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A NONRECIPROCAL n-PORT BRUNE SYNTHESIS

by

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ABSTRACT

This work sets forth n-port designs
Including gyrators, as one soon finds.

Minim in element number reactive,
Bypassing complex constituents resistive,

In two cases, McMillan's model revised
Yields the skew type Brune sections emphasized.

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I. INTRODUCTION

Presently several nonreciprocal synthesis methods exist; [1], [2], [3]. The first of these, due to Oono, proceeds from the scattering matrix and gives an indirect proof of the fact that an impedance matrix can be realized by a number of reactive elements equal to the degree of the matrix. The third of these gives a simpler synthesis method, but no control is directly available over the number of reactive elements. The second method, due to Belevitch, gives a Brune synthesis which uses the minimum number of reactive elements. This synthesis relies heavily upon imaginary resistors and transformers, which to us are aesthetically unsatisfying. That such elements can be used is justified by Belevitch's definition of passivity, [2, p. 284], which, however, is not the same as the one due to Desoer and Kuh, [4, p. 417], which we would prefer to use.

Because of our desire to have a direct synthesis based upon only real-valued elements, but using the minimum number of inductors and capacitors, we have developed the Brune synthesis of this report. Since the essential step of the synthesis follows closely the ideas of McMillan, [5], the method is best thought of as a generalization of that of McMillan. One of the essential features of the synthesis is that it allows the exhibition of the typical non-reciprocal Brune section; as yet such has not been possible with the use of complex valued elements, [2, p. 294].

The reader is assumed familiar with basic matrix theory operations and n-port ideas. The properties of PR matrices, which are heavily relied upon, are covered in detail in an accompanying report, [6].

We will adhere to the following conventions. The complex frequency variable is taken as $p = \sigma + j\omega$, $j = \sqrt{-1}$; a superscript asterisk, *, denotes complex conjugation; a subscript asterisk, $_*$, denotes Hurwitz conjugation, that is, replacement of p by -p; a superscript tilde, $\sim$, denotes matrix transposition; a subscript H denotes the Hermitian part, that is, $2A_H = \tilde{A}^* + A$; 0_n and 1_n denote the n x n zero and identity matrices, respectively, while

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad (1)$$

$\dot{+}$ denotes the direct sum of matrices and $\delta(A)$ is McMillan's degree of the matrix A, which is discussed in Appendix 1. Inductors and capacitors are called reactive elements.

II. PRELIMINARY STEPS

Consider a given $n \times n$ PR impedance matrix, $Z(p)$. $Z(p)$ then, almost by definition, [6, def.], is rational in p with real coefficients and has $Z_H(p)$ positive semi-definite in the open right half-plane $\sigma > 0$; Z need not be symmetric. The problem is then to synthesize a network, N, consisting of a finite number of passive elements, having Z as its terminal impedance matrix and using $\delta(Z)$ reactive elements.

The procedure is analogous to the one-port Brune process and can be outlined as follows. All poles on the imaginary axis of Z and its remainder are removed and realized by a lossless network; this is repeated until an impedance matrix Z_k is obtained for which neither Z_k nor its inverse Y_k have any poles on the imaginary axis. A resistance is then extracted from one of the ports to yield a reduced matrix Z_m which has its Hermitian part singular at some point on the imaginary axis; Z_m is however, itself not singular. A Brune section is then extracted from Z_m and the process repeated until the degree falls to zero. The Brune extraction is the difficult part; the entire next section is devoted to obtaining it.

By a partial fraction expansion, or any other suitable method, we write

$$Z(p) = {}_{\ell}Z_1(p) + Z_1(p) \tag{2a}$$

where $Z_1(p)$ is PR and analytic on $\sigma = 0$ with $Z_1(\infty) \neq -\tilde{Z}_1(\infty)$, while ${}_{\ell}Z_1 = -{}_{\ell}\tilde{Z}_{1*}$. ${}_{\ell}Z_1$ is lossless, PR, and can be realized by $\delta({}_{\ell}Z_1)$ reactive elements; such a synthesis is given in [1, p. 155], but simpler ones are given in [8, p. 113]. This realization for ${}_{\ell}Z_1$ is then connected in series with a realization for Z_1 to obtain Z. If Z_1 is nonsingular, we form $Y_1 = Z_1^{-1}$ and extract the poles of Y_1 on the imaginary axis to get

$$Y_1(p) = {}_{\ell}Y_2(p) + Y_2(p) \tag{2b}$$

where Y_2 has all the properties of Z_1, and ${}_{\ell}Y_2$ has all the properties of ${}_{\ell}Z_1$. ${}_{\ell}Y_2$ is realized in a manner dual to ${}_{\ell}Z_1$, and the lossless network so obtained is connected in parallel with a realization for Y_2. If Z_1 is singular, we can form, [6, T-11],

$$Z_1(p) = \tilde{T}_1[Z_1'(p) \dotplus 0_{n-r_1}]T_1 \tag{3a}$$

$$= \tilde{T}_1' Z_1'(p) T_1' \tag{3b}$$

where r_1 is the rank of $Z_1(p)$ and T_1 is a nonsingular, real constant matrix with T_1' being the first r rows of T_1. Y_1 of (2b) is then replaced by $Y_1' = [Z_1']^{-1}$. Similarly, if Y_2 is singular, we can write $Y_2(p) = T_2' Y_2'(p) \tilde{T}_2'$ where T_2' is $n \times r_2$, r_2 being the rank of Y_2. Z_1 is realized by terminating a transformer (n+r)-port in a realization of $Z_1'(p)$, which is PR, similarly for a realization of Y_2 in terms of one for Y_2', [9, p. 307]. We then repeat this process until a PR matrix $Z_k(p)$ is obtained which is nonsingular in p with Z_k and $Y_k = Z_k^{-1}$ both analytic on $\sigma = 0$. We have

$$\delta(Z) = \delta({}_{\ell}Z_1) + \delta(Z_1) \tag{4a}$$

$$= \delta({}_{\ell}Z_1) + \delta({}_{\ell}Y_2) + \dots + \delta({}_{\ell}Z_k) + \delta(Z_k) \tag{4b}$$

by properties 3) and 4) of δ, Appendix 1, (in some cases ${}_{\ell}Z_k$ should be ${}_{\ell}Y_k$). If any singular matrices are met, then properties 6) and 7) of δ show that $\delta(Z_j') = \delta(Z_j)$, and clearly (4b) is still valid. Since the individual lossless subnetworks can be realized by $\delta({}_{\ell}Z_j)$ or $\delta({}_{\ell}Y_j)$ reactive elements, (4b) shows that at most $\delta(Z)-\delta(Z_k)$ reactive elements need be used in these lossless extractions.

Considering Z_k, which we assume to be $n \times n$, we now form the minimum matrix

$$Z_m(p) = Z_k(p) - [r \dotplus 0_{n-1}] \tag{5a}$$

where, [6, C-12],

$$r = \min_{0 \le \omega \le \infty} \Delta(\omega)/\Delta_{11}(\omega) \tag{5b}$$

with Δ and Δ_{11} the determinant and (1,1) minor, respectively, of $Z_{kH}(j\omega)$. $Z_m(p)$ is PR, and, if Z_m is nonsingular, then $Y_m = Z_m^{-1}$ is also minimum, that is, since $Y_{mH} = \tilde{Y}_m^* Z_{mH} Y_m$ the Hermitian parts of Z_m and Y_m are both singular at the $p_o = j\omega_o$ for which the minimum of (5b) occurs. We further have

$$\delta(Z_m) = \delta(Z_k) \qquad (6)$$

by properties 2) and 4) of δ, Appendix 1, while a network for Z_k results from that for Z_m by connecting a resistor in series with port one; clearly no reactive elements are used.

All the steps so far described can then be repeated until one obtains a zero immittance matrix, at which point the synthesis is complete, or a minimum matrix, also called Z_m, which is nonsingular at every point on the imaginary axis, and consequently also in the entire right half-plane. Up to this point the procedure is quite similar to that of Belevitch, except we believe that the resistance extraction of (5a) was originally due to Oono, [7, p. 168], in the symmetric case.

III. THE BRUNE SECTION

Attention is now turned to the realization of a minimum impedance matrix. We therefore assume that an n x n PR impedance matrix $Z_m(p)$ is given, which is analytic and nonsingular for each p on $\sigma = 0$ and for which $Z_{mH}(j\omega_o)$ is singular. Of course this means $\omega_o \neq 0$ or ∞.

