and (20a) follows. With the appropriate changes in (A2) and (A6), the same argument leads to the remaining cases in (18) and (20).

References

[1] R. Fletcher and M. J. D. Powell, "A rapidly convergent descent method for minimization," *Brit. Comput. J.*, vol. 6, pp. 163–168, June 1963.
[2] R. Fletcher and C. M. Reeves, "Function minimization by conjugate gradients," *Brit. Comput. J.*, vol. 7, pp. 149–154, July 1964.
[3] C. G. Broyden, "A class of methods for solving nonlinear simultaneous equations," *Math. Comput.*, vol. 19, pp. 577–593, 1965.
[4] S. W. Director and R. A. Rohrer, "Automated network design—the frequency-domain case," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 330–337, Aug. 1969.
[5] J. F. Pinel and M. L. Blostein, "Computer techniques for the frequency analysis of linear electrical networks," *Proc. IEEE*, vol. 55, pp. 1810–1826, Nov. 1967.
[6] T. Downs, "An approach to the computation of network sensitivities," *Marconi Rev.*, third quarter, pp. 204–213, 1972.
[7] E. C. Neu, "Obtaining partial derivatives for a general linear network," in *Proc. 1972 Midwest Symp. Circuit Theory*.
[8] G. A. Richards, "Second derivative sensitivity using the concept of the adjoint network," *Electron. Lett.*, vol. 5, pp. 398–399, Aug. 1969.
[9] D. A. Calahan, *Computer-Aided Network Design.* New York: McGraw-Hill, 1972, pp. 132–135.
[10] G. C. Temes, "Optimization methods in circuit design," in *Computer Oriented Circuit Design* (F. F. Kuo and W. G. Magnuson, Jr., Eds. Englewood Cliffs, N.J.: Prentice Hall, 1969, ch. 5.
[11] S. W. Director, "LU factorization in network sensitivity computations," *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 184–185, Jan. 1971.
[12] F. H. Branin, Jr., "Network sensitivity and noise analysis simplified," IBM, Kingston, N.Y., Rep. TR 21-472, Apr. 21, 1972.
[13] M. L. Blostein, "Sensitivity analysis of linear systems in the frequency domain," Notes for Short Course on Computer Aided Network Analysis and Design, IEEE Conv. Expo., New York, Mar. 20, 1972.
[14] J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* Oxford, England: Clarendon Press, 1965.
[15] I.B.M. Corp., "System/360 scientific subroutine package (360A-CM-03X) version III," pp. 221–225, 1968.

Narrow-Band Multiple-Coupled Cavity Synthesis

A. E. ATIA, MEMBER, IEEE, A. E. WILLIAMS, AND R. W. NEWCOMB, FELLOW, IEEE

Abstract—**A synthesis procedure of identical narrow-band high-Q cavities is presented. The procedure handles multiple couplings in a noncascaded form by synthesizing coupling impedances in one step. Necessary and sufficient conditions for realizability of singly and doubly terminated networks are obtained. An example of a sixth-order elliptic function waveguide filter is included.**

I. Introduction

In some practical applications [1], particularly at microwave frequencies, bandpass filters are conveniently constructed using interconnections of identical high-Q cavity resonators. Commonly, these cavities are coupled in a simple cascade connection [2] from which, however, only all-pole insertion loss functions can be produced. When more general characteristics are desired, coupling between nonadjacent cavities can be considered. For this latter situation, Saito [3] has presented a possible synthesis, though his technique may require an unnecessarily large number of coupling elements raising inconveniencies for production and adjustment. A different approach is presented by Rhodes [4].

Here an improved synthesis of identical multiple-coupled high-Q cavities is presented in which the minimum number of coupling elements is easily obtained. Realizability conditions are presented for both doubly and singly terminated networks, these being derived from the necessary and sufficient conditions of Theorem 1 on the admittance matrix of the cavities. In essence, this paper presents in complete form the theory behind designs upon which experimental results have been reported [1], [5].

II. Equivalent Circuit and Fundamental Definitions

Fig. 1 shows the n-loop equivalent circuit to be assumed for a structure composed of multiple-coupled, but otherwise identically tuned, cavities. The cavities being identical, and hence called synchronously tuned, are all assumed tuned to the normalized center frequency $\omega_o = 1/\sqrt{LC} = 1$ and to have normalized characteristic impedance $z_o = \sqrt{L/C} = 1$. We make the narrow-band approximation by using a lumped element representation of a cavity and by assuming that the $n \times n$ symmetric coupling impedance matrix jM (having zero diagonal entries but otherwise arbitrary signs on the entries) is purely imaginary and frequency independent for frequencies near ω_o.

Introducing the bandpass frequency variable

$$P = p + 1/p \tag{1a}$$

allows us to write the loop impedance matrix as

$$Z_l(P) = P1_n + jM \tag{1b}$$

where 1_n is the $n \times n$ identity matrix.

Manuscript received May 14, 1973; revised November 29, 1973. This paper is based upon work performed at the COMSAT Laboratories under the sponsorship of the International Telecommunications Satellite Organization (INTELSAT). Any views expressed in this paper are not necessarily those of INTELSAT.
A. E. Atia and A. E. Williams are with the COMSAT Laboratories, Clarksburg, Md. 20734.
R. W. Newcomb is with the Electrical Engineering Department, University of Maryland, College Park, Md. 20742.

