The Realization of n-Port Networks Without Transformers—A Panel Discussion*

This is the first in a series of panel discussions of the status of unsolved problems in circuit theory, first suggested to the PGCT by Lotfi A. Zadeh. The problem of realizing an n-port network without transformers is one which has received considerable attention and now appears to be near solution. It is therefore a problem which should be clearly summarized and defined to avoid repetition of effort and to speed the time of the presentation of a complete solution. Each discussant was asked to define the problem, to outline methods of solution, and to cite key difficulties.—*The Editor*

Statements By Discussants

E. A. Guillemin:† This title may mean different things to different people according to their specific interests and their interpretation of the term "port" as a point of access or merely as an alternate designation for a node pair. Both interpretations may be appropriate in the same problem at different stages of its solution.

My own interest in the n-port problem arises from a long-standing desire to place synthesis procedures on a much more general basis that, unlike the Foster, Brune, Darlington, etc., methods, does not lead to specific topological configurations unalterably fixed by the theoretical aspects of these methods. These conventional methods of realizing driving-point and/or transfer functions are coupled with certain structural forms: lattices, tee's, pi's, ladders, or these things in parallel or in tandem—sometimes with and sometimes without terminal resistances. We pick a method, and we are stuck with a certain topological configuration whether we like it or not. Frequently we do not like it but there is little we can do about it. The ability to absorb parasitic elements, for example, is tied in with this situation. Sometimes the resulting configuration permits us to absorb such elements, sometimes it doesn't. Sometimes we can do some horse-trading, like absorbing some particularly obnoxious parasite, if we are willing to relax some feature in the desired response. It takes a lot of experience with realization techniques to recognize such possibilities, but even so we often find ourselves frustrated. Whoever says that passive network synthesis has reached maturity in the sense that we now understand how to do all the essential things and only a little minor polishing remains to be done has not really tried to solve practical problems with this stuff. Actually the accomplishment to date represents only a few beacons in a sea of darkness. We are still waiting for the big lights to be turned on, and a recently revived interest in the "n-port problem" as I see it is an encouraging move that promises ultimately to achieve that end.

In the normal synthesis problem we are given a rational function or set of functions (impedances, admittances, or dimensionless ratios) and a network with an appropriate number of access points realizing these functions is sought. A method of solution having the desired generality consists essentially of two major steps: 1) construction of appropriate parameter matrices from the given rational function or functions, and 2) construction of a network appropriate to these parameter matrices. Each of these steps present a huge task; and the two are, of course, interrelated since construction of parameter matrices must not be done without consideration of their realizability conditions.

In step 1) the concept of normal coordinates is vital since there is an intimate connection between these coordinates and the partial fraction expansion of desired rational functions (their residues, except for an appropriate normalization, are the pertinent direction cosines) and, moreover, because normal coordinate transformations are intimately linked with parameter matrices. Step 2) involves the "n-port problem" with which the present discussion is concerned. This indeed is the setting in which that problem is of major importance.

As is well understood by all who have worked on this problem, we distinguish between procedures appropriate to the realization of a short-circuit admittance matrix and an open-circuit resistance matrix. On the first of these problems the most important part of the solution has been accomplished. On the second practically nothing has so far been accomplished.

One can, of course, invert a given open-circuit resistance matrix, thus converting it into a short-circuit conductance matrix the realization of which is likewise appropriate to the given resistance matrix; but this is not useful in the general synthesis problem outlined above where two or all three kinds of elements are involved. In this same sense the realization of an nth order conductance matrix in a network involving more than $(n + 1)$ nodes is not useful

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either; and while this more general realization problem is of academic or collateral interest, it lacks the motivation given above.

This motivation incidentally has a collateral aspect. Once we succeed in solving the synthesis problem in this way, we will no longer be plagued with the equivalent network problem, for it will become an integral part of the total procedure. All possible networks will be included in the end results.

Ronald M. Foster:† We assume that in all networks under consideration exactly n ports, or pairs of terminals, are specified; and that all access to the network is through these specified ports only. In the general case this would involve a total of $2n$ accessible terminals, but these are not necessarily all distinct since various pairs of ports may have a terminal in common. There is considerable latitude with respect to the number and type of elements constituting the network, which may be active or passive, reciprocal or nonreciprocal, finite or infinite, etc. One problem however of very considerable importance, underlying the treatment of all such networks, is that of the single-element-kind network. That is, the network is composed of elements which are all positive multiples of some fundamental building block (including zero and infinite values), finite in number, with no gyrators and with no transformers, ideal or otherwise. This particular part of the discussion will be limited exclusively to this particular case,—the single-element-kind network with exactly n ports.

Accordingly, in treating this somewhat specialized case of the problem (elements of a single kind) there is no loss in generality in assuming all the elements to be ordinary resistors. Of course negative resistors are excluded. The main questions to be studied concern 1) a canonical form for such a network, and 2) necessary and sufficient conditions on the short-circuit admittances Y_{ij} of the network with respect to the specified n ports, referred to as the Y matrix (or occasionally as the G matrix, in view of this restriction to resistors); or the open-circuit impedances Z_{ij} of the network with respect to the specified n ports, referred to as the Z matrix (or occasionally as the R matrix).

By a canonical form we mean a set of resistors connected together in some fixed structural arrangement so that by suitably adjusting the values of the various resistances (including zero and infinite values), this network may be the complete equivalent with respect to the n ports of any resistive n-port network whatsoever; and furthermore that this be accomplished with the minimum possible number of elements.

