*A Multivariable Bounded-reality Criterion**

by N. K. BOSE†

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley, California

K. ZAKI *and* R. W. NEWCOMB

Department of Electrical Engineering
University of Maryland, College Park, Maryland

ABSTRACT: *The multivariable approach to the synthesis of networks composed of a finite number of uniform lossless transmission lines, commensurable or incommensurable, and lumped passive elements is verified by showing that the multivariable rational matrix, $W(\lambda_0, \lambda_1, .., \lambda_n)$ is bounded real in $(n+1)$ complex variables if and only if W is bounded real in p after substituting, $\lambda_0 = \alpha_0 p + \beta_0$ and $\lambda_i = \tanh(\alpha_i p + \beta_i)$ for λ_i $(1 \leq i \leq n)$ where all the α's and β's are nonnegative and arbitrary, except for being not simultaneously zero in like indexed pairs. Consequences of this result are discussed.*

I. Introduction

Large-scale use of integrated circuits, along with a demand for microminiaturization, have necessitated the introduction of disciplines for systematic analysis and synthesis of lumped-distributed networks composed of a finite number of commensurate and/or incommensurate transmission lines along with lumped elements. The multivariable approach has been popularized for this purpose, as the study of single variable transcendental functions in one complex variable is converted to the study of multivariable rational functions in several independent complex variables. Such study necessarily rests on the subject of analytic functions of several variables, a topic which has been reasonably well explored by mathematicians. With reference to particular classes of lumped-distributed networks (the distributed network elements are assumed to be uniform, commensurate or incommensurate transmission lines with prescribed nonnegative real characteristic impedances and one-way time delays), Koga (**1**) gave his proof of his multivariable bounded-reality criteria. This result has recently been discussed by Youla (**2**) and Csurgay (**3**). The proof, originally given, dwells heavily on the properties of algebraic and algebroidal functions. The complexity, along with attendant lack of clarity, makes it desirable to find a

* This research was supported in part by the Air Force Office of Scientific Research under Grant No. AFOSR 70–1910.

† N. K. Bose is a visiting faculty member from the Department of Electrical Engineering, University of Pittsburgh, Pittsburgh, PA 15261, U.S.A.

simple proof for the validity of the multivariable approach to the synthesis of networks composed of a finite number of uniform lossless transmission lines, commensurable or incommensurable, and lumped passive elements. This is achieved here by establishing a necessary and sufficient condition for multivariable bounded reality. The proof for this condition is simple and suitable for presentation to a regular class, where multivariable theory is expected to play an important future role, especially because of its equally important applications in areas outside network theory.

II. Main Result

The main theorem to follow dwells heavily on the result given and proved in Theorem I, the essential ingredients of which were presented very recently (4).

Theorem I. The multivariable polynomial $g(\lambda_0, \lambda_1, \ldots, \lambda_n)$ relatively prime to $(\lambda_k - 1)$, $k = 1, 2, \ldots, n$, has no zeros in the open right-half polydomain $\mathrm{Re}\,\lambda_i > 0$ $(i = 0, 1, \ldots, n)$ if and only if the function

$$g[\alpha_0 p + \beta_0, \tanh(\alpha_1 p + \beta_1), \ldots, \tanh(\alpha_n p + \beta_n)]$$

has no zeros in the open region $\mathrm{Re}\,p > 0$ for all nonnegative α_i and β_i which are not simultaneously zero in like indexed pairs $(0 \leqslant i \leqslant n)$.