We first write, with $p_o = j\omega_o$,

$$Z_m(p_o) = R + jX \tag{7}$$

where R and X are Hermitian; R is then singular as it is $Z_{mH}(j\omega_o)$. We can diagonalize the real part of R, R_{sy}, to $0_{n-\rho} \dotplus 1_\rho$ by a real congruency transformation, where ρ is the rank of R_{sy}, [10, p. 298]. Note that R_{sy} is positive semi-definite. The imaginary part of the transformed R, R'_{ss}, has its first n-ρ rows and columns zero, as a consequence of the PR character of R. By a real orthogonal congruency transformation on the already transformed R, which operates on only the last ρ rows and columns, we can skew diagonalize R'_{ss} to the direct sum of zeros and multiples of E, (1), [10, p. 285]. Consequently we can find a (nonsingular) real T_m such that

$$Z'_m(p) = \tilde{T}_m Z_m(p) T_m \tag{8a}$$

$$Z'_m(p_o) = \{0 \dotplus \ldots \dotplus 0 \dotplus \begin{bmatrix} 1 & j\alpha_1 \\ -j\alpha_1 & 1 \end{bmatrix} \dotplus \ldots \dotplus \begin{bmatrix} 1 & j\alpha_q \\ -j\alpha_q & 1 \end{bmatrix} \dotplus 1 \dotplus \ldots \dotplus 1\} + jX' \tag{8b}$$

If $\rho = n$, which is possible even though R has rank less than n, then the first zeros are absent and we can assume $\alpha_1 = 1$. In this case we further transform Z'_m. Considering (8b) with $\rho = n$, we look at the real part of X'. If some element in the second row and column, except for the (1,2), (2,1) and (2,2) elements, is nonzero it can be permuted to the (2,3), and (3,2) position by elementary operations. By adding the third row and column to higher numbered rows and columns, the corresponding elements of the second row and column can be brought to zero, while the upper left 2 x 2 submatrix in the Hermitian part of $Z_m'(p_o)$ is preserved; one can use elementary row and column operations for this. A similar set of operations on the (1,4) and (4,1) terms brings higher numbered elements of the first row and column (still of the real part of X') to zero. One can then get

$$\tilde{T}_m' Z_m'(p_o) T_m' = \left\{ \begin{bmatrix} 1 & j1 \\ -j1 & 1 \end{bmatrix} \dotplus R_{n-2} \right\} + j \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & 0 \ldots 0 \\ b_{12} & b_{22} & b_{23} & 0 & \ldots \quad 0 \\ b_{13} & b_{23} & & & \\ b_{14} & 0 & & & \\ 0 & \vdots & & X_{n-2} & \\ \vdots & & & & \\ 0 & 0 & & & \end{bmatrix} + G_m \tag{8c}$$

where G_m is a real, skew-symmetric matrix and R_{n-2} and X_{n-2} are (n-2)x(n-2) Hermitian matrices. Consequently we define

$$Z^{(0)}(p) = \tilde{T}_m' \tilde{T}_m Z_m(p) T_m T_m' = \tilde{T}_o Z_m(p) T_o \tag{9}$$

where $T_m' = 1_n$ if $\rho < n$ and T_m' is as needed to get (8c), if $\rho = n$. Thus $Z^{(0)}(p_o)$ takes the form of (8b), if $\rho < n$, and the form of (8c), if $\rho = n$. A realization for Z_m results by terminating a transformer network in a realization for $Z^{(0)}$. In either of these two cases we will form

$Z^{(1)}(p) = Z^{(0)}(p) + Z_1(p)$	(10a)
$Z_1(p) = G_1 + pL_1 + D_1/p$	(10b)

where G_1 is a real, constant, skew-symmetric matrix, while L_1 and D_1 are real, constant, symmetric and positive semi-definite. G_1, L_1 and D_1 are chosen such that $Z^{(1)}(p_o)$ has rank n-1, however, their choice depends upon whether $\rho < n$ or $\rho = n$, and we therefore have two cases to consider.

Case I, $\rho < n$:

Here we choose G_1, L_1 and D_1 such that $Z^{(1)}(p_o)$ has rank $n-1$ with its first row and column zero. This can be done in many ways. One method is to let G_1 equal the first row and column of the skew-symmetric part of jX', and then diagonalize the symmetric part of X', X'_{sy}, by writing

$$X'_{sy} = \tilde{T}'[1_\alpha \dotplus (-1_\beta) \dotplus 0_{n-\alpha-\beta}]T' \tag{11a}$$

and choosing

$$\omega_o L_1 = \tilde{T}'[0_\alpha \dotplus 1_\beta \dotplus 0_{n-\alpha-\beta}]T' + [0 \dotplus 1_{n-1}] \tag{11b}$$

$$D_1 = \omega_o \tilde{T}'[1_\alpha \dotplus 0_{n-\alpha}]T' \tag{11c}$$

where we can of course assume $\omega_o > 0$.

Case II, $\rho = n$:

Here we first choose G_1 such that

$$Z^{(0)}(p_o)+G_1 = \left\{\begin{bmatrix} 1 & j1 \\ -j1 & 1 \end{bmatrix} \dotplus R_{n-2}\right\} \dotplus j \begin{bmatrix} b_{11} & b_{12}+jg_{12} & b_{13}+jg_{13} & b_{14} & 0 \\ b_{12}-jg_{12} & b_{22} & b_{23}+jg_{23} & jg_{24} & \\ b_{13}-jg_{13} & b_{23}-jg_{23} & & & \\ b_{14} & -jg_{24} & & X_{n-2} & \\ & 0 & & & \end{bmatrix} \tag{12}$$

Of course we can do this with entries only in the first two rows and columns of G_1. At this point we choose $D_1 = 0_n$ and L_1 such that $Z^{(1)}(p_o)$ has rank $n-1$. To see that this is possible, we add j times the following real, symmetric, positive semi-definite matrix to (12).

$$B_1 = \begin{bmatrix} b'_{11} & -b_{12} \\ -b_{12} & b'_{22} \end{bmatrix} \dotplus B_{n-2} \tag{13}$$

To show that B_1 and G_1 can be chosen to force the resulting matrix singular, we form

$$\left\{\begin{bmatrix}1 & -j\\ 0 & 1\end{bmatrix}\dot{+}1_{n-2}\right\}\left\{Z^{(0)}(p_o)+G_1+jB_1\right\}\left\{\begin{bmatrix}1 & 0\\ j & 1\end{bmatrix}\dot{+}1_{n-2}\right\} = \left\{\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\dot{+}R_{n-2}\right\} +$$

$$j\left[\begin{array}{cc|ccc}
b_{11}+b'_{11}+b_{22}+b'_{22}-2g_{12} & j(g_{12}-b_{22}-b'_{22}) & (b_{13}+g_{23})+j(g_{13}-b_{23}) & b_{14}+g_{24} & 0\\
-j(g_{12}-b_{22}-b'_{22}) & b_{22}+b'_{22} & b_{23}+jg_{23} & jg_{24} & \\
\hline
(b_{13}+g_{23})-j(g_{13}-b_{23}) & b_{23}-jg_{23} & & & \\
b_{14}+g_{24} & -jg_{24} & & X'_{n-2} & \\
\hline
0 & & & &
\end{array}\right] \tag{14}$$

Choosing

$$g_{24} = -b_{14} \tag{15a}$$

$$g_{13} = b_{23} \tag{15b}$$

$$g_{23} = -b_{13} \tag{15c}$$

$$g_{12} = b_{22} + b'_{22} \tag{15d}$$

$$b_{11} + b'_{11} = b_{22} + b'_{22} \tag{15e}$$

then forces the first row and column of (14) to zero. Now the semi-definite constraint on B_1 is that B_{n-2} be positive semi-definite and

$$b'_{11} \geq 0 \tag{16a}$$

$$b'_{22} \geq 0 \tag{16b}$$

$$b'_{11}b'_{22} \geq b_{12}^2 \tag{16c}$$

We will show that (16) is consistent with (15). Of course (15a)-(15d) are satisfied by properly choosing G_1, and, hence, we only need check (15e) against (16). Assuming $b'_{11} > 0$, (16c) gives $b'_{22} \geq b_{12}^2/b'_{11}$, which is consistent with (16b). Substituting this into (15e) gives $b_{11} + b'_{11} \geq b_{22} + b_{12}^2/b'_{11}$ or

$$(b'_{11})^2 - (b_{22}-b_{11})b'_{11} - b_{12}^2 \geq 0 \tag{17a}$$

Consequently any positive

$$b'_{11} \geq \frac{(b_{22}-b_{11})}{2} + \frac{1}{2}\sqrt{(b_{22}-b_{11})^2+4b_{12}^2} \geq 0 \tag{17b}$$

can be chosen. B_{n-2} can then be chosen to yield (14) of rank $n-1$. Of course L_1 is then given by

$$\omega_o L_1 = B_1 \tag{18}$$

In summary, then, given $\rho = n$, (8c) is formed and any positive b'_{11} satisfying (17b) is chosen. Next $b'_{22} \geq 0$ is obtained by solving (15e); if $b_{12} \neq 0$ then $b_{22} > 0$. Finally, L_1 and G_1 are chosen to satisfy (18) and (15a)-(15d), respectively. The reason for choosing $D_1 = 0_n$ will become apparent later.

In either Case $Z^{(1)}$ is PR, being the sum of two PR matrices, non-singular in p, [6, T-9], but singular of rank $n-1$ at $p = p_o$. Its inverse can then be formed and we get

$$[Z^{(1)}]^{-1} = Y^{(1)}(p) = Y_2(p) + Y^{(2)}(p) \tag{19a}$$

$$Y_2(p) = \frac{pA + B}{p^2 + \omega_o^2} \tag{19b}$$

where $Y^{(2)}$ is analytic at $p = p_o$ and A is real, constant, symmetric and positive semi-definite, while B is real, constant, and skew-symmetric, [6, (13a)].