It is known that any n-port resistive network with $2n$ terminals may be replaced by a directly-connected network with one resistor connecting every pair of the terminals. If no ports have a terminal in common, then this directly-connected network would consist of $n(2n-1)$ elements. This general network can represent all possible cases, since certain of the elements can be reduced to zero or infinite values to care for those situations where two or more of the ports have terminals in common. But, although this general network can represent any possible resistive n-port, it is not necessarily a canonical network, as outlined above, since it may not be composed of the minimum number of elements. All that we can be certain of at the moment is that such a canonical network contains no more than $n(2n-1)$ elements; and that it contains at least $n(n+1)/2$ elements, since there are that many conditions to be met in the general case.

Let us consider the very special case of a resistive 2-port. Perhaps it would be natural to expect a canonical network to consist of a simple wye or delta connecting the ports. Unfortunately this is not canonical in the sense discussed above since no such single structure can represent all cases without rearranging the connections, since any fixed connection will represent only one fixed sign of the mutual admittance or impedance. We would have to include some sort of switching device, and this is not included in our set of admissible elements. On the other hand, a six-element network directly connecting the four terminals would certainly be capable of representing all possible cases; but this is presumably not canonical because it contains too many elements. A canonical network is afforded by a four-element general lattice, not a symmetrical lattice. That is, without changing any connections such a lattice can represent any possible resistive 2-port; and hence may be regarded as a true canonical network for the resistive 2-port. Furthermore any one of the four component resistors may be limited to two values, zero and infinite, if so desired, with the other three resistors capable of assuming all values.

Thus in the general case of n-ports we expect a canonical network to have a number of elements somewhere between the two extremes of $n(2n-1)$ and $n(n+1)/2$ elements. Actually very little seems to have been done up to the present in studying this question.

The other major question is this: Given a matrix (real, square, symmetric, of order n) what are necessary and sufficient conditions that it be realizable as the Y matrix of a resistive n-port network (of the type outlined above), or as the Z matrix of such a network?

Something must first be said about this subject of necessary and sufficient conditions. Any logically equivalent statement of the problem can of course be cited as a necessary and sufficient condition for the original problem. It is thus relatively easy to give many different sets of necessary and sufficient conditions for the same problem. They are all logically equivalent. The real question relates to the utility of the conditions. Are they easy to apply? Do they involve essentially different operations from those normally envisaged in the original problem? Are the proposed sufficiency conditions simply a program of

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synthesis which may or may not work out? As examples of the more desirable sort of necessary and sufficient conditions, consider the following: A necessary and sufficient condition for the realizability of the kind of matrix discussed above as the admittance (or impedance) matrix of a resistive n-port network in which ideal transformers are permitted is simply that it be a positive definite matrix (singular or nonsingular). This is a condition which can be tested directly by well-established mathematical means. Or a necessary and sufficient condition for a function of the complex radian frequency s to be realizable as the driving-point impedance of a finite network composed of all three kinds of ordinary elements (resistors, inductors, capacitors) is that it be a rational positive real function. Again this is a condition which can be tested directly, and by a choice among several different methods.

Up to the present no quite satisfactory set of necessary and sufficient conditions for our present problem seems to have been formulated. Certain necessary conditions and certain sufficient conditions are known.

A reasonably well-known necessary condition on a matrix for it to be either the Y or the Z matrix of a resistive n-port is that it be a paramount matrix, that is, a real symmetric square matrix in which any principal minor is greater than or equal to the absolute value of any other minor drawn from the same rows as the principal minor [15], [16], [36], [41].

Various sufficient conditions are also known:

1) For realization as either Y or Z, the matrix be paramount and the number of ports n be not greater than 3. [Thus in this case, satisfactory necessary and sufficient conditions are known] [37], [41].

2) For realization as Y, the matrix be dominant, that is, any element on the main diagonal is greater than or equal to the sum of the absolute values of all the other elements in the same row as the element on the main diagonal [36], [41].

3) For realization as Y, the matrix to have elements which are all positive or zero and which are uniformly tapered [6], [26], that is, the rows and columns can be renumbered (if necessary) so that

$$Y_{ij} + Y_{i-1,j+1} \geqq Y_{i-1,j} + Y_{i,j+1}$$

for all $i \leq j$, with the additional notation

$$Y_{i,n+1} = Y_{0,j} = 0.$$

Sufficient conditions have been developed for certain other cases, particularly when the structure is limited to $n + 1$ nodes; but these conditions seem to be largely operational in nature [7], [8], [11], [17], [26], [27].

It is also definitely known that paramountcy is not sufficient when the number of ports n is greater than 3. This has been shown simply by the citation of an appropriate numerical example [17].

It is also known that complete necessary and sufficient conditions must differ for realization as a Y matrix or as a Z matrix. Again this has been shown by the citation of numerical examples: 1) of a matrix realizable as a Y matrix but not as a Z matrix, and 2) of a matrix realizable as a Z matrix but not as a Y matrix [17], [18]. This last statement of fact may necessitate a change in some of our preconceived ideas as to the nature of the principle of duality as applied to networks.

Louis Weinberg:† In this note we attempt to give a precise statement of a network problem, indicate its importance, and suggest possible methods for solving it. Because of the limited space available the discussion must perforce be concise. It is assumed the reader knows what voltages and currents are, but since there are some mathematicians among the writer's friends who are dismayed by such an assumption, it is immediately stated for their benefit that we also include a purely mathematical formulation of the problem in matrix terms.

