SUBHARMONICS IN FUNCTIONAL DIFFERENTIAL SYSTEMS OF FIRST ORDER*

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Abstract

Simple subharmonics in first order time-invariant functional differential systems are investigated, it being shown that under the presence of delay or non-linearities coupled into the dynamics such subharmonics can exist though not in more realizable circumstances.

1. INTRODUCTION

The use of subharmonics in engineering systems is well recognized [1][2] with interesting studies on various aspects available [2][3]. However, almost exclusively these studies are limited to second order differential equations. But because first order systems are normally more easily realized physically, requiring only one dynamical element instead of two or more, there is some merit in an investigation of those first order systems which can generate subharmonics. Here we show instances in which a scalar first order differential system can and can not generate simple subharmonics.

First we define the class of first order systems under investigation along with the meaning of a simple subharmonic. Then we show that no simple subharmonics exist for the systems most naturally realized, say by electronic circuitry. Following this we show that by the use of other characteristics, such as delay or nonlinearities on the dynamics, simple subharmonics of any order can be obtained.

2. SIMPLE SUBHARMONICS

Consider a scalar time-invariant system with input u and output y described through the functional equation

$$F(y,\dot{y},y_\tau,\dot{y}_\tau)=u \ , \quad \dot{y}=\frac{dy}{dt} \ , \quad y_\tau=y(t-\tau), \ \tau>0 \qquad (1)$$

If $u(t)=U\cos(n\omega t+\varphi_u)$ yields $y(t)=Y\cos(\omega t+\varphi_y)$ for real numbers $U, Y, \omega, \varphi_u, \varphi_y$ and integer $n>1$, then this input output pair is called a <u>simple subharmonic</u> of order n for the first order differential system F. For a simple subharmonic we will assume, as we can by a linear scaling and shifting of time t, that

$$u(t)=U\cos(nt+\varphi) \qquad (2a)$$

$$y(t)=Y\cos t \qquad (2b)$$

We now investigate various classes of F.

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Case 1 (No Simple Subharmonics):

If (1) takes the form

$$\dot{y}+f(y)=u \tag{3a}$$

$$f(y)=\sum_{k=0}^{\infty} a_k y^k \tag{3b}$$

we immediately see the impossibility of simple subharmonics by direct substitution of (2) into (3) since $u-f(y)= \alpha \sin nt+\sum_{k=0}^{\infty} \alpha_k \cos kt$ has no possibility to equal $\dot{y}= -Y \sin t$. Since (3) probably represents the simplest first order differential systems for realization in terms of transistor electronics, the generation of simple subharmonics by electronic means with only one dynamic element necessitates some type of complication. The following three cases illustrate the types of complications under which simple subharmonics result.

Case 2:

If (1) takes the form

$$\dot{y}g(y)=u \tag{4a}$$

$$g(y)= \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} y^{n-(2k+1)} (1-y^2)^k \tag{4b}$$

where $[x]$ denotes the integer part of x, that is the greatest integer less or equal to x, and $\binom{n}{r}= \frac{n!}{(n-r)!\, r!}$, then a simple subharmonic of order n results when $\varphi =\pi/2$, U=Y=1, in (2). This is seen by direct substitution of (2) into (4) and the use of standard trigonometric identities [4, p. 75]. Here g(y) was determined by expanding $(e^{j\theta})^n =(\cos\theta +j \sin\theta)^n= e^{jn\theta} = \cos n\theta + j \sin n\theta$ in a binomial series as powers of $\cos\theta$ and $\sin\theta$ and then taking the imaginary part.

Case 3:

If (1) takes the form

$$(\dot{y})^2 + h(y)=u \tag{5a}$$

$$h(y)=(1-y^2)+ \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} y^{n-2k} (1-y^2)^k \tag{5b}$$

where $[x]$ and $\binom{n}{r}$ have the same meaning used in Case 2, then a simple subharmonic solution of order n results when $\varphi=0$, U=Y=1 in (2). Also this can be verified by the direct substitution of (2) into (5) and use of trigonometric formulae. Here h is found as was g, except by taking the real part and adding $\sin^2\theta$.

The next two cases deal with two types of scalar differential equations with delay, in which the function y(t) and its derivative enter in different manners.