At this point we investigate the form of Y_2, which differs for Cases I and II. As we show here, the residue matrix has rank one in both Cases, but (only) in Case I, $B = 0_n$. In either Case we write by the use of Taylor and Laurent series

$$Z^{(1)}(p) = (p-p_o)P + Q(p) \tag{20a}$$

$$Y^{(1)}(p) = \frac{K}{p-p_o} + U(p) \tag{20b}$$

where K is the positive semi-definite Hermitian residue matrix and $Q(p_o) = Z^{(1)}(p_o)$; P is a complex constant matrix. As a consequence of $Z^{(1)}Y^{(1)} = 1_n = Y^{(1)}Z^{(1)}$, we have

$$PK + (p-p_o)PU(p) + \frac{Q(p)K}{p-p_o} + Q(p)U(p) = 1_n \tag{20c}$$

as well as a similar expression with all terms commuted. Multiplying by $p-p_o$ and letting $p = p_o$ gives

$$Z^{(1)}(p_o)K = KZ^{(1)}(p_o) = 0_n \tag{21}$$

K is then partitioned into $K = \begin{bmatrix} k_{11} & k_{12} \\ \tilde{k}_{12}^{*} & k_{22} \end{bmatrix}$ with k_{11} one by one and $Z^{(1)}(p_o)$ written as

$$Z^{(1)}(p_o) = \begin{cases} 0 \dotplus Z_I & \text{Case I} \quad (22a) \\ \left(\begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \dotplus 1_{n-2}\right)\left(0 \dotplus Z_{II}\right)\left(\begin{bmatrix} 1 & 0 \\ -j & 1 \end{bmatrix} \dotplus 1_{n-2}\right) & \text{Case II} \quad (22b) \end{cases}$$

where Z_I and Z_{II} are (complex) constant, nonsingular matrices of order n-1. Substituting (22) into (21), performing the indicated multiplications, and using the nonsingularity of Z_I and Z_{II} gives

$$K = k_{11} \dotplus 0_{n-1} \qquad \text{Case I} \qquad (23a)$$

$$K = \left\{ \begin{bmatrix} k_{11} & -jk_{11} \\ jk_{11} & k_{11} \end{bmatrix} \dotplus 0_{n-2} \right\} \qquad \text{Case II} \qquad (23b)$$

Consequently, we have, since $A = K + \tilde{K}$ and $B = j\omega_o[K-\tilde{K}]$, [6, (13)],

$A = [2k_{11} \dotplus 0_{n-1}]$, $B = 0_n$	Case I	(24a)
$A = [2k_{11}1_2 \dotplus 0_{n-2}]$, $B = [2\omega_o k_{11} E \dotplus 0_{n-2}]$	Case II	

One easily convinces oneself that $Y^{(2)}$ is nonsingular in p. To see this, note that

$$Z^{(0)}+\tilde{Z}_*^{(0)} = Z^{(1)}+\tilde{Z}_*^{(1)} = \tilde{Z}_*^{(1)}[Y^{(1)}+\tilde{Y}_*^{(1)}]Z^{(1)} = \tilde{Z}_*^{(1)}[Y^{(2)}+\tilde{Y}_*^{(2)}]Z^{(1)}$$

For any nonzero n-vector x we have, on the imaginary axis,

$$\tilde{x}^*\tilde{Z}^{(1)*}[Y^{(2)}+\tilde{Y}^{(2)*}]Z^{(1)}x = \tilde{x}^*[Z^{(0)}+\tilde{Z}^{(0)*}]x > 0 \qquad (25)$$

Using familiar arguments, based upon the maximum modulus theorem, [6, T-5, L-3], (25) holds for all p in $\sigma > 0$ and $Y^{(2)}$ is non-singular. Inverting $Y^{(2)}$ gives

$[Y^{(2)}]^{-1} = Z^{(2)}(p) = Z^{(3)}(p) + Z_3(p)$	(26a)
$Z_3(p) = pL_3 + D_3/p$	(26b)

where $Z^{(3)}$ is PR and analytic at $p = 0$ and ∞ and L_3 and D_3 are real, constant, symmetric and positive semi-definite. A realization for $Z^{(0)}$ results from one for $Z^{(3)}$ by the use of the coupling network N_c of Figure 1. Here all blocks are n-ports, the series arms being described by their impedance matrices, while the shunt arms are described by the admittance matrix.

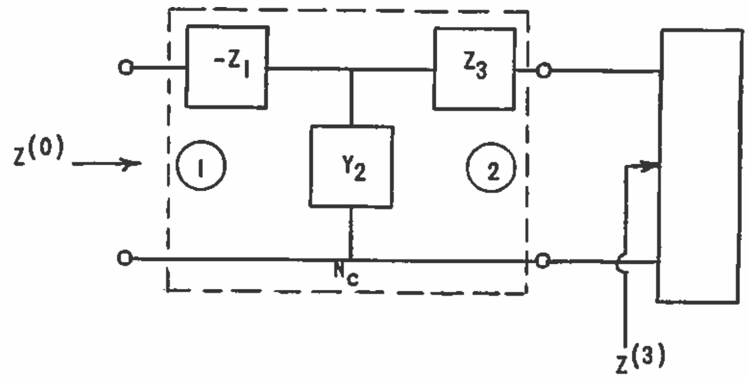


FIG. 1. ORIGINAL COUPLING NETWORK.

The problem is then to show that N_c can be realized by using only passive elements, of which $\delta(Z^{(0)}) - \delta(Z^{(3)})$ are reactive.

The first term to investigate is Y_2. In Case I, by observing (19b) and (24a) we see that this represents a capacitor and inductor in series, shunting port one, with open circuits shunting the remaining ports. This can then be described as the series connection of two n-ports of admittance matrices A/p and pA/ω_o^2. For Case II, $Y_2 = \frac{2k_{11}}{p^2+\omega_o^2} \{p1_2+\omega_o E\} \dotplus 0_{n-2}$. The upper left 2 x 2 submatrix in this can be inverted to give $\frac{1}{2k_{11}} \{p-\omega_o E\}$. This shows that, in Case II, Y_2 represents the series connection of an inductor 2-port and a gyrator, shunting the first two ports, with the remaining n-2 ports shunted by open circuits. This can then be described as the series connection of two n-ports of admittance matrices A/p and B/ω_o^2. With this the coupling network can be redrawn in the forms shown in Figure 2. Since the elements outside the dashed lines of Figure 2 are clearly realizable with passive elements (gyrators for $\tilde{G}_1$ and B/ω_o^2, inductors for D_3/p), we concentrate on the subnetworks labelled N_c'.

First consider N_c' for Case II, which is identical in form to the upper tee subnetwork of N_c' for Case I. We have from (19a), $Y^{(1)} - Y^{(2)} = Y_2$ or

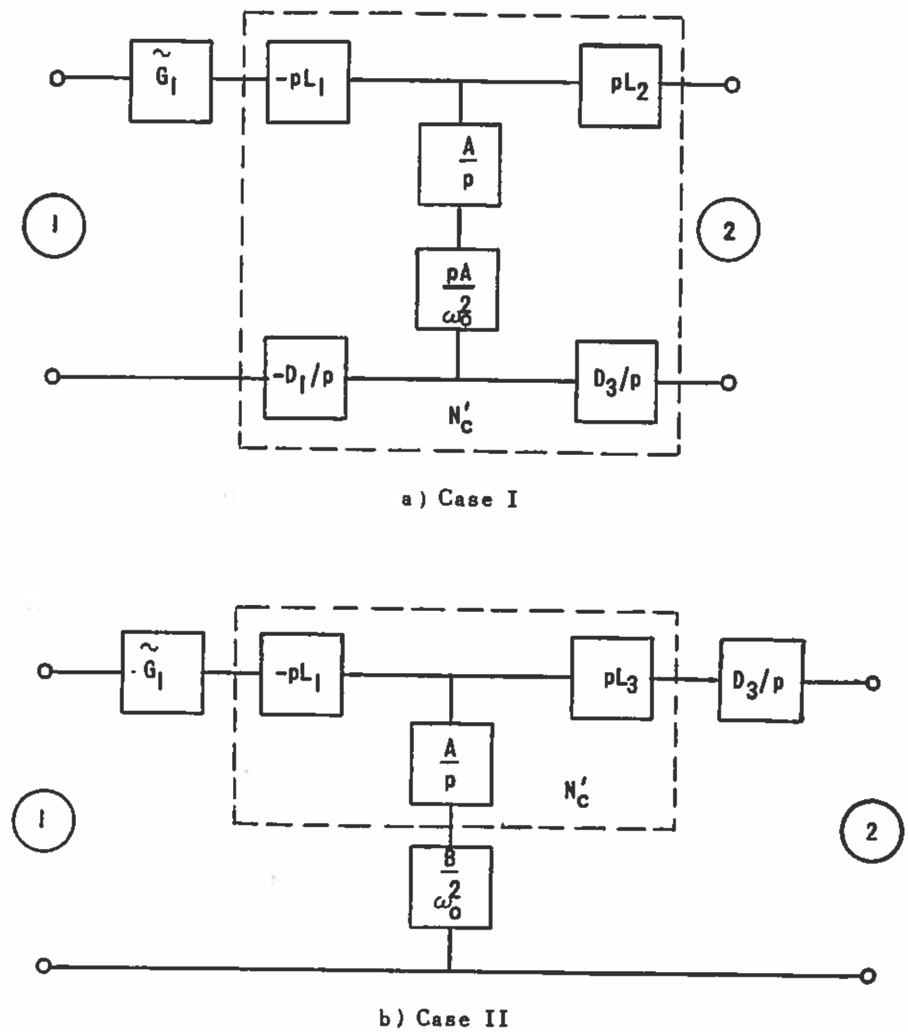


FIG. 2. STRUCTURE OF COUPLING NETWORKS.