Consider the electrical network N shown in Fig. 1, which is a representation of a $2n$-terminal passive network; all our knowledge of the internal structure of the network must be obtained from measurements of the voltages and currents of the terminals. Frequently we specialize the network by specifying that certain pairs of terminals are to be considered together, that is, as ports, and thus the $2n$-terminal network is converted into an n-port network. For example in Fig. 2 terminals 1 and 2 have been paired as port 1, terminals 3 and 4 as port 2, etc.; it is assumed

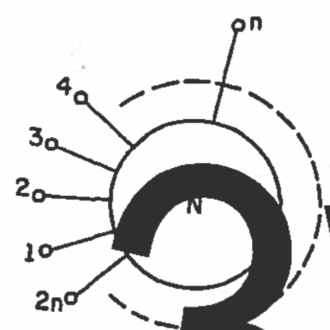

Fig. 1

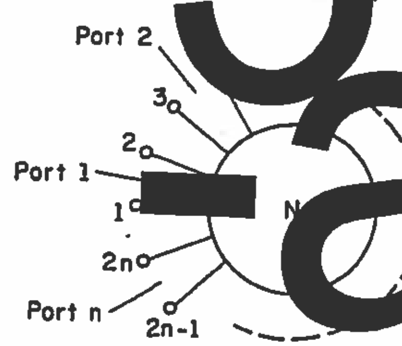

Fig. 2.

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in this n-port network that all measurements are made only at ports and that no connections are made between terminals of different ports; for example, an impedance can be connected between terminals 1 and 2 but not between terminals 2 and 3.

Because of the above restrictive assumption on the n-port only $n(n+1)/2$ independent functions are needed for its complete specification, if N is a passive network obeying reciprocity; thus for a two-port a complete description is given by a symmetric matrix of the second order. For the $2n$-terminal network, however, $n(2n-1)$ independent functions provide a complete specification; thus for $n = 2$, we require six independent functions, which is to be contrasted with the requirement of three functions for the two-port.

A solution is known for the problem when ideal transformers are permitted in the network. In fact the two problems of the realization of a network specified by its ports and by its terminals are equivalent in this case; for example, realization of a three-port can be converted to the realization of a corresponding four-terminal network. Without transformers two different problems exist.

In the following we consider networks without real or ideal transformers. We first discuss the network formed purely from resistances and then extend this to the case of a network containing resistances, capacitances, and inductances.

Suppose an nth-order symmetric matrix H with real elements is given; the problem that is unsolved is to determine the necessary and sufficient conditions on H for it to be realizable as the open-circuit resistance matrix or the short-circuit conductance matrix of an n-port containing only resistances.

Some necessary conditions are known. First of all H must be positive semidefinite. An even more restrictive condition is that H must be a paramount matrix, where a symmetric matrix of order n is defined as paramount if each principal minor of order r, where $r = 1, 2, \cdots, n-1$, is not less than the absolute value of any rth-order nonprincipal minor formed from the same rows. This condition is sufficient for $n \leq 3$ but is not sufficient for $n > 3$.

A sufficient condition is also known for the short-circuit conductance case and can be stated in terms of dominant matrices. A real symmetric matrix is defined as a dominant matrix (or a matrix with dominant main diagonal) if each of its main-diagonal elements is not less than the sum of the absolute values of all the other elements in the same row. The given matrix H may be realized as a short-circuit conductance matrix if it is dominant.

A less general form of network than the n-port containing $2n$ terminals is the n-port containing only $(n+1)$ terminals. Two cases must be distinguished for this case of $(n+1)$ terminals: one is an n-port in which all ports have the same common ground terminal with the positive direction of each voltage rise measured from this terminal, that is, the ports correspond to a tree in a star configuration. The second is an n-port the ports of which, when an element is drawn between terminals of each port, form an arbitrary tree on the $(n+1)$ terminals.

An nth-order real matrix is realizable as the short-circuit conductance matrix of an n-port containing only $(n+1)$ terminals, one of which is a ground terminal for all the ports, if and only if the matrix is dominant and each of the off-diagonal terms is nonpositive. Furthermore the $(n+1)$-terminal network may be synthesized to have only $(n+1)$ nodes.

When the ports do not correspond to a starlike tree, that is, there is no common ground for all the ports, then a linear tree may be used for characterizing the graph, where by a linear tree we mean a tree that can be ordered to have successive branches each with a single common node with the preceding branch; thus the linear tree can be drawn without lifting one's pencil from the paper. A short-circuit conductance matrix is realizable if and only if the transformed matrix corresponding to the ports forming a linear tree is uniformly tapered. The definition of a uniformly tapered matrix is given in the preceding discussion by Professor Foster. Methods have been given for transforming the given matrix to one corresponding to a linear tree.

The preceding method introduces some computational difficulty when the given matrix has some zero elements, for it may be necessary to try a large number of different matrices in forming the sign matrix that is required.

A method that uses matrix algebra is also available for realizing the $(n+1)$-terminal network. The given matrix is decomposed by an algorithmic procedure into a congruent transformation of a diagonal matrix

$$H = ADA'$$

in which D is an nth-order diagonal matrix with positive elements on the main diagonal, A is a unimodular matrix and A' is its transpose. A necessary and sufficient condition for realizability of H as a short-circuit conductance matrix is that A be realizable as a cut-set matrix of a linear graph with $(n+1)$ nodes, and for realizability as an open-circuit resistance matrix A must be realizable as the loop matrix of a graph containing precisely n independent loops. In the preceding we have used the term unimodular matrix; by this term we mean a matrix each of whose minors is equal to ± 1 or 0.

The conditions for the general n-port problem can also be given a matrix formulation in terms of matrices defined above. A necessary and sufficient condition for a matrix H to be realizable as the short-circuit conductance matrix of an n-port network is that H be a principal submatrix of the inverse of a congruent transformation ADA', where the order of each of these square matrices is $> n$. Even though we have stated a necessary and sufficient condition, this does not mean the problem is solved; what is unknown is a procedure for determining whether a given matrix may be represented in the above form.

Thus if we solve the n-port problem by a technique other than the matrix decomposition, we will at the same time have solved the pure mathematics problem stated above.