Case 4:

Consider the linear scalar differential equation with delay

$$\dot{y}(t)+ay(t)+b\dot{y}(t-\tau)+c\, y(t-\tau)=U\cos(n\omega t+\varphi) \tag{6}$$

This equation can only have a periodic solution with the same period as the input function $2\pi/\omega$ and therefore no simple subharmonics exist for this equation. This impossibility of simple subharmonics is clear by a direct substitution of (2) into (6) for with only linear terms on the left side of (6) no terms of the same frequency as the input force will be generated when $y=Y\cos \omega t$ is substituted.

Case 5:

Consider the nonlinear scalar differential equation with delay

$$\dot{y}+ y(t-\tfrac{\pi}{2}) + T_n(y) = \cos nt \tag{7a}$$

where $T_n(y)$ is the Chebychev polynomial defined by [5, p. 508].

$$T_n(y) = \cos(n \cos^{-1} y) \tag{7b}$$

Equation (7a) has a periodic solution $y= \cos t$ which is a simple subharmonic solution of order n.

This can be verified immediately in the following two steps:

a) The homogeneous linear differential equation with retarded argument:

$$\dot{y}+y(t-\frac{\pi}{2}) = 0 \quad (7c)$$

has a periodic solution of the form $y = \cos t$ as has been shown by Yorke [6], and as is easily seen by direct substitution.

b) Since $T_n(\cos t) = \cos nt$, adding T_n to both sides of (7c) gives, for (7a),

$$\dot{y}+ y(t-\frac{\pi}{2}) + T_n(y) = T_n(y) = \cos nt .$$

3. DISCUSSION

In the above we have shown that time-invariant first order systems are capable of generating simple subharmonics. However, in order to do this the devices needed to construct the system require characteristics which are not readily available. For example the nonlinearities of (4b), (5b) and (7b) have negative as well as positive coefficients in the polynomials necessitating active components. Likewise the presence of delay or the manner of coupling the nonlinearity with the dynamics requires some sophistication in circuit construction.

Using similar techniques to the last three cases we can derive many other first order systems to yield simple subharmonics, for example $ym(\dot{y})=u$. Likewise if we are willing to consider time-variable systems many first order systems would become available, for example,

$$-2\dot{y} \sin(n-1)t + T_{n-2}(y) = u = \cos nt \quad (8)$$

It is valuable also to mention here a general theorem on the possible occurence of subharmonics in a system of differential equations with delay given in [7].

Theorem [El'sgol'ts and Norkin]

The equation

$$\dot{y}(t) = F_1(y(t), y(t-\tau_1), \ldots, y(t-\tau_m), \dot{y}(t-\tau_1), \ldots, \dot{y}(t-\tau_m), \mu t) + F_2(t)$$

where x(t), F_1, and F_2 are vector functions, and all τ_1 are constants, $0< \tau_1<\ldots<\tau_m$, for sufficiently small μ, may have a periodic solution only in the case of a periodic vector function $F_2(t)$, where the period of the solution may only be equal to or a multiple of the period of the function $F_2(t)$.

However, actual conditions for the existence of these possible subharmonics are missing.

At present these ideas are being developed toward hearing improvement circuits. However, prior to going on to physical realization, it is important to investigate the stability of the systems. Preliminary investigation has shown, however, that this investigation will be a lengthy one.

4. REFERENCES

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5. BIOGRAPHIES

1. Mrs. Ferial El-Mokadem was born in Egypt in 1944. She attended the College of Engineering, Cairo University, and received her B. S. degree in Electrical Engineering in 1966. In the period 1966-72 she worked as a demonstrator in the Department of Mathematical and Physical Sciences at the College of Engineering, Cairo University. She received her M. S. degree in 1972 from the same University in the field of Synthesis of Passive Filters.

Presently, she is a Ph. D. candidate at the University of Maryland working on her dissertation under the supervision of Prof. Newcomb in the theory of subharmonic generation.

2. Professor Newcomb was born in Glendale California in 1933 and received degrees from Purdue, Stanford and the University of California, Berkeley. Presently he is Professor and Director of the E. E. Office of Graduate Studies at the University of Maryland. Besides his interest in subharmonics for hearing improvement his research is in the area of the operator theory of networks and computer structure number bases and devices.