$$Z^{(2)} - Z^{(1)} = Z^{(2)}Y_2Z^{(1)} = Z^{(1)}Y_2Z^{(2)} \tag{27}$$

Substituting the values of $Z^{(1)}$, Y_2, $Z^{(2)}$ from (10), (19b) and (26), dividing by p, and letting $p \to \infty$, we easily find from (27)

$$L_3 - L_1 = L_3AL_1 = L_1AL_3 \tag{28a}$$

By a simple manipulation, this gives

$$L_3 = L_1[1_n - AL_1]^{-1} \tag{28b}$$

which shows that

$$\text{rank } L_1 = \text{rank } L_3 = r_\ell \tag{29}$$

Equation (29) serves to define r_ℓ. Equation (28a) can be rewritten as $-L_1 - L_1AL_3 + L_3 = 0_n$. This can be multiplied on the right by A and 1_n added to the resultant to get $-L_1A[1_n + L_3A] + 1_n[1_n + L_3A] = 1_n$ which gives, using the symmetry of L_1, L_3 and A,

$$1_n + AL_3 = [1_n - AL_1]^{-1} \tag{30}$$

Besides being of further use, this justifies the inverse used in (28b). Now compute the 2n-port chain (transmission) matrix, Ψ, for N_c' of Case II. For any tee structure, as shown in Figure 3, one has

$$\Psi = \begin{bmatrix} 1_n & Z_1 \\ 0_n & 1_n \end{bmatrix} \begin{bmatrix} 1_n & 0_n \\ Y_2 & 1_n \end{bmatrix} \begin{bmatrix} 1_n & Z_3 \\ 0_n & 1_n \end{bmatrix} \tag{31a}$$

$$= \begin{bmatrix} 1_n + Z_1Y_2 & Z_1 + Z_1Y_2Z_3 + Z_3 \\ Y_2 & 1_n + Y_2Z_3 \end{bmatrix} \tag{31b}$$

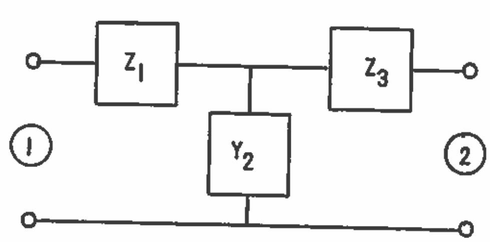


FIG. 3. GENERAL TEE 2n-PORT.

Using the values given in Figure 2b), in conjunction with (28a), (31b) gives for N_c'

$$\mathcal{Y}_\ell = \begin{bmatrix} 1_n - L_1 A & 0_n \\ A/p & 1_n + AL_3 \end{bmatrix} \tag{32a}$$

$$= \begin{bmatrix} \widetilde{(1_n - AL_1)} & 0_n \\ A/p & (1_n - AL_1)^{-1} \end{bmatrix} \tag{32b}$$

$$= \begin{bmatrix} 1_n & 0_n \\ (A + AL_3 A)/p & 1_n \end{bmatrix} \begin{bmatrix} \widetilde{(1_n - AL_1)} & 0_n \\ 0_n & (1_n - AL_1)^{-1} \end{bmatrix} \tag{32c}$$

Here (30) has been used to obtain (32b) and (32c). Since an ideal transformer 2n-port is described by $V_1 = \tilde{T}V_2$, $I_2 = -TI_1$, where T is the turns ratio matrix, the right term of (32c) describes a transformer 2n-port of turns ratio matrix

$$\boxed{T_\ell = 1_n - AL_1} \tag{33a}$$

The left term of (32c) describes a shunt n-port of admittance matrix

$$\boxed{Y_\ell(p) = (A + AL_3 A)/p} \tag{33b}$$

The residue matrix in (33b) is clearly positive semi-definite, as A and L_3 are. Since $A + AL_3A = [1_n + AL_3]A$, the residue matrix has its rank equal to that of A, by the non-singularity of $[1_n + AL_3]$. Thus in Case I, one inductor is used for Y_ℓ, while in Case II, two are needed. By (32c), $\mathcal{Y}_\ell$ describes the cascade connection of this inductor network with the transformers of (33a). Before drawing the physical picture, we state similar results for the lower tee subnetwork of Figure 2a).

Using (27) with $B = 0_n$, in conjunction with (10), (19b), and (25), multiplying by p, and letting $p \to 0$, we now get

$$D_3 - D_1 = D_3 \frac{A}{\omega_o^2} D_1 = D_1 \frac{A}{\omega_o^2} D_3 \tag{34a}$$

which, as before, shows (in Case I)

$$\text{rank } D_3 = \text{rank } D_1 = r_c \tag{34b}$$

Letting

$$\boxed{C_2 = A/\omega_o^2} \tag{35}$$

we have

$$1_n + C_2 D_3 = [1_n - C_2 D_1]^{-1} \tag{36}$$

giving

$$\mathcal{Y}_c = \begin{bmatrix} 1_n & 0_n \\ (C_2 + C_2 D_3 C_2)p & 1_n \end{bmatrix} \begin{bmatrix} \widetilde{(1_n - C_2 D_1)} & 0_n \\ 0_n & (1_n - C_2 D_1)^{-1} \end{bmatrix} \tag{37}$$

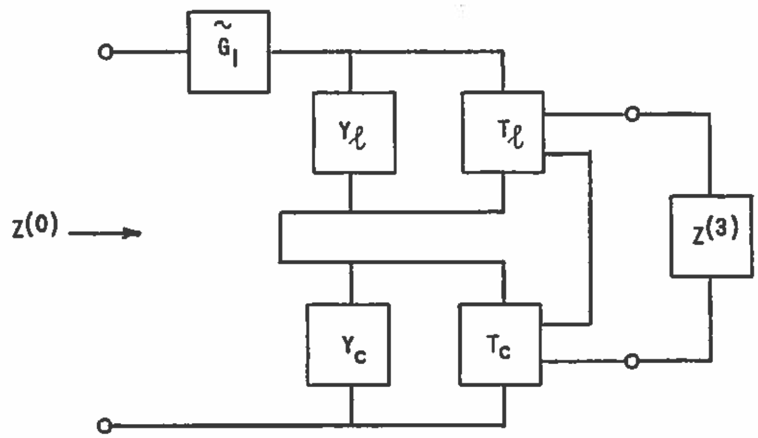
Then

$$\boxed{T_c = 1_n - C_2 D_1} \tag{38a}$$

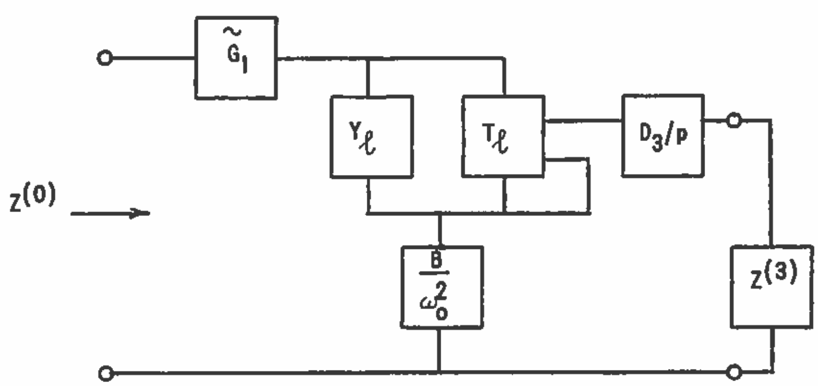
$$\boxed{Y_c = p(C_2 + C_2 D_3 C_2)} \tag{38b}$$

$\mathcal{Y}_c$ is realized by a capacitive network, using one capacitor in cascade with a transformer 2n-port.

As these arguments show, the coupling networks of Figure 2 have the completely passive equivalents used in Figure 4.



a) Case I



b) Case II

FIG. 4. COMPLETELY PASSIVE BRUNE SECTIONS.