Proof: (a) *If part.* We wish to show $g(\lambda_0, \lambda_1, \ldots, \lambda_n) \neq 0$ in $\mathrm{Re}\,\lambda_i > 0$, if $g[\alpha_0 p + \beta_0, \tanh(\alpha_1 p + \beta_1), \ldots, \tanh(\alpha_n p + \beta_n)] \neq 0$ in $\mathrm{Re}\,p > 0$ for arbitrary α_i, β_i satisfying the nonnegative constraints. Thus assume the contrary, that is, for $g[\alpha_0 p + \beta_0, \tanh(\alpha_1 p + \beta_1), \ldots, \tanh(\alpha_n p + \beta_n)] \neq 0$ in $\mathrm{Re}\,p > 0$ for all suitable nonnegative α_i, β_i there is a set λ_{i0} with $\mathrm{Re}\,\lambda_{i0} > 0$ such that $g(\lambda_{00}, \lambda_{10}, \ldots, \lambda_{n0}) = 0$. As the zeroes of polynomials of two or more variables lie on continuous algebraic curves, it is always possible to select the λ_{j0}'s such that $\lambda_{j0} \neq 1$, for $j = 1, 2, \ldots, n$, in cases when $g(\lambda_0, \lambda_1, \ldots, \lambda_n) = 0$ with at least two of the λ_j's assuming the value unity. For example,

$$g(\lambda_0, \lambda_1, \lambda_2) = (\lambda_1 \lambda_2 - 1)\lambda_0$$

has a zero at $(2, 1, 1)$ as well as at $(2, c, 1/c)$ for $c \neq 1$. The only case when the choice mentioned above is not possible is when factors of the form $\prod_{j=1}^{n}(\lambda_j - 1)^{m_j}$ for integer m_j occur and these are ruled out in the statement of the theorem. Set $\gamma_0 = \lambda_{00}$, $\gamma_k = \tanh^{-1}\lambda_{k0}$, $k = 1, \ldots, n$ and choose $p_0 = \sigma_0 + j\omega_0$ such that

$$\angle p_0 = \tan^{-1}(\omega_0/\sigma_0) \geqslant \tan^{-1}(\mathrm{Im}\,\gamma_i/\mathrm{Re}\,\gamma_i) = \angle \gamma_i \quad \text{for all } i = 0, 1, \ldots, n.$$

Prior to this we use the periodicity and positive-real properties of tanh to insure $\mathrm{Im}\,\gamma_i \geqslant 0$, $\mathrm{Re}\,\gamma_i > 0$. This choice of p_0 guarantees that we can find nonnegative α_i, β_i such that

$$\gamma_i = \alpha_i p_0 + \beta_i, \quad i = 0, 1, \ldots, n,$$

as a graphical comparison γ_i with p_0 shows. But this choice of p_0 and α_i, β_i gives $g[\alpha_0 p_0+\beta_0, \tanh(\alpha_1 p_0+\beta_1), \ldots, \tanh(\alpha_n p_0+\beta_n)] = 0$ contrary to assumption.

For clarity and later purposes, we comment that, if γ_k is the γ_i with the largest angle, we can choose $p_0 = \gamma_k$, in which case $\alpha_k = 1$, $\beta_k = 0$.

(*b*) *Only if part*. For this we wish to show that $g(\lambda_0, \lambda_1, \ldots, \lambda_n) \neq 0$ in $\operatorname{Re}\lambda_i > 0$ implies $g[\alpha_0 p+\beta_0, \tanh(\alpha_1 p+\beta_1), \ldots, \tanh(\alpha_n p+\beta_n)] \neq 0$, $\operatorname{Re} p > 0$. Assume $g[\alpha_0 p_0+\beta_0, \tanh(\alpha_1 p_0+\beta_1), \ldots, \tanh(\alpha_n p_0+\beta_n)] = 0$ for some p_0 with $\operatorname{Re} p_0 > 0$ and nonnegative α_i, β_i which are not simultaneously zero in like indices. Choosing $\lambda_0 = \alpha_0 p_0+\beta_0$, $\lambda_i = \tanh(\alpha_i p_0+\beta_i)$ shows, however, the impossibility since $g(\lambda_0, \lambda_1, \ldots, \lambda_n)$ would then be zero with $\operatorname{Re}\lambda_i > 0$ by the positive reality of tanh.

The theorem covers $\operatorname{Re}\lambda_i > 0$, but it is clear that the same result holds for $\operatorname{Re}\lambda_i \geqslant 0$ by considering $\operatorname{Re} p \geqslant 0$ for all nonnegative α_i, β_i which may now be simultaneously zero.