A recent attack on the n-port problem that almost succeeded was given by Biorci in an unpublished paper. It failed because he could not prove that if a given nth-order matrix is realizable by an n-port having M resistances, where $M > n(n + 1)/2$, it is also realizable with $M - 1$ resistances. If this theorem is true for all n, as it is for $n \leq 3$, it implies that $n(n + 1)/2$ resistances are sufficient to realize any matrix, if it is realizable at all. The theorem stands as a conjecture at present, having been neither proved nor disproved.

We now can consider the RLC problem. Here the matrix has elements that are rational functions of the frequency variable $s = \sigma + j\omega$, and the network may contain resistances, inductances and capacitances, but no mutual inductance or ideal transformer. Some necessary conditions are known. The given matrix $H(s)$ must be a positive real matrix; furthermore it must be a paramount matrix for all s in the range $0 < s < \infty$.

The necessary and sufficient conditions for the realization of the RLC n-port are the same as for the resistance case, except that the diagonal matrix D now has elements on the main diagonal given by a, b/s, cs, with $a, b, c > 0$.

To guide the solution of this problem some further properties of the matrices are available. If we group together in D the elements of the same kind—that is, we renumber the rows and columns so that constants are in the diagonal elements of the first group of columns, terms of the form $b_\nu s$ in the second group, and terms of the form c_μ/s in the third group—then D must be decomposable in the direct sum

$$D = D_1 \dotplus s\, D_2 \dotplus \frac{1}{s} D_3,$$

where $\dotplus$ indicates direct sum and D_1, D_2, D_3 are diagonal matrices with positive elements in their main diagonals.

This is a much more formidable problem than the resistance case. In fact we do not even know a solution when the network is an RC quadripole, that is, $n = 2$ and only resistances and capacitances are permitted in the network. Again some necessary conditions are known for $n = 2$, but perhaps it is possible to leapfrog this case by a general approach.

It will be found that the conditions for realizability of the impedance matrix and the admittance matrix are different. In fact our ideas of duality have been modified by some of the results already obtained in work on the resistance network; for example we know that there are matrices realizable as impedance matrices but not as admittance matrices, and vice versa.

This is a crucial problem in network synthesis; it promises to enlarge our concept of duality and throw light on the basic problem of equivalent networks. It also has applicability in other branches of engineering; for example, having realized a resistance network we have also realized a communication network, where nodes represent stations and branches represent communication channels of specified capacity.

A number of suggestions could be given on approaches for solving the problem. Since this note is already too long, only one suggestion will be given here. It might yield useful insights to assume that the n-port possesses terminals rather than n ports, or it may be convenient to switch between the two representations. There is a simple formula relating system functions in one representation to system functions in the other. This formula, which is given below, is not so widely known as it should be; its first appearance and proof in the literature are somewhat in doubt, and it is continually being rediscovered. One of the conceptual advantages of the $2n$-terminal network representation is that only driving-point measurements need be made; these characterize the n-port uniquely. Thus an obvious necessary condition on each measurement is that it is a non-negative number.

Consider a resistance n-port with an open-circuit resistance matrix $R = [R_{ik}]$. Of course since the n-port obeys reciprocity, of the n^2 driving-point and transfer resistances only $n(n + 1)/2$ are independent, that is, the matrix is symmetrical. Now consider this network as a $2n$-terminal network with the terminals numbered from 1 to $2n$, and with the ports so numbered that port 1 comprises terminals 1 and $(n + 1)$, the assigned positive direction being from terminal 1 to terminal $(n + 1)$. In general port k will run from terminal k to terminal $(n + k)$

For the representation of the $2n$-terminal network let $S_{i,k}$ denote the measured driving-point impedance between terminals i and k, all other terminals being left free. Then we define $S_{k,k} = 0$, since this measurement corresponds to both of the measuring leads connected to the same terminal. It is clear that

$$R_{kk} = S_{k,n+k}.$$

The general formula for the elements of matrix R is

$$R_{ik} = \tfrac{1}{2}[S_{i,n+k} + S_{k,n+i} - S_{i,k} - S_{n+i,n+k}],$$

which reduces to the previous formula when $i = k$.

Thus we see that a great deal is known about the problem. We should be on the brink of a solution. If this note helps to push a reader over the brink, then the writer will get off the hook, since he predicted in the 1960 report on Circuit Theory to the URSI General Assembly that the resistance n-port problem would be solved before the next General Assembly in 1963.

I. Cederbaum:† The determination of the conditions under which a matrix may represent the specifications of an n-port network without ideal transformers remains an outstanding problem in network theory.

The admission of ideal transformers (or ideal gyrators) removes the topological constraints inherent in any actual

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configuration of network elements. Other constraints inherent, for example, in the linear, passive and time-invariant character of elements have been successfully dealt with, and the pertinent realization theory has achieved quite a high level of mathematical sophistication.

The topological constraints however have not yet been overcome. They meet us directly in their unveiled form when dealing with the problems of synthesis of pure resistive or R-networks. Only the case of so-called minimum realization of a given immitance matrix over the field of real numbers may be here considered as solved. However if the requirements of the minimum possible rank or nullity of the resulting network are dropped the broader family of realizable matrices is not yet well-defined.

It is my intention to present the problem of realization of a pure resistive n-port from its immitance matrix in the language of mathematical programming.

Take the given Y matrix (with constant real elements, symmetric and paramount) and visualize it as derived from an *s.c.* admittance matrix $\bar{Y}$ of a $(2n - 1)$ port described on a $2n$-vertex network N, completely connected, *i.e.*, having $n(2n - 1)$ edges. Without impairing the generality of the discussion, the $(2n - 1)$ port may be assumed to have the linear port structure, and the n-port looked for may be considered as derived from this $(2n - 1)$ port by open-circuiting each alternate port. If the numeration of the ports is such that the open-circuited ports have the last $n - 1$ order numbers, then after partitioning $\bar{Y}$ in the form

$$\bar{Y} = \underbrace{\begin{bmatrix} Y_1 & Y_{12} \\ Y'_{12} & Y_2 \end{bmatrix}}_{n \quad n-1} \begin{matrix} n \\ n-1 \end{matrix}, \tag{1}$$

we have the relation

$$Y = Y_1 - Y_{12}Y_2^{-1}Y'_{12}. \tag{2}$$

Of course there is

$$\bar{Y} = CGC', \tag{3}$$

where C is the given fundamental cut-set matrix of order $\{(2n - 1) \times [n(2n - 1)]\}$ corresponding to the linear tree and G is the diagonal branch admittance matrix of the $n(2n - 1)$ unknown edge conductances.