It remains to show that $\delta(Z^{(0)})-\delta(Z^{(3)})$ reactive elements are used. We have, using the properties of δ of Appendix 1 and (10), (19), (26) and (29),

$$\delta(Z^{(1)}) = \delta(Z^{(0)}) + r_\ell + \text{rank } D_1 \tag{39a}$$

$$= 2 \text{ rank } K + \delta(Y^{(2)}) \tag{39b}$$

$$= 2 \text{ rank } K + \delta(Z^{(3)}) + r_\ell + \text{rank } D_3 \tag{39c}$$

or, from (39a) and (39c)

$$\delta(Z^{(0)})-\delta(Z^{(3)}) = 2 \text{ rank } K + \text{rank } D_3 - \text{rank } D_1 \tag{40a}$$

From (34b), the fact that $D_1 = 0_n$ for Case II, and since rank K = 1,

$$\delta(Z^{(0)})-\delta(Z^{(3)}) = \begin{cases} 2, \text{ Case I} & (40b) \\ 2 + \text{rank } D_3, \text{ Case II} & (40c) \end{cases}$$

By the comments made concerning Ψ_ℓ and Ψ_c, and observing Figure 4, it is clear that this many reactive elements are used. Since the procedure can be repeated on the PR matrix $Z^{(3)}$, the procedure can be continued until a matrix of degree zero is obtained. Such a matrix must be a PR constant and can be realized by resistors and gyrators. Consequently, we have given a Brune synthesis which uses as many reactive elements as the degree of the original PR matrix.

In the above, it appears as $D_3 = 0_n$ in Case II. However, we have been unable to definitely prove this as yet. We can show, using the arguments leading to (34a), when $B \neq 0_n$, that D_3 is zero except at most for its upper left 2 x 2 submatrix (in Case II). In Case II, if we had allowed $D_1 \neq 0_n$, there would in general be no D_3 present to cancel the active capacitors.

IV. STRUCTURE OF THE COUPLING NETWORK

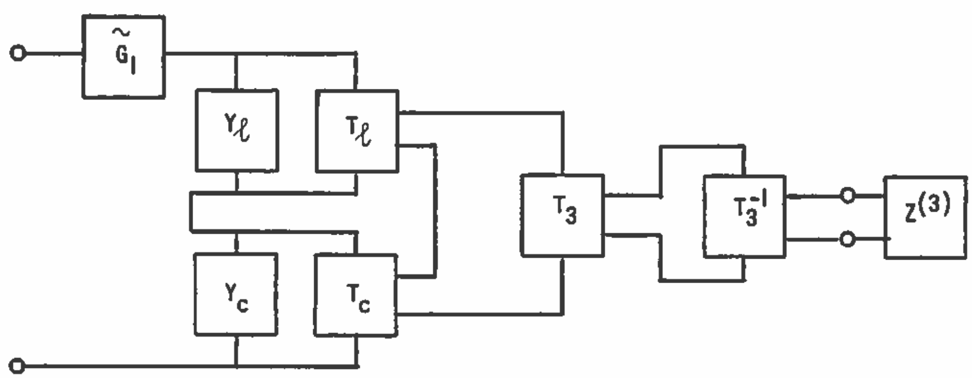
Here we investigate possible canonical forms for the Brune sections. The Case I structure is investigated first, since it is the simplest.

In Figure 4a) we can insert a transformer 2n-port of turns ratio matrix T_3 in cascade with one of turns ratio matrix T_3^{-1} between N_c and the realization of $Z^{(3)}$, as shown in Figure 5a). This leaves $Z^{(\delta)}$ unchanged, but T_3 can be incorporated in N_c and T_3^{-1} used to transform $Z^{(3)}$. Using the equivalence of Figures 2-1 and 2-2 of Appendix 2, Figure 5b) results when

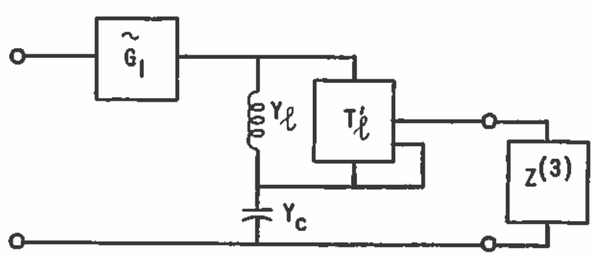
$$T_3 = T_c^{-1} = 1_n + C_2 D_1 \tag{41a}$$

$$T'_\ell = T_c^{-1} T_\ell = [1_n + C_2 D_1][1_n - AL_1] \tag{41b}$$

$$Z_o^{(3)}(p) = \tilde{T}_3^{-1} Z^{(3)}(p) T_3^{-1} \tag{41c}$$



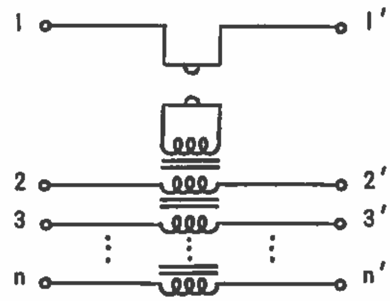
a) Insertion of cascade transformers



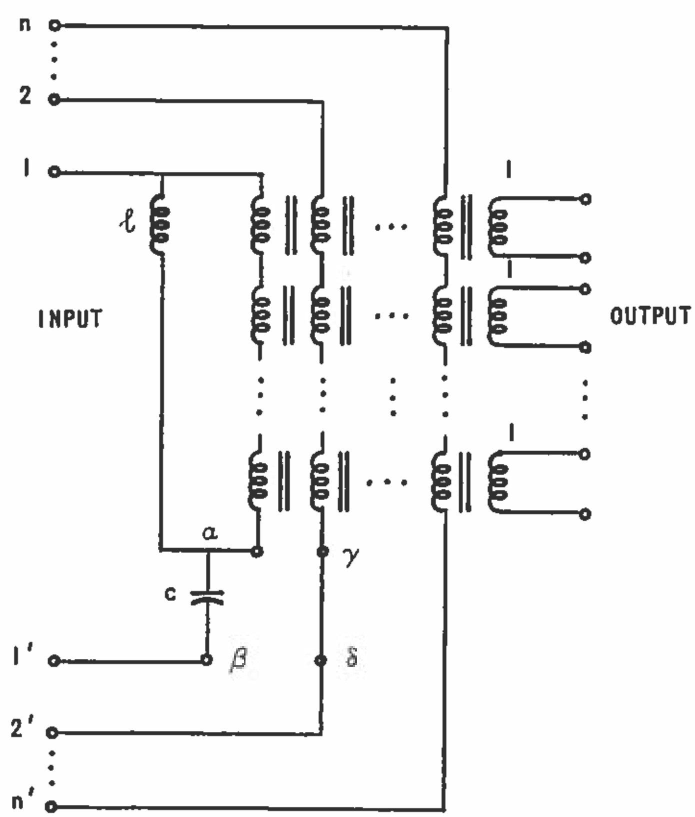
b) Equivalent using (41)

FIG. 5. CANONICAL CASE I BRUNE SECTION.

In Figure 5b), $\tilde{G}_1$ has all but its first row and column zero and can then be realized by the network of Figure 6a). Since Y_ℓ and Y_c have only their (1,1) term nonzero, the middle circuit of Figure 5b) is realized by the network of Figure 6b), where the transformer turns are in one to one agreement with the entries of (41b); ℓ and c are determined from (33b) and (38b).



a) Realization of $\tilde{G}_1$



b)

FIG. 6. DETAILS FOR FIG. 5.

Considering Case II, we insert transformers T_1^{-1}, T_1 at the immediate left of $\tilde{G}_1$ in Figure 4b). T_1 is split into two equal sub-networks, T_1, without any change, to give Figure 7a) where D_3/p is associated with $Z^{(3)}$ for convenience.

