Fibonacci Numbers as a Computer Structure Base

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Abstract:

The concept of a "Fibonacci-Lucas computer" is introduced. This is based upon advantageous properties of the Fibonacci and Lucas number systems. Some of these properties are reviewed, such as completeness with redundancy, nonadjacent form representation, and interrelationships between the two number systems.

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"¡ Que dulce y bello el viaje
por una tierra asi! " [1]

I. Introduction

By now the versatility of computers in the service of society is well recognized [2]. And in more and more applications, as in on site stroke patient care [3, p. 53], the desire for error free operation is obvious. Because present day computers are far from error free, it is my contention, and that also felt in other quarters [4], that the major problem facing future computer technology is that of error free computer operation. Also in order to strive toward error free operation it has seemed best to me to concentrate not only on the improvement of present day hardware and development of superior coding theory [5] but also upon the number base in which the computer does its calculation. One effort in this direction is a development within the p-adic number system of one of my present students [6] while another is that in terms of Fibonacci and Lucas numbers as proposed in conjunctionwith one of my former students [7]. Here I take this opportunity to present some of the ideas associated with the latter.

II. The Fibonacci and Lucas Numbers

Knowledge of the Fibonacci numbers apparently stems from the 13th century while presently there is a journal exclusively devoted to them (The Fibonacci Quarterly). An excellent and concise available treatment is the 92 page paperback of Hoggatt [8].

The Fibonacci numbers, F_i, are defined through the recurrance relation

$$F_i = F_{i-1} + F_{i-2}, \qquad F_1 = F_2 = 1, \qquad i = 0, \pm 1, \pm 2, \ldots \tag{1}$$

whereas another set of numbers, the Lucas numbers, L_i, are similar, differing only in initial conditions

$$L_i = L_{i-1} + L_{i-2}, \qquad L_1 = 1,\ L_2 = 3, \qquad i=0, \pm 1, \pm 2, \ldots \tag{2}$$

Examples are then:

...	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	...
	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	
...	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	...
...	L_{-6}	L_{-5}	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	L_5	L_6	...
	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	‖	
...	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	...

III. Properties of Computer Interest

A number of properties and relations on Fibonacci and Lucas numbers are available [8, p. 59]. Here we present a few of most interest to computer construction.

One of these properties of considerable importance to us is that any positive interger N can be written as a finite sum of Fibonacci and Lucas numbers, [8, p. 70]

$$N = \sum_{i=0}^{m_F} \alpha_i F_i = \sum_{i=0}^{m_L} \beta_i L_i \qquad (3)$$

where the α_i and β_i are binary numbers, that is either zero or one. In other words, as with the presently used radix -2 sequence $\{2^i\}$, the sequences $\{F_i\}$ and $\{L_i\}$ are, for $i \geq 0$, complete in the positive integers. However, the Fibonacci numbers possess a property not available in the radix -2 system, this being that the sequence $\{F_i\}$, $i \geq 0$, is complete when any one arbitrary F_j is deleted from the sequence; this property essentially distinguishes the Fibonacci sequence from any other complete sequence [9, Thm. 5].

For addition we note that, by (1),

$$F_i + F_i = F_i + F_{i-1} + F_{i-2} = F_{i+1} + F_{i-2} \qquad (4)$$

and hence there are carries in two directions. For multiplication the formula [8, p. 59] $F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$ can be used to obtain, for $n \geq m$,

$$F_m F_n = \sum_{i=0}^{m-2} (-1)^i F_{m+n-(2i+1)} \qquad (5)$$

The property which initially led us to the Fibonacci numbers for computer use is the existence of a non-adjacent form representation for any positive integer [8, p. 74], called the minimal representation. That is, every N in (3) can be written with

$$\alpha_j \alpha_{j+1} = 0 \quad \text{for } j = 0, \ldots, m_F - 1 \tag{6}$$

and this representation is unique. This property is available, and useful in coding, in the radix -2 system [10, p. 276] but there the coefficients are no longer binary, as -1 is required along with zero and one.

If we denote the "greatest common divisor" by (.,.) then we have [8, p. 39]

$$(F_m, F_n) = F_{(m,n)} \tag{7}$$

while

$$F_{-n} = (-1)^{n+1} F_n, \quad L_{-n} = (-1)^n L_n \tag{8}$$

and the Fibonacci and Lucas numbers are interconnected by various relationships such as [8, p. 59]

$$L_{m+n+1} = F_{m+1} L_{n+1} + F_m L_n \tag{9a}$$

$$F_{n+p} + F_{n-p} = \begin{cases} F_n L_p, & p \text{ even} \quad (9b) \\ L_n F_p, & p \text{ odd} \quad (9c) \end{cases}$$

$$2F_m = L_n F_{m-n} + F_n L_{m-n} \tag{9d}$$

Finally, since present computer circuitry works in a modular ring of integers, $Z_M = \{1, 2, \ldots, M\}$, it is of interest to note that [8, p. 52]

$$F_{n+2} - F_1 = \sum_{i=0}^{n} F_i, \qquad n \geq 1 \tag{10}$$

in which case we have

$$\sum_{i=0}^{n} F_i \equiv 0 \bmod M, \quad M = F_{n+2} - F_1 \quad : \tag{11}$$

Similarly one has

$$F_n F_{n+1} = \sum_{i=0}^{n} F_i^2, \qquad n \geq 1 \tag{12}$$

IV. Computer Construction Discussion

Because of the representations of any positive integer in terms of Fibonacci or Lucas numbers, (3), these complete number bases can be used as the bases for construction of a computer, which we might call a Fibonacci-Lucas computer [7]. Because only binary coefficients occur in these representations, binary logic using available integrated circuit modules can be taken over almost completely. Such circuitry has been presented for obtaining the minimal representation [11] while the Boolean expressions have been developed for addition incorporating the carries of (4) [7].

As a result of the redundancy of the Fibonacci numbers as a basis system, for which any one F_j can be deleted and completeness still maintained, one could conceivably design a computer such that if any one unit, depending only upon on F_j, fails, accurate calculations can still be processed with the remaining units. Of equal interest is the fact that economical calculations can occur directly in terms of the non-adjacent forms, thus giving faster results less subject to error than when more terms are present. The non-adjacent form allows standard coding to take place [10] should that be desired. Indeed since such coding occurs within a modular ring the choice of modulus M of (11) appears of significance for avoiding multiple end around carries. Too, since the Fibonacci recursion relation, (1), holds for negative indices, and, since, by (8), some of these numbers are negative, there may be advantages in using negative indexed numbers, as at (3), in developing a complete theory. For this various results are available [12] but apparently yet untapped.

However it appears to me that new means of coding and fault tolerability could best be developed by taking into account relationships, such as (9), between the Febonacci and Lucas numbers. For example, a reasonably accurate means of coding is through duplication. This could be improved upon, it seems to me, by building separately a Fibonacci and a Lucas computer and having monitoring of each other through the relationships, as (9), which appear capable of isolating the sources of error. This is why we call such a machine by the hyphonated title "Fibonacci-Lucas" and why we hope to pursue in considerably more depth the concepts.

"Bien sabe Dios que ese era
mi más hermoso sueño." [1]

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