The relation (2) presents a system of $n(n + 1)/2$ equations of order n in these $n(2n - 1)$ unknowns. The synthesis problem may now be looked on as finding a solution of the system (2) under the constraint on all the unknowns to be real and non-negative numbers.

If such a solution exists, then it is in general non-unique. The objective function which is to be minimized in order to obtain the optimal alternative may be either the number of the vertices or the number of edges of N. The first criterion may be met if we maximize the number of unknown edge-conductances which increase without bounds, since each such edge represents a short circuit unifying a pair of vertices. The second criterion implies maximizing the combined number of edges with either infinite or zero conductances.

Of course the difficulty with this technique lies in the prohibitive amount of computational work implicit in testing the realizability of a given Y matrix. However the simplicity of the objective function which need to be minimized and the non-negativity constraint on the variables indicate that the methods of mathematical programming might here be advantageously applied.

In conclusion some words on the importance of this problem. With respect to network theory the realization of R n-ports may be looked on as presenting the first step in realization of RLC n-port networks without mutual inductance. Since such a network displays for positive real values of the complex frequency s the properties of an R-network, a necessary condition for its synthesis turns out to be evidently the R-realizability of its immitance matrix for all positive real values of s.

Pursuing further the same argument we are able to synthesize a subclass of RLCM-networks. This class may be obtained from a linear graph after replacing each of its edges by a two-terminal box with an arbitrary configuration of elements inside of the boxes [19].

One of the most interesting facts which may presently be observed is that some related fields like contact, communication and probabilistic networks or sequential machines are able to apply a number of network-theoretic results. Since the weights assigned to edges of the pertinent graphs in these applications are normally non-negative real numbers, it is only natural that the border line between these fields and network theory passes along the region of pure resistive structures. It is to be hoped that any progress made in one of these fields will produce interesting consequences in the whole neighborhood.

Giuseppe Biorci:† The problem can be stated as follows: given a real nth-order paramount matrix, to find (if it exists) a network composed of positive resistors only, which shows among n terminal pairs (ports), properly chosen, a short-circuit conductance matrix identical to the assigned matrix. The dual statement is omitted for brevity.

The number of nodes of the network cannot be lower than $(n + 1)$ since the n voltages across the ports must be independent. If the network turns out to have more than $2n$ nodes, some nodes do not belong to any port, and they may be suppressed through star-mesh transformations. Therefore the number of nodes may be any number between $(n + 1)$ and $2n$.

The solution of the problem is extremely important.

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First, it is a basic theoretical problem; in fact it is the inverse problem of the analysis of resistive networks solved by Kirchhoff and Maxwell. Weinberg[1] has indicated that a few years ago many scientists could not believe that it was an unsolved problem. Second, as Guillemin [24] has pointed out, its solution may lead to the topological synthesis of multiterminal networks made of inductances and capacitors, as well as resistors in the frequency domain, and such a synthesis has a great theoretical and practical interest. Third, the problem has implications in switching circuits, the importance of which is well-known.

The present status of research on the problem is the following: if we restrict the investigation to those networks which have $(n + 1)$ nodes exactly, we know several methods of solving the problem [8], [15], [26], [45]. I believe that none of the proposed solutions is fully satisfactory and that some work should be done to reach, in this particular section of the problem, a necessary and sufficient condition which does not require a full process of building up the network. I hope that the other participants in this discussion will give their opinion on this point.

If the nodes are more than $(n + 1)$ the problem becomes much more difficult for two main reasons: 1) the number of possible branches of the network becomes larger than the number of elements of the assigned matrix, whereas such numbers are identical if the nodes are $(n + 1)$, and 2) the functional relationship between branch conductances and matrix elements is rational but not linear as it is in a network with $(n + 1)$ nodes.

Guillemin devised a technique, which can be derived from the theory of Howitt transformations, to augment the given matrix in such a way as to find out the conductance matrix of the $(n + p)$-node network on which the n-port is based. This is equivalent to putting two networks in parallel both having positive and negative resistors but such that the resulting one is realizable. Although theoretically the second network, and hence the final one, can be constructed in infinitely many ways, it is quite difficult to find out even a single solution since the problem is to solve a set of so many quadratic inequalities, as many as the branches in the complete graph on $(n + p)$ nodes, while the variables are less in number. In the approach proposed by Guillemin, no indication is given to make the method systematic, and hence the procedure cannot be applied since even a trial-and-error procedure must be limited to a finite number of trials.

Cederbaum [16], [17] seems to hope that a solution can be reached by making use of the paramount condition. I believe that the probability of success along this line is very low since paramountcy is not a sufficient condition, as Cederbaum [17] and Piglione [33] have proved. Furthermore the physical meaning of paramountcy has been found recently by Biorci and Meo [10] and Civalleri [21]: it is nothing but a very particular consequence of the no-amplification condition.

I propose a new approach. It should be investigated whether the following statement is true or false: Given a network of positive resistors with n ports and m branches [m larger than $n(n + 1)/2$, which is the number of elements of the short circuit conductance matrix], there exists another network of positive resistors with n ports and $(m - 1)$ branches having the same short-circuit conductance matrix as the first network. If the statement is true it follows that, if a short-circuit conductance matrix is realizable with m branches, it is also realizable with $n(n + 1)/2$ branches (note that this is true for $n = 2$ and $n = 3$).