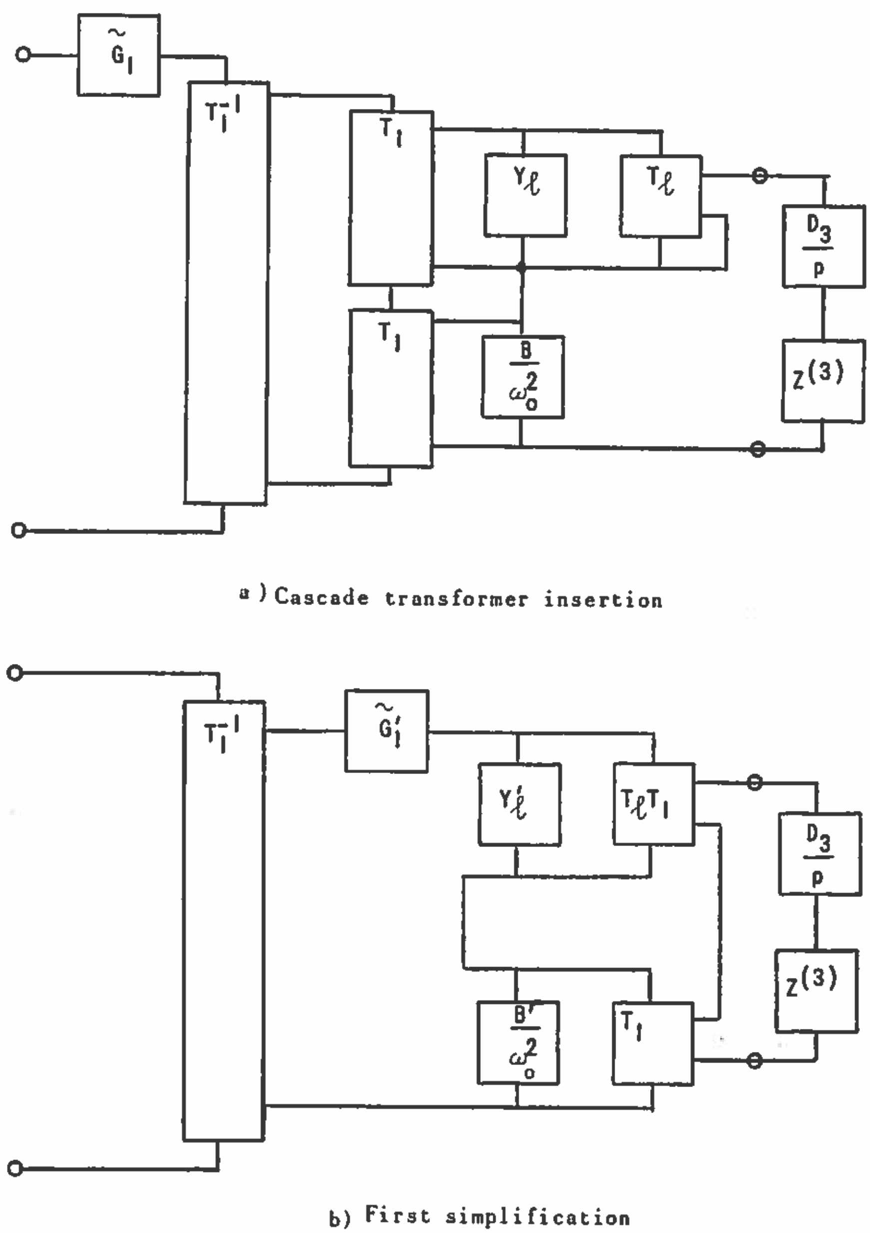


FIG. 7. EQUIVALENT CASE II REALIZATIONS.

T_1 and Y_ℓ are then commuted and the equivalences of Figures 2-3 and 2-4 used. Similarly T_1 and B/ω_o^2 are commuted and we have

$$Y'_\ell = T_1^{-1} Y_\ell \tilde{T}_1^{-1} \tag{42a}$$

$$B' = T_1^{-1} B \tilde{T}_1^{-1} \tag{42b}$$

Figure 2-5 is also used to commute $\tilde{G}_1$ and T_1^{-1}, giving

$$\tilde{G}'_1 = \tilde{T}_1 \tilde{G}_1 T_1 \tag{42c}$$

At this point, T_1 can be chosen to diagonalize Y'_ℓ. Since Y_ℓ is zero except in its upper left 2 x 2 corner, this can be done by a T_1 of the form $T_1 = T'_1 \dotplus 1_{n-2}$ where T'_1 is 2 x 2. Such a T_1 gives B' of the same form as B, that is $B' = bE \dotplus 0_{n-2}$. As with Case I, we now insert T_3 and T_3^{-1} to the left of D_3/p. The equivalences of Figures 2-3 and 2-4 are then used to choose $T_3 = T_1^{-1}$, which deletes the transformers on the lower gyrators, B'/ω_o^2. The final equivalence is then as shown in Figure 8, where

$$Z_o^{(3)} = \tilde{T}_1 Z^{(3)} T_1 \tag{43a}$$

$$D'_3 = \tilde{T}_1 D_3 T_1 \tag{43b}$$

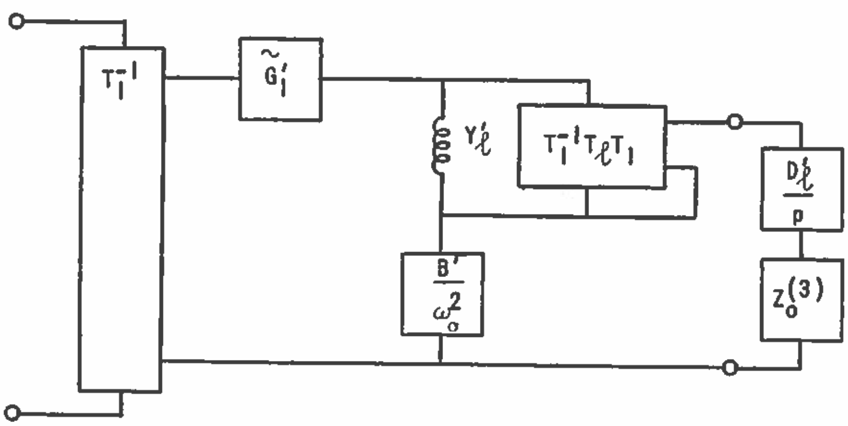


FIG. 8. CANONICAL CASE II BRUNE SECTION.

Since $\tilde{G}_1$ has only its first two rows and columns nonzero, this will be true of $\tilde{G}_1'$ by the form of T_1. $\tilde{G}_1'$ is then realized by connecting two networks of the type of Figure 6a) in series; one with the gyrator at port two. The inductor-transformer-gyrator middle portion of Figure 8 can be obtained from Figure 6b) as follows. One first places an inductor across the second column of transformer turns (connected at points 2 and γ). The capacitor and the short between γ,δ are removed and a gyrator with input at α,β and output at γ,δ, is finally inserted. The final structure is somewhat difficult to draw and it is hoped that this description will suffice.

V. EXAMPLE

Here we carry out a simple Case II synthesis to illustrate the basic concepts involved. Consider

$$Z(p) = \frac{1}{p+1}\begin{bmatrix} p+5 & 6(p+1) \\ -6p & p+2 \end{bmatrix}$$

this is PR by the PR test, [6, T-1], since

$$Z_H(j\omega) = \frac{1}{\omega^2+1}\begin{bmatrix} \omega^2+5 & 3(1+j\omega) \\ 3(1-j\omega) & \omega^2+2 \end{bmatrix} \quad \text{which has det } Z_H(j\omega) = \left(\frac{\omega^2-1}{\omega^2+1}\right)^2 .$$

At $p_o = j1$ we have

$$R+jX = \begin{bmatrix} 3 & \frac{3}{2}+j\frac{3}{2} \\ \frac{3}{2}-j\frac{3}{2} & \frac{3}{2} \end{bmatrix} - j\begin{bmatrix} 2 & \frac{3}{2}+j\frac{9}{2} \\ \frac{3}{2}-j\frac{9}{2} & \frac{1}{2} \end{bmatrix}$$

The real part of R is then diagonalized. This has

$$\tilde{T}_m(R+jX)T_m = \begin{bmatrix} 1 & j1 \\ -j1 & 1 \end{bmatrix} - j\,\frac{1}{3}\begin{bmatrix} 2 & 1+j9 \\ 1-j9 & -2 \end{bmatrix}$$

$$T_m = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

Consequently, we use a Case II synthesis with

$$Z^{(0)}(p) = \tilde{T}_m Z(p) T_m = \frac{1}{3(p+1)}\begin{bmatrix} p+5 & 11p+7 \\ -13p-5 & 5p+1 \end{bmatrix}$$

We wish to add to $Z^{(0)}(p)$

$$Z_1(p) = \begin{bmatrix} 0 & g_1 \\ -g_1 & 0 \end{bmatrix} + p \begin{bmatrix} b'_{11} & \frac{1}{3} \\ \frac{1}{3} & b'_{22} \end{bmatrix}$$

For the positive semi-definiteness of the last matrix we require

$$b'_{11} b'_{22} \geq \frac{1}{9}$$

and we are obliged to choose, by (17b),

$$b'_{11} \geq \frac{2+\sqrt{5}}{3}$$

For convenience let

$$b'_{11} = 2$$

then, from (15e),

$$b'_{22} = 2/3$$

Clearly the semidefinite requirement is met; by (15)

$$g_1 + 3 = -g_{12} = -(b_{22} + b'_{22}) = -4/3$$
$$g_1 = -13/3$$

Consequently,

$$Z_1(p) = \begin{bmatrix} 0 & -13/3 \\ 13/3 & 0 \end{bmatrix} + p \begin{bmatrix} 2 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$Z^{(1)}(p) = Z^{(0)}(p) + Z_1(p)$$

$$= \frac{1}{3(p+1)} \begin{bmatrix} 6p^2+7p+5 & p^2-p-6 \\ p^2+p+8 & 2p^2+7p+1 \end{bmatrix}$$

This has det $Z^{(1)}(p) = (p^2+1)(11p^2+56p+53)/[9(p+1)^2]$ and

$$Y^{(1)}(p) = \frac{3(p+1)}{(p^2+1)(11p^2+56p+53)} \begin{bmatrix} 2p^2+7p+1 & -p^2+p+6 \\ -p^2-p-8 & 6p^2+7p+5 \end{bmatrix}$$

$$= \frac{\frac{3}{7}\begin{bmatrix} p & 1 \\ -1 & p \end{bmatrix}}{p^2+1} + \frac{3/7}{11p^2+56p+53} \begin{bmatrix} 3p+7 & -(7p+11) \\ -(7p+3) & 31p+35 \end{bmatrix}$$

where the second term on the right of this last expression is $Y^{(2)}$. This has det $Y^{(2)} = 36/49(11p^2+56p+53)$, or

$$Z^{(2)}(p) = \frac{7}{36} \begin{bmatrix} 31p+35 & 7p+11 \\ 7p+3 & 3p+7 \end{bmatrix} = \frac{7p}{36} \begin{bmatrix} 31 & 7 \\ 7 & 3 \end{bmatrix} + \frac{49}{36} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} + \frac{7}{9} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