The necessity of imposing the restriction that $g(\lambda_0, \lambda_1, \ldots, \lambda_n)$ be devoid of factors of the form, $\prod_{k=1}^{n}(\lambda_k-1)^{m_k}$, with integer m_k, can be further substantiated by the simple example given next.

Example I. $H(\lambda_0, \lambda_1) = \lambda_0(\lambda_1-1)$ has zeros in the domain $\operatorname{Re}\lambda_0 > 0$, $\operatorname{Re}\lambda_1 > 0$, but $H[\alpha_0 p+\beta_0, \tanh(\alpha_1 p+\beta_1)] = (\alpha_0 p+\beta_0)[\tanh(\alpha_1 p+\beta_1)-1])$ has no zeros in $\operatorname{Re} p > 0$, for all nonnegative values of $\alpha_0, \beta_0, \alpha_1, \beta_1$, because the tanh function cannot assume the value 1 in any finite domain.

In Theorem II a multivariable bounded reality criterion that verifies the application of multivariable theory in lumped-distributed network synthesis is established. To facilitate reading the following result is included (**5**).

Bounded reality. A rational multivariable matrix, $W(\lambda_0, \lambda_1, \ldots, \lambda_n)$ is bounded real if and only if

(i) W is real for $\lambda_i > 0$ (i $= 0, 1, 2, \ldots, n$) (actually with the rational restriction, W is real for any real λ_i),

(ii) the least common denominator of W is Hurwitzian, i.e. it has no zeros in the open polydomain, $\operatorname{Re}\lambda_i > 0$ $(i = 0, 1, 2, \ldots, n)$,

(iii) $I - W^* W$ is nonnegative Hermitian for $\operatorname{Re}\lambda_i = 0$ $(i = 0, 1, 2, \ldots, n)$, except at possible singularities of indeterminacy, where I is an identity matrix of appropriate order and the "star" superscript denotes "conjugate transpose".

When transcendental functions are present in the elements of the matrix, condition (iii) should hold over $\operatorname{Re}\lambda_i > 0$ (**6**), as the following simple example illustrates.

Example II. Consider the scalar function

$$w(\lambda_0) = \frac{\cosh \alpha\lambda_0}{\cosh \alpha\lambda_0 - \sinh \alpha\lambda_0}, \quad \alpha > 0.$$

Clearly all conditions (i)–(iii) in Definition 1 are satisfied by $w(\lambda_0)$. Yet, it is not bounded real, as $1-w^*w$ is not nonnegative definite in $\text{Re}\,\lambda_0>0$. Actually continuity considerations and holomorphicity in $\text{Re}\,\lambda_0>0$ by condition (ii) suggest that for the scalar function, $w(\lambda_0)$, condition (iii) must be tested for nonnegative definiteness of the appropriate function in $\text{Re}\,\lambda_0 = 0$, as well as at any one point in $\text{Re}\,\lambda_0>0$. Theorem II is stated and proved exact.

Theorem II. A multivariable rational matrix, $W(\lambda_0,\lambda_1,\ldots,\lambda_n)$ is bounded real in $(n+1)$ variables, if and only if W is bounded real in p after the substitution of $\lambda_0=\alpha_0 p+\beta_0$, $\lambda_i=\alpha_i p+\beta_i$ for $i=1,2,\ldots,n$, where all the α's and β's are non-negative and arbitrary, except for being not simultaneously zero in like indexed pairs.

Proof: Condition (i) is evidently satisfied. Condition (ii) is also satisfied before and after the substitution given in the statement of the theorem. This results after applying Theorem I and noting that factors of the form $\prod_{k=1}^{n}(\lambda_k-1)^{m_k}$ cannot be present in the least common denominator of $W(\lambda_0,\lambda_1,\ldots,\lambda_n)$ if this is to be bounded real (it is also seen that presence of such a factor will at once lead to violation of condition (iii), modified for the transcendental case). That condition (iii) is satisfied before and after the aforementioned substitution can be directly verified using a constructive argument similar to that advanced in the proof for Theorem I .