Therefore the synthesis problem, if the property is true, becomes much simpler. In fact the number of possible structures of a network with $n(n + 1)/2$ is finite, although large, and since for each of them the problem is defined (there are as many unknowns as equations) it is possible to find out, by systematic trials, if one of them is composed of positive resistors only. If the statement is false, from the proof of its incorrectness it should be possible to find some indications about the real minimum number of branches and possibly new ways of approaching the problem.

I hope that the other participants in this discussion will kindly give their opinion about the above approach.

Evaluation and Questions

Paul Slepian:† The panelists are to be commended for their delineation of the problems of resistive network synthesis. However, no member of the panel has suggested that an insight into the synthesis problem can be obtained by a more critical examination of the processes of network analysis.

In particular suppose that n is a positive integer and N is an n-port resistive network. The processes of analysis yield an $(n \times n)$ symmetric matrix of real numbers $Z(N)$, which we call the open-circuit impedance matrix of N. Thus Z is a function on the set of all such resistive n-port networks into the set of all $(n \times n)$ symmetric matrices of real numbers.

Network synthesis is merely concerned with an investigation of the inverse of the function Z. It appears to this observer that an investigation of the inverse of a function should be preceded by a thorough investigation of the properties of the function itself. Thus I suggest that our problems of network synthesis will not be solved by an investigation of isolated synthesis techniques. I fear that Dr. Weinberg will remain impaled on his hook until such a time as the investigators turn their directions to the processes of analysis.

As a concrete suggestion suppose that A is an n-port resistive network and B is an n-port resistive network.

[1] L. Weinberg, private communication.

† Dept. of Mathematics, University of Arizona, Tucson, Ariz.

First, it is a basic theoretical problem; in fact it is the inverse problem of the analysis of resistive networks solved by Kirchhoff and Maxwell. Weinberg[1] has indicated that a few years ago many scientists could not believe that it was an unsolved problem. Second, as Guillemin [24] has pointed out, its solution may lead to the topological synthesis of multiterminal networks made of inductances and capacitors, as well as resistors in the frequency domain, and such a synthesis has a great theoretical and practical interest. Third, the problem has implications in switching circuits, the importance of which is well-known.

The present status of research on the problem is the following: if we restrict the investigation to those networks which have $(n + 1)$ nodes exactly, we know several methods of solving the problem [8], [15], [26], [45]. I believe that none of the proposed solutions is fully satisfactory and that some work should be done to reach, in this particular section of the problem, a necessary and sufficient condition which does not require a full process of building up the network. I hope that the other participants in this discussion will give their opinion on this point.

If the nodes are more than $(n + 1)$ the problem becomes much more difficult for two main reasons: 1) the number of possible branches of the network becomes larger than the number of elements of the assigned matrix, whereas such numbers are identical if the nodes are $(n + 1)$, and 2) the functional relationship between branch conductances and matrix elements is rational but not linear as it is in a network with $(n + 1)$ nodes.

Guillemin devised a technique, which can be derived from the theory of Howitt transformations, to augment the given matrix in such a way as to find out the conductance matrix of the $(n + p)$-node network on which the n-port is based. This is equivalent to putting two networks in parallel both having positive and negative resistors but such that the resulting one is realizable. Although theoretically the second network, and hence the final one, can be constructed in infinitely many ways, it is quite difficult to find out even a single solution since the problem is to solve a set of so many quadratic inequalities, as many as the branches in the complete graph on $(n + p)$ nodes, while the variables are less in number. In the approach proposed by Guillemin, no indication is given to make the method systematic, and hence the procedure cannot be applied since even a trial-and-error procedure must be limited to a finite number of trials.

Cederbaum [16], [17] seems to hope that a solution can be reached by making use of the paramount condition. I believe that the probability of success along this line is very low since paramountcy is not a sufficient condition, as Cederbaum [17] and Piglione [33] have proved. Furthermore the physical meaning of paramountcy has been found recently by Biorci and Meo [10] and Civalleri [21]: it is nothing but a very particular consequence of the no-amplification condition.

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It would be nice if, with certain restrictions on A and B, we could add these networks to obtain a new n-port resistive network which we denote by

$$A \oplus B.$$

It would be even nicer if, after such addition,

$$Z(A \oplus B) = Z(A) + Z(B),$$

where the addition on the right is the usual matrix addition. Thus I ask what the conditions on A and B must be in order that addition of the networks will satisfy the above equation.

Such thoughts are not original with this writer. Indeed Kron [31] and his loyal disciples have concerned themselves for many years with this problem and claim some powerful results. Unfortunately this writer cannot understand these results.

Note however that if such an addition of networks can be defined, then the synthesis of a complicated matrix can be attacked by a decomposition of the complicated matrix into the sum of simpler matrices, each of which is realizable in such a way that the realizing networks are mutually additive. Furthermore the networks realizing these simpler matrices will probably be related to the canonical networks discussed by Prof. Foster.

S. L. Hakimi:† The problem is the following: Given an $(n \times n)$ symmetric matrix A with real entries, what are the necessary and sufficient conditions for this matrix to be the open-circuit impedance matrix Z_{oc} (or the short-circuit admittance matrix Y_{sc}) of an n-port resistive network.

It is known that if A is realizable as Z_{oc} (or Y_{sc}), it is not necessarily also realizable as Y_{sc} (or Z_{oc}); and furthermore if the question of realizability of A as Z_{oc} (or Y_{sc}) is completely answered, this by itself will not solve the problem of realizability of a given matrix as Y_{sc} (or Z_{oc}). This stems from the fact that Z_{oc}^{-1} (or Y_{sc}^{-1}) may not exist. Even if the above problem (as stated in the first paragraph) is solved, there may remain another interesting and important problem, that of realizability of A as a grounded n-port resistive network.