From (33)

$$T_\ell = 1_n - AL_1 = \frac{1}{7} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Y_\ell = (A+AL_3A)/p = \frac{1}{28p} \begin{bmatrix} 43 & 7 \\ 7 & 15 \end{bmatrix}$$

Further, for Figure 4

$$D_3 = 0_2,\quad \tilde{G}_1 = \frac{13}{3}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad B/\omega_o^2 = \frac{3}{7}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad Z^{(3)}(p) = \frac{49}{36}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} + \frac{7}{9}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Although a realization could now be drawn, we continue to obtain the canonical structure. We can write

$$Y'_\ell = \begin{bmatrix} 1 & 0 \\ -7/43 & 1 \end{bmatrix} Y_\ell \begin{bmatrix} 1 & -7/43 \\ 0 & 1 \end{bmatrix} = \frac{1}{28p}\begin{bmatrix} 43 & 0 \\ 0 & 596 \end{bmatrix}$$

and T_1 is

$$T_1 = \begin{bmatrix} 1 & 0 \\ 7/43 & 1 \end{bmatrix},\quad T_1^{-1} = \begin{bmatrix} 1 & 0 \\ -7/43 & 1 \end{bmatrix}$$

Thus

$$\tilde{G}'_1 = \tilde{G}_1 = \frac{13}{3} E$$

$$B' = B = \frac{3}{7} E$$

$$Z_o^{(3)} = \frac{49}{36}\begin{bmatrix} 9896/(43)^2 & 50/43 \\ 50/43 & 1 \end{bmatrix} + \frac{7}{9} E$$

$$= \frac{49}{36}\begin{bmatrix} 1 & 50/43 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 50/43 & 1 \end{bmatrix} + \frac{7}{9} E$$

$$T_1^{-1} T_\ell T_1 = \frac{1}{7}\begin{bmatrix} 36/43 & -1 \\ -1800/(43)^2 & 50/43 \end{bmatrix}$$

The realization following Figure 8 is given in Figure 9, where the 1:1 transformers could be deleted.

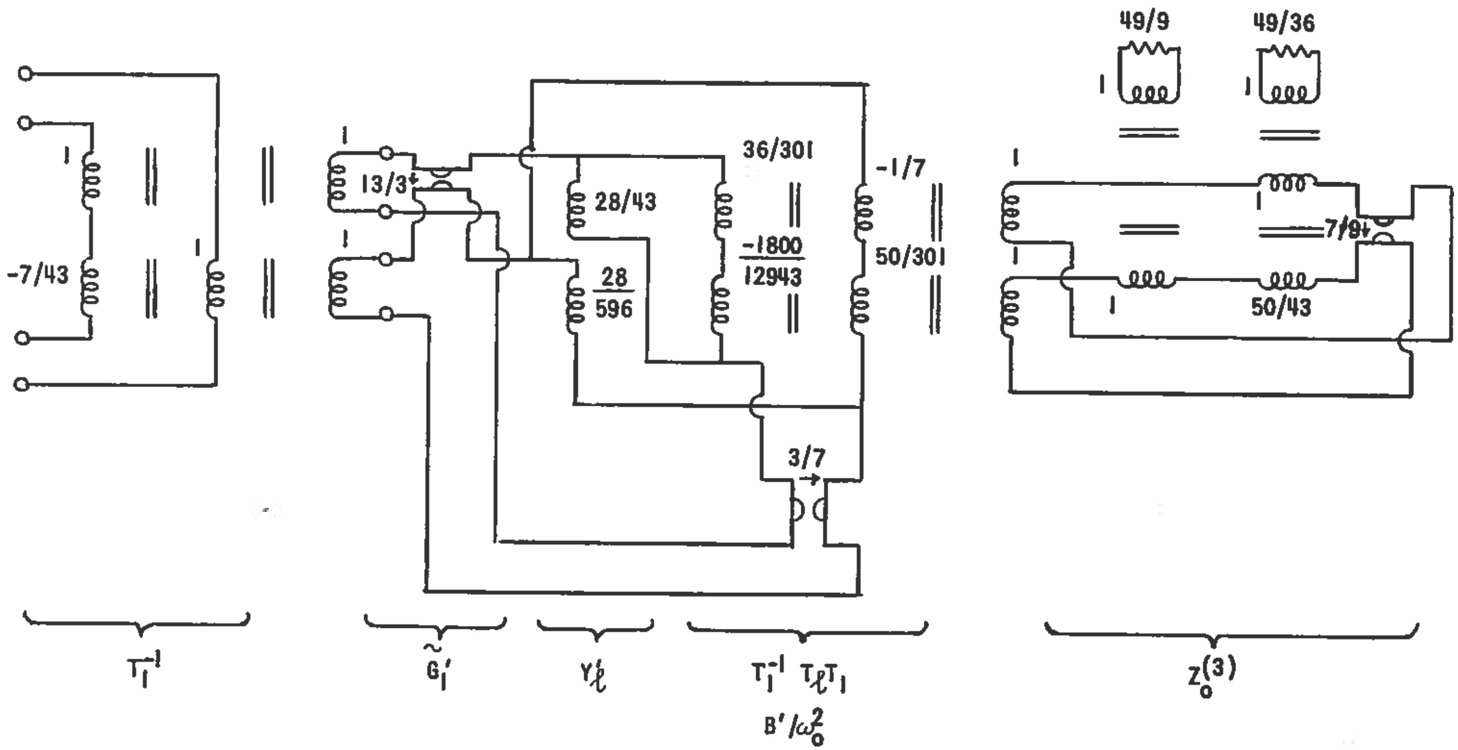


FIG. 9. EXAMPLE REALIZATION, ELEMENT VALUES IN HENRIES AND OHMS.

VI. CONCLUSIONS

By following a more classical approach than that used by Belevitch, this report presents an n-port Brune synthesis. The key idea used is to insert a pole with residue of rank one in $Y^{(1)}(p)$. This is done by, perhaps, first extracting many inductors and capacitors which then later are cancelled by further extraction. The final Brune section then uses only two inductors or one inductor and a capacitor (which can be replaced by an inductor and a gyrator in a known manner). Because the residue matrix is chosen of rank one, the canonical forms of Figures 5 and 8 are easily obtained. In many cases, one need not hold the rank one constraint of the above mentioned residue matrix. However, as yet, nothing can be said about the generality of this possibility.

It should be quite clear that the method yields a reciprocal network, if the given matrix is symmetric. In this case the method is a simple modification of that of McMillan. It is also quite similar to that of Tellegen, but easily treats the case of $x'_{11} = 0$ in (8b), for which Tellegen's method seems to fail, [12, p. 13]; see also [1, p. 150]. The $x'_{11} = 0$ case can also be handled by extracting a cascade gyrator from port one, but this would put gyrators in a reciprocal network.

Since it is known that no fewer than $\delta(Z)$ reactive elements can be used to realize Z, [8, p. 184], the method gives a concrete way of obtaining networks using a minimum number of reactive elements. Of course, as with any general PR synthesis, many transformers are required.

We believe that the method of Oono, [7], can also be used (here $D_1 = 0_n$ in all cases), but, although it has worked on all examples to which we have applied it, as yet no general proof of its validity exists.

> Des rêves! toujours des rêves! et plus l'âme
> est ambitieuse et délicate, plus les rêves
> s'éloignent du possible.
>
> Baudelaire, "L'invitation au voyage"

ACKNOWLEDGEMENTS

The support of the National Science Foundation under NSF G-18945 is gratefully acknowledged, as is the assistance of Yvonne Kilfoil who devoted her spare time to obtaining the excellent format of this report.

APPENDIX 1: McMILLAN'S DEGREE, δ

The degree of a rational matrix, $F(p)$, as defined by McMillan, [5, p. 543], is an extension of the concept of the degree of a rational scalar, $f(p)$, which is defined as

$$\text{degree } f(p) = \sum_{p_j} [\text{order of the pole of } f(p) \text{ at } p_j] \tag{1-1}$$

the sum being taken over all poles including infinity.

Consider an n x n matrix $F(p)$ whose elements are rational in p with complex coefficients. Using the theory of equivalent matrices and invariant factors, [11], we can write F as

$$F(p) = C\tilde{1}_{r,n}\Lambda 1_{r,n}D \tag{1-2}$$

where C and D are polynomial matrices with constant nonzero determinants, $1_{r,n}$ is the r x n matrix whose first r columns are 1_r with all other entries zero, and

$\Lambda = \text{diag}[e_1/\psi_1, \ldots, e_r/\psi_r]$
$r = \text{rank } F$
e_i divides e_{i+1}, $i = 1,\ldots,r-1$
ψ_{i+1} divides ψ_i, $i = 1,\ldots,r-1$
e_i and ψ_i are polynomials with unit leading coefficients
e_i/ψ_i is in lowest terms

Here ψ_1 is the least common multiple of the denominators of F.