The fact that Theorem II serves to verify the application of the multivariable approach in lumped distributed synthesis becomes apparent from the realization of

$$\tanh(\alpha_i p+\beta_i)=\frac{\tanh\alpha_i p+\tanh\beta_i}{1+\tanh\alpha_i p\tanh\beta_i},\quad \alpha_i\geqslant 0,\quad \beta_i\geqslant 0,$$

with uniform lossless transmission lines and non-negative resistors.

III. Some Special Results

In this section, some special results that follow from the theorem, stated and proved above, are given. Rational functions and polynomials of two independent complex variables play very significant roles in network theory (7) as well as in other areas, such as bidimensional recursive filtering (8). The following assertion can be justified.

Assertion 1. The two-variable polynomial, $g(\lambda_0,\lambda_1)$, has no zeros in the open polydomain, $\text{Re}\,\lambda_0>0$, $\text{Re}\,\lambda_1>0$, if and only if both of the following conditions are satisfied:

(a) $g[p,\tanh(\alpha_1 p+\beta_1)]$ has no zeros in $\text{Re}\,p>0$ for all $\alpha_1\geqslant 0$, $\beta_1\geqslant 0$ (not both simultaneously zero), and

(b) $g(\alpha_0 p+\beta_0,\tanh p)$ has no zero in $\text{Re}\,p>0$, for all $\alpha_0\geqslant 0$, $\beta_0\geqslant 0$ (not both simultaneously zero).

Again, factors of the form $(\lambda_1-1)^{m_1}$ are assumed to be absent in $g(\lambda_0,\lambda_1)$.

Outline of proof: Observing the comment to part (a) of the proof of Theorem I, we see that condition (a) holds for those $\lambda_{00}, \lambda_{10}$ with $\angle\gamma_0 \geqslant \angle\gamma_1$ while condition (b) holds for $\angle\gamma_0 \leqslant \angle\gamma_1$. In order to test for all λ_0 and λ_1 in $\mathrm{Re}\,\lambda_0 > 0$, $\mathrm{Re}\,\lambda_1 > 0$, we need to consider simultaneously both cases. Q.E.D.

The result stated in the above assertion enables one to convert the test for location of zeros of a transcendental function of one complex variable and four real variables to a test for location of zeros of two transcendental functions of one complex variable and two real variables each. Similar results as stated in the assertion can be written down for n-variable polynomials, when $n > 2$.

As shown in (**9**) the results of (**1**) are in error. The results of (5) in turn appear to rest upon the validity of the following statement. Consequently, it appears important to obtain a clear and concise verification, or a counter-example, for the statement, perhaps by using ideas of Theorem I.

Statement (*true or false?*)

The $(n+1)$ variable polynomial $g(p, \lambda_1, \lambda_2, \ldots, \lambda_n)$ has no zeros in the region $\mathrm{Re}\,p > 0$, $\mathrm{Re}\,\lambda_i > 0$, $1 < i < n$, if and only if the function

$$g(p, \tanh\alpha_1 p, \tanh\alpha_2 p, \ldots, \tanh\alpha_n p)$$

has no zeros in the region $\mathrm{Re}\,p > 0$, $\alpha_i > 0$, $1 \leqslant i \leqslant n$. It is assumed that factors of the form $\prod_{k=1}^{n}(\lambda_k - 1)^{m_k}$ are absent in $g(p_0\lambda_1, \lambda_2, \ldots, \lambda_n)$, so that the 'if' part is not automatically violated as explained before.

The question is valid because the number of degrees of freedom in the multivariable polynomial, $g(p, \lambda_1, \lambda_2, \ldots, \lambda_n)$ for $n \geqslant 1$ is $2n+2$ and thus larger than the number of degrees of freedom, $n+2$, in the generated function, $g(p, \tanh\alpha_1 p, \tanh\alpha_2 p, \ldots, \tanh\alpha_n p)$. However, as the objectives of this paper are different, the statement above will not be elaborated upon any further, but is left for consideration by others. The comment made at the end of part (a) in the proof of Theorem I and the result of Assertion 1 serve to substantiate the fact that there is no inconsistency arising out of the degree-of-freedom considerations of the type just mentioned when applied to our Theorems I and II.