As it has been pointed out, a number of extremely interesting papers on this subject have already been published. In spite of this fact it might be advisable for *some* research workers to ignore the present literature on this subject, for it seems improbable that a complete solution is merely a generalization of the presently known results.

In any realizability problem, one may find 1) a set of necessary and sufficient conditions for realizability of the given specification without having a finite scheme for arriving at the realization (the network), or 2) a finite scheme for arriving at the realization without having a set of conditions to examine the given specification prior to the realization. Neither of these two by itself is a com-completely satisfactory solution. It is the feeling of this writer that a "satisfactory" solution to "the n-port problem" is not within reach and that the problem most likely will remain unsolved for some time to come.

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Robert W. Newcomb:† If I understand correctly the approach suggested by Biorci and commented upon by Weinberg, it consists of an *analysis* of all structures of $n(n+1)/2$ branches. Supposedly this will result in a catalogue of such networks as concerns their applicability for realizations. It is not clear if Biorci has carried out this analysis, which certainly must be tedious for most n of interest, $n > 3$. In any event one unsolved problem is to reduce the network to the $n(n+1)/2$ branches. If the above interpretation is correct, it appears to me that this latter could be avoided by analyzing all networks with the maximum necessary number of $2n$ nodes. These would still contain a finite number of branches, something like $n(2n-1)$ of them, which could be catalogued in the same manner. If I understand the method, it seems unlikely that any conclusions could be drawn for very large n because of the computational difficulties. However it does seem worth pursuing since an inductive proof could possibly be found.

Some comments on duality seem appropriate. First, one wonders if the duality problems are real or just apparent; that is, can they be by-passed by making a proper choice of variables or by the use of some transformation (such as that of Puckett [34]). Second, it seems that a search for planar realizations (including sources) would seem worthwhile. For instance the nonplanar realization described by Foster in his previous example B can be replaced by a planar one [22]. The importance of this lies in the fact that if A can be realized as $A = Y$ but not $A = Z$ (as a resistive transformerless n-port with the same polarity conventions for Y = admittance and Z = impedance), then the realization of $A = Y$ must be nonplanar. It is not clear however that if $A = Y$ and $A = Z$ are both realizable, there is a planar realization. A thorough investigation of duality concepts certainly seems in order.

In equivalent network studies one of the most important concepts is the scattering matrix S. Has anyone investigated the use of S or even the constraints on S for transformerless n-ports? Likewise Richard's theorem is useful for 1-ports and has been generalized by Bayard to n-ports [1]. Although Bayard's result doesn't seem to assist much in a transformerless synthesis, perhaps further investigations of generalizations of the Bott-Duffin synthesis would be worthwhile.

† Stanford University, Stanford, Calif.

Yilmaz Tokad:† For the realization of a real symmetrical matrix $\mathcal{G}$ of order n, as an $(n + 1)$ terminal R-network, an approach which does not suffer from the existance of 0 entries is as follows: Consider the set of matrices $\mathcal{Q}_i$ which transforms a given tree with $(n + 1)$ vertices into another. This set of matrices can be generated easily from a vertex matrix [38]. If $\mathcal{G}$ is realizable as an $n + 1$-terminal R-network, one of the matrices in this set transforms $\mathcal{G}$ into a dominant matrix $\mathcal{Q}\mathcal{G}\mathcal{Q}'$. The determination of $\mathcal{Q}_i$ can be realized through a digital computer search program.

As for the canonical or minimum element realization, it is not always necessary to use a $2n$-terminal R-network with n ports having no common terminals. It may be possible to determine a terminal graph (port structure) which contains less than $2n$-terminal and more than $n + 1$. In general this terminal graph contains more than one part. Even though it may be possible to establish such a terminal graph, the problem of determinating the element values remains unsolved. To determine this separated terminal graph one can first partition the given matrix as

$$\begin{array}{c} \\ (k_1) \\ (k_2) \end{array}\begin{array}{c} (k_1) \quad (k_2) \\ \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} \end{array}, \qquad k_1 + k_2 = n.$$

If $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ are realizable as $k_1 + 1$ and $k_2 + 1$ terminal R-networks for any $1 \leq k_1, k_2 \leq n - 1$, then the terminal graph is in two parts if not more partitioning in the $\mathcal{G}$ matrix is necessary [11].

For a given realizable paramount p.r. matrix of order n, assume that the terminal graph is found to be connected, *i.e.*, the synthesis is possible as an $n + 1$-terminal RLC network. The realization therefore must be possible by increasing the number of nodes inside the network by at least p where $p = k/2$ if k is even, $p = (k - 1)/2$ if k is odd, and k is the degree of the common denominator [38]. An alternate procedure is to create one internal node at a time by the use of polygon to star transformations.

If realization is not possible with $n + 1$ terminals, the realization as an $n + k$ $(k > 1)$ terminal network may be possible, but represents a much more complex problem.

S. D. Bedrosian:‡ The panel has done an admirable job of defining the problem and indicating its importance in network theory. This writer heartily concurs with the emphasis on single-element-kind treatment in efforts to solve the n-port problem. It appears that a somewhat broader view should be taken of the over-all multiterminal network problem of which this is a special case. This is particularly important due to the apparently justified optimism about an impending solution to this special subclass of n-port networks without transformers.

The more general case of multiterminal networks involves the recognition of three types of nodes, namely, accessible, partially accessible and inaccessible [4]. These are defined as

Accessible nodes: the usual external terminals;
Partially accessible nodes: terminals restricted to application or measurement of voltage;
Inaccessible nodes: internal or "concealed" nodes.