Now let F have an m^{th} order pole at $p = p_j$, where p_j may be infinity, and define

$$\gamma_{ij} = \text{order of the pole at } p_j \text{ in } e_i/\psi_i \tag{1-3}$$

Thus, $\gamma_{1j} = m$ if p_j is finite and $\gamma_{rj} = m$ if $p_j = \infty$. We then define the degree of F at p_j, $\delta(F,p_j)$, by

$$\delta(F,p_j) = \sum_{i=1}^{r} \gamma_{ij} \qquad (1\text{-}4)$$

These degrees are now summed over all the poles of F.

Definition: Let F have poles at p_j, $j = 1,\ldots,q$, then the degree of F, $\delta(F)$, is defined by

$$\delta(f) = \sum_{j=1}^{q} \delta(F,p_j) = \sum_{j=1}^{q} \sum_{i=1}^{r} \gamma_{ij} \qquad (1\text{-}5)$$

We note that a pole at infinity must be included if it is present. $\delta(F)$ is merely the degree of the polynomials $\prod_{i=1}^{r} \psi_i$ or $\prod_{i=1}^{r} e_i$, whichever is largest.

This method of defining the degree allows the following property 4) to be easily proven, something which is very difficult to do with the "order" of Tellegen, [12, p. 1]. The following properties are merely stated, as all but the last two are proven in detail by McMillan. The last two properties are easily obtained by noting that no major change occurs in (1-2).

Properties:

1) $\delta(F)$ is an integer ≥ 0
2) $\delta(F) = 0$ if and only if F is a constant matrix
3) If F^{-1} exists, then $\delta(F^{-1}) = \delta(F)$
4) If $F = F_1+F_2$ where F_1 is finite at every pole of F_2, then $\delta(F) = \delta(F_1)+\delta(F_2)$
5) If $F(p) = f(p)G$ where $f(p)$ is a rational scalar and G is a constant matrix, then $\delta(F) =$ [degree $f(p)$][rank G]
6) If A and B are constant matrices, then $\delta(F) \leq \delta(AFB)$ where equality holds if A and B are nonsingular (this is true even if A and B are polynomial matrices with constant nonzero determinants)
7) If F is formed from an m x m matrix F_1 by bordering with zeros, then $\delta(F) = \delta(F_1)$

8) $\delta(\tilde{F}) = \delta(F)$

9) $\delta(F_*) = \delta(F)$

Although det F = det Λ, when F is nonsingular, it is not always true that $\delta(F)$ is equal to the degree of det F, as cancellation of terms may occur. This is illustrated by $F = p \dotplus 1/p$ which has $\delta(F) = 2$, degree det F = 0, as well as by Cauer's example, [13, p. 551]. Bayard defines the degree of F as the degree of det Λ, [14, p. 381], and claims that the properties listed above hold. However, the example given above shows that property 4) fails in this case.

APPENDIX 2: TRANSFORMER EQUIVALENTS

a) Consider the transformer network of Figure 2-1 where all turns ratio matrices are assumed nonsingular.

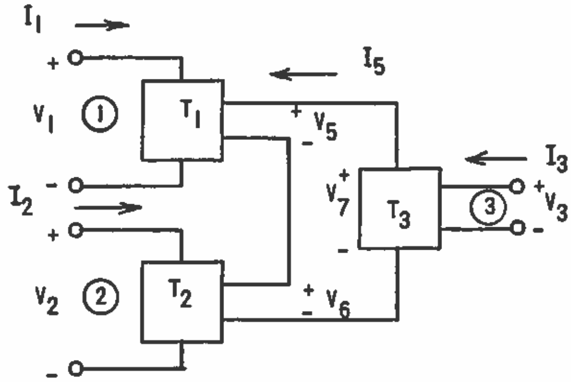


FIG. 2-1. TRANSFORMER CONNECTIONS.

Here, by the definition of a transformer 2n-port,

$$V_1 = \tilde{T}_1 V_5, \qquad I_5 = -T_1 I_1 \tag{2-1}$$

$$V_2 = \tilde{T}_2 V_6, \qquad I_5 = -T_2 I_2 \tag{2-2}$$

$$V_7 = V_5 + V_6 = \tilde{T}_3 V_3, \quad I_3 = T_3 I_5 \tag{2-3}$$

Therefore, solving the voltage equations of (2-1), (2-2) and substituting into (2-3), solved for V_3, gives

$$V_3 = \tilde{T}_3^{-1}\tilde{T}_1^{-1} V_1 + \tilde{T}_3^{-1}\tilde{T}_2^{-1} V_2 \tag{2-4}$$

Similarly solving the current equations

$$I_1 = -T_1^{-1} T_3^{-1} I_3 \tag{2-5}$$

$$I_2 = -T_2^{-1} T_3^{-1} I_3 \tag{2-6}$$

Equations (2-4), (2-5), and (2-6) describe the transformer network of Figure 2-2, which is, as a consequence, equivalent to the one of Figure 2-1.

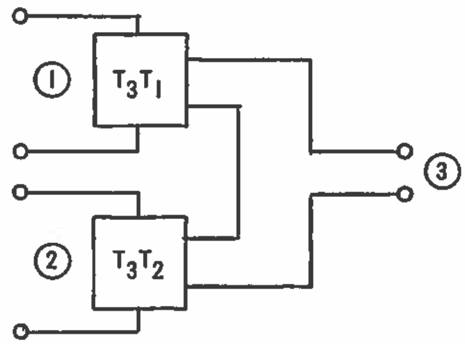


FIG. 2-2. EQUIVALENT OF FIG. 2-1.

b) We now show the equivalence of Figure 2-3.

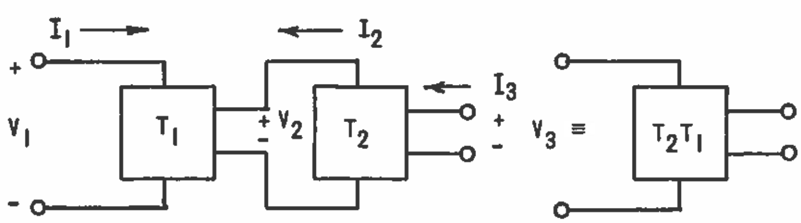


FIG. 2-3. CASCADE EQUIVALENCE

Here

$$V_1 = \tilde{T}_1 V_2, \; I_2 = -T_1 I_1 \tag{2-7}$$

$$V_2 = \tilde{T}_2 V_3, \; I_3 = T_2 I_2 \tag{2-8}$$

By direct substitution

$$V_1 = \tilde{T}_1 \tilde{T}_2 V_3, \; I_3 = -T_2 T_1 I_1 \tag{2-9}$$

which proves the equivalence.

c) Here we determine the relationship between parameters for the equivalence of Figure 2-4 to be valid.

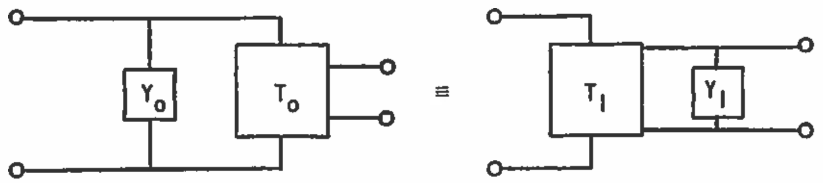


FIG. 2-4. COMMUTATION OF SHUNT CASCADE CONNECTION.

The left side is described by the chain matrix

$$\psi_L = \begin{bmatrix} 1_n & 0_n \\ Y_o & 1_n \end{bmatrix} \begin{bmatrix} \tilde{T}_o & 0_n \\ 0_n & T_o^{-1} \end{bmatrix} = \begin{bmatrix} \tilde{T}_o & 0_n \\ Y_o\tilde{T}_o & T_o^{-1} \end{bmatrix} \qquad (2\text{-}10)$$

while the right side is described by its chain matrix

$$\psi_R = \begin{bmatrix} \tilde{T}_1 & 0_n \\ 0_n & T_1^{-1} \end{bmatrix} \begin{bmatrix} 1_n & 0_n \\ Y_1 & 1_n \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 & 0_n \\ T_1^{-1}Y_1 & T_1^{-1} \end{bmatrix} \qquad (2\text{-}11)$$

Since for equivalence $\psi_L = \psi_R$, this requires

$$T_1 = T_o \qquad (2\text{-}12)$$

$$Y_1 = T_oY_o\tilde{T}_o \qquad (2\text{-}13)$$

In a similar manner the equivalence of Figure 2-5 requires

$$T_1 = T_o \qquad (2\text{-}14)$$

$$Z_o = \tilde{T}_1Z_1T_1$$

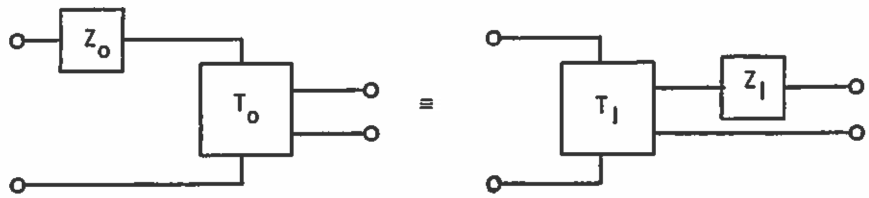


FIG. 2-5. COMMUTATION OF SERIES CASCADE CONNECTION.

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