Assertion 1 and Theorem II lead to the next assertion.

Assertion 2. A two-variable rational matrix, $W(\lambda_0, \lambda_1)$ is bounded real if and only if the following two conditions are satisfied:

(a) $W[p, \tanh(\alpha_1 p + \beta_1)]$ is bounded real in p for all nonnegative α_1, β_1 not simultaneously zero, and

(b) $W[(\alpha_0 p + \beta_0), \tanh p]$ is bounded real in p for all nonnegative α_0, β_0 not vanishing simultaneously.

The proof is simple and is omitted, for brevity.

IV. Discussion

Results leading to verification of the multivariable approach to the synthesis of networks composed of a finite number of commensurate and incommensurate uniform lossless transmission lines (with real characteristic impedances and one-way time delays) and lumped passive elements have been established. The proofs for the results are simple and constructive and are very suitable for presentation in a regular class, where multivariable theory is expected to play an important future role. The special results cited in Section III only serve to link tests on certain multivariable polynomial or rational functions with those on certain transcendental functions of a single complex variable. It is recognized that tests for zero distribution of both multivariable polynomials and single-variable transcendental functions are difficult, though results applicable to particular cases are known to exist (**7, 10–12**). Therefore, the contents of Section III are to be viewed as certain, possibly interesting, consequences of the main results established in this paper, rather than necessarily as preludes to some workable computational algorithms.

References

(**1**) T. Koga, "Multivariable bounded-reality criteria", Polytechnic Institute of Brooklyn, Report MRI-1381-67, pp. 1–21, Oct. 1967.

(**2**) D. Youla, "A review of some recent developments in the synthesis of rational multivariable positive-real matrices", in "Mathematical Aspects of Electrical Network Analysis", ed. by H. S. Wilf and F. Harary, pp. 161–190, Am. Math. Soc., *SIAM-AMS Proc.*, 1972.

(**3**) Private communication, J. D. Rhodes to N. Bose at N.A.T.O. Advanced Studies Institute, Bournemouth, England, Sept. 1972.

(**4**) N. K. Bose, K. Zaki and R. W. Newcomb, "A theorem in lumped-distributed network synthesis", presented at the Sixteenth Midwest Symposium on Circuit Theory, Waterloo, Canada, April 1973.

(**5**) T. Koga, "Synthesis of a resistively terminated cascade of uniform lossless transmission lines and lumped passive lossless two-ports", *IEEE Trans. Circuit Theory*, Vol. CT-18, No. 4, pp. 444–455, July 1971.

(**6**) R. W. Newcomb, "Linear Multiport Synthesis", p. 94, McGraw-Hill, New York, 1966.

(**7**) H. J. Ansell, "On certain two-variable generalizations of circuit theory with applications to networks of transmission lines and lumped reactances", *IEEE Trans. Circuit Theory*, Vol. CT-11, No. 2, pp. 214–223, June 1946.

(**8**) T. S. Huang, W. J. Schreiber, O. J. Tretiak, "Image processing", *Proc. IEEE*, Vol. 59, No. 11, pp. 1586–1609, Nov. 1971.

(**9**) J. D. Rhodes and P. C. Marston, "Cascade synthesis of transmission lines and lossless lumped networks", *Electron. Lett.*, Vol. 7, No. 20, pp. 621–622, Oct. 1971.

(**10**) N. K. Bose and E. I. Jury, "Positivity and stability tests for multidimensional filters (discrete-continuous)", *IEEE Trans. Audio Electro-acoustics*, to be published, 1974.

(**11**) N. G. Cebotarev, "On Hurwitz's problem for transcendental functions", *C. R. (Doklady) Acad. Sci. URSS* (N.S.), Vol. 33, pp. 479–481, 1941.

(**12**) L. S. Pontryagin, "On zeroes of some transcendental functions", *Am. Math. Soc. Transl.*, Ser. 2, Vol. 1, pp. 95–110, 1955.