T_1-pI)^{-1}\Gamma_1+\underset{\sim}{p}_2(T_2-pI)^{-1}\Gamma_2$$

$$[I + J\Gamma_1^a(T_1-pI)^{-1}\Gamma_1] \tag{9c}$$

$$=R_1+R_2S_1 \tag{10}$$

where

$$R_1=\underset{\sim}{p}_1(T_1-pI)^{-1}\Gamma_1 \tag{9d}$$

$$R_2=\underset{\sim}{p}_2(T_2-pI)^{-1}\Gamma_2 \tag{9e}$$

$$S_1=I+J\Gamma_1^a(T_1-pI)^{-1}\Gamma_1 = I+J\Gamma_1^a R_1 \tag{9f}$$

$$S_2=I+J\Gamma_2^a(T_2-pI)^{-1}\Gamma_2 = I+J\Gamma_2^a R_2 \tag{9g}$$

Equation (10) is the main one which allows the factorization, for

$$S=I+J\Gamma^a[T-pI]^{-1}\Gamma \tag{11a}$$

$$=I+J(\Gamma_1^a+\Gamma_2^a)[R_1+R_2S_1] \tag{11b}$$

$$=I+J\Gamma_1^a R_1+J\Gamma_2^a R_2 S_1 \quad (\text{as } \Gamma_2^a R_1=0) \tag{11c}$$

$$=S_1+J\Gamma_2^a R_2 S_1 = (I+J\Gamma_2^a R_2)S_1 \tag{11d}$$

$$=S_2S_1 \tag{12}$$

Having obtained the desired factorization, (12), we comment that if the degree $\delta[S]$, [6, p. 195] is m, that is the state-space realization of $S(p)$ for (1a) is minimal, then

$$\delta[S] = \delta[S_1] + \delta[S_2] \quad (13)$$

as seen from (9) using $\underset{\sim}{P}_1 + \underset{\sim}{P}_2 = I$.

IV. FACTORIZATION - IRRATIONAL CASE

If $S(p)$ is irrational in p then the above procedure still works whenever T can be put into the lower triangular form (5) by an isometry Q. The disucssion in [7, p. 73] indicates that this will always be the case if S is a contraction. See also [2, Chap. 6].

V. DISCUSSION

Given an $S(p)$ in the form of (1a) we have presented the ideas of Livsic which lead to its factorization, as we have seen always in the rational case. This has the advantage that optimal degree reduction occurs. Too, if one desires degree one factors then such can be obtained by the use of a partition at (5) in which T_{11} is 1×1. However, it should be pointed out that the triangularization of T required may lead to complex valued T and Γ.

The $S(p)$ under consideration all satisfy

$$J - S^a(jw)JS(jw) = 0 \quad (14)$$

in which case they are often called J-lossless[8]. Such correspond physically to lossless network structures when $S(p)$ has appropriate analyticity properties to guarantee possivity.

"A few more strokes! The bank was now closed" [1].

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INTERNATIONAL SYMPOSIUM

ON

OPERATOR THEORY OF NETWORKS AND SYSTEMS

Volume I

AUGUST 12-14, 1975

MONTREAL