The partially accessible node provides a direct method of treating subnetworks within a larger network. This is also applicable to the n-port case. A systematic matrix method has been given for formulating equations representing the externally observable behavior of such multiterminal networks [3]. This approach has been developed in connection with a different but somewhat related problem area, that of network element value solvability. Solvability theory, if one may coin a name, is distinguished by emphasizing treatment of networks wherein all the nodes are not accessible. In network solvability one seeks to determine the specific element values of a specific configuration by means of its externally observable behavior. It develops that there is great dependence on network topological conditions.

An intimate connection may exist between the compound matrix formulation and generalized reduction formulas for solvability and Cederbaum's (1) and (2). The concepts and results being obtained should also be useful in providing additional insight into the topics of equivalent networks and canonical networks as discussed by Foster.

Wan H. Kim:† The summary of the synthesis problems of the ordinary RLC multiport networks by Weinberg and Foster is clear and up to date. There is however one school of approach which is very much worth looking into, although it is covered only by Biorci and even then just briefly. We refer to the use of the development of the theory of linear graphs for the synthesis of multiport networks. This will now be explored further.

The immittance matrix of a multiport network with $n + 1$ nodes H is given by

$$H = ADA^t, \tag{1}$$

where A is a unimodular matrix. Furthermore A is a basic cut-set or loop matrix if the network has n ports, and if the network has less than n ports A is a modified basic cut-set or loop matrix including a fictitious branch(es) or loop(s).

It is then clear that, as Weinberg stated, the necessary and sufficient conditions for a matrix H to be realizable are 1) that matrix H be decomposable into the form

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of (1), and 2) that A be realizable as a basic cut-set or loop matrix of a connected graph. A presently known necessary condition for the decomposition of H is paramountcy. However we need much stronger condition(s) than paramountcy before we can attempt to decompose H because of the fact that if A is unimodular then H is always paramount but *not vice versa*. Thus the following question should be answered: Find the implication or influence of the unimodular character of A on the immittance matrix H.

Tutte's theorem [39] gives the necessary and sufficient condition for the realization of A. Let us assume that A is in a basic form $[UA_{12}]$, where U is the unit matrix of rank equal to A. Then A must be unimodular and must not contain a submatrix which is a loop or cut-set matrix of either one of Kuratowski's basic nonplanar graphs. From this theorem it follows directly that a matrix A of rank three is always realizable if it is unimodular, since the minimum number of independent cut-sets or loops is four in the Kuratowski graph. As a consequence of this observation and of the fact that we have an available algorithm for the decomposition of A into a unimodular congruence (if such a decomposition exists), the following question may be answered strictly in terms of the theory of linear graphs.

2) All (2×2) and (3×3) paramount matrices are decomposable into the form ADA^t such that A is always realizable in terms of linear graphs. For the case of a paramount matrix of (2×2) the proof is straightforward, but this may not be so in the case of a (3×3) paramount matrix. (Note that there exist two different networks proposed by Slepian and Weinberg [36] and Tellegen [37], respectively, but these are not based on the realizability of graphs.)

Our discussion up to this point has lead to the following question:

1) What are the constraints of H if A satisfies the conditions of Tutte's theorem? In other words how can one make sure that the decomposition algorithm always gives a realizable A matrix? Once this question is answered (that is, once A is obtained such that it satisfies Tutte's theorem), then we know how to construct a graph corresponding to A [28].

Let us now turn to a problem of a more general nature: the realization of H with a multiport of at most $2n$ nodes. For this case it is known [17] that if $H = [h_{ij}]$ satisfies the condition that $h_{kk} = h_{kl}$ for $k = 1, 2, \cdots, n$ and for some l, then H is realizable *only if* it can be realized by an n-port with $n + 1$ nodes. This result should be extended to the case where $k = 1, 2, \cdots, n - m$ and $m \geq 1$. Thus the following conjecture is made:

2) If the elements of H, h_{ij}, satisfy the condition that $h_{kk} = h_{kl}$ for $k = 1, 2, \cdots, n - m$, $m \geq 1$, and for some l, then H is realizable *only if* it can be realizable with an n-port having $n + m$ nodes. The worst case would be the one in which $k = l$ only. Then H may be realizable *only* by an n-port with $2n$ nodes.

The proof or disproof of this conjecture may indicate the minimum number of resistors required for the realization of H which Biorci attempts to solve.

The last comment which I would like to make in support of further work in this area is that some of the results obtained for a resistive network are also useful in the study of active networks [29]. Moreover the decomposition algorithm may be modified to yield a realization of a multiport with the minimum number of negative resistors. Because of limited space an example is given to support our statements. Given H such that

$$H = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 12 & 4 & 5 \\ 2 & 4 & 15 & 6 \\ 3 & 5 & 6 & 18 \end{bmatrix}. \tag{2}$$

Then it can be shown that Cederbaum's decomposition algorithm will result in ADA^t, but A is *not* unimodular. Hence it is not realizable with a resistive four-port with five nodes. However it can also be shown that

$$H = ADA^t, \tag{3}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{4}$$

$$D = \begin{bmatrix} 4 & & & & & & & & \\ & 11 & & & & & & \text{O} & \\ & & 10 & & & & & & \\ & & & 8 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & \text{O} & & & & 1 & & \\ & & & & & & & 4 & \\ & & & & & & & & -1 \end{bmatrix}. \tag{5}$$

And the realization of H with one negative resistor is shown in Fig. 3.

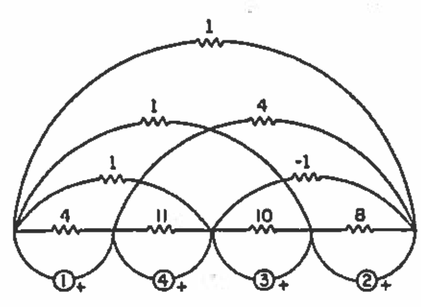

Fig. 3.