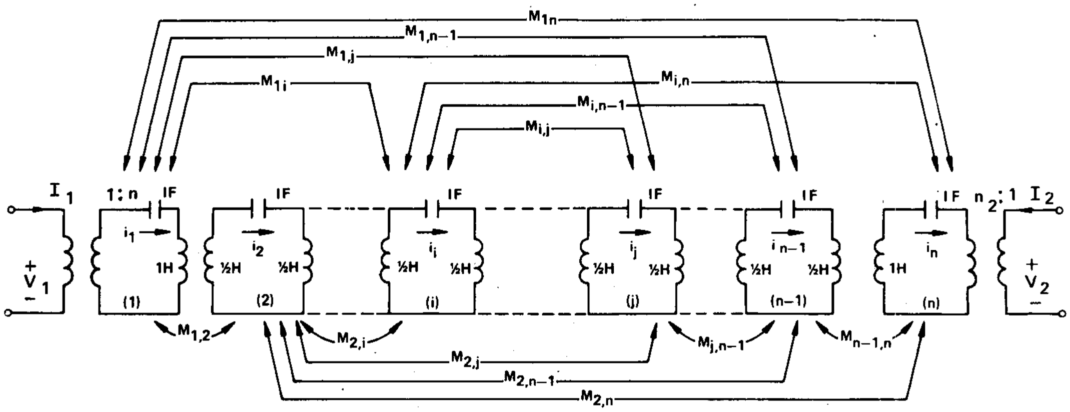

Fig. 1. The general 2-port composed of synchronously tuned coupled cavities.

Considering the structure as a 2-port with input and output ports introduced through ideal transformers inserted in loops 1 and n, yields (using $I_1 = n_1 i_1, v_1 = n_1 V_1$, $I_2 = -I_n = -n_2 i_n$, and $V_2 = -n_2 V_n = -v_n$)

$$Z\mathscr{I}_l = \mathscr{E}$$

$$\mathscr{I}^t = \left[\frac{1}{n_1} I_1, i_2, \cdots, i_{n-1}, \frac{1}{n_2} I_n\right]$$

$$\mathscr{E}^t = [n_1 V_1, 0, \cdots, 0, n_2 V_n] \tag{2a}$$

where the superscript t denotes matrix transposition.

The 2-port (short-circuit) admittance matrix is then found from $Z_l^{-1} = Y_l$ as

$$Y = \begin{bmatrix} y_{11} & y_{21} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} n_1^2 Y_{l11} & -n_1 n_2 Y_{ln1} \\ -n_1 n_2 Y_{ln1} & n_2^2 Y_{lnn} \end{bmatrix}. \tag{2b}$$

III. Admittance Matrix Properties

The properties of Y of interest can now be determined. Since M is a real symmetric matrix, it can be diagonalized to its (real) eigenvalues by a real orthogonal matrix T, as seen by

$$M = T\Lambda T^t \text{ where } TT^t = T^t T = I,$$

$$\Lambda = \text{diag}\,[\lambda_1, \cdots, \lambda_n]. \tag{3a}$$

Consequently,

$$Y_l(P) = Z_l^{-1}(P) = (P1_n + jT\Lambda T^t)^{-1}$$

$$= T\,\text{diag}\left[\frac{1}{P + j\lambda_1}, \cdots, \frac{1}{P + j\lambda_n}\right] T^t. \tag{3b}$$

Using (3), we can conclude the following.

Theorem 1

The necessary and sufficient conditions that the rational 2×2 matrix $Y(P)$ be the admittance matrix of the 2-port of Fig. 1 are that $Y(P)$ be expressible as

$$Y(P) = \sum_{k=1}^{n} \frac{K^{(k)}}{P - P_k} \tag{4a}$$

where:

a) the poles $P_k = -j\lambda_k$ are simple and purely imaginary, and satisfy

$$\sum_{k=1}^{n} P_k = 0; \tag{4b}$$

b) the 2×2 residue matrices $K^{(k)}$ are real and symmetric and positive semidefinite with zero determinant (compact);

c)

$$\sum_{k=1}^{n} \lambda_k K_{11}^{(k)} = \sum_{k=1}^{n} \lambda_k K_{22}^{(k)} = 0 \tag{4c}$$

$$\sum_{k=1}^{n} K_{21}^{(k)} = 0. \tag{4d}$$

Proof for necessity: The form of $Y(P)$ exhibited in (4a) follows from the relation (2b), of Y to Y_l of (3b). The latter shows that the poles are simple and purely imaginary. Since the trace of M is zero, and the trace of M is the sum of its eigenvalues λ_k [see [6], p. 87], $\sum \lambda_k = j \sum P_k = 0$, giving (4a). From (3b), we have

$$Y_{lij} = \sum_{k=1}^{n} \frac{T_{ik} T_{jk}}{P - P_k} \tag{5a}$$

or, using (2b),

$$K^{(k)} = \begin{bmatrix} n_1^2 T_{1k}^2 & -n_1 n_2 T_{nk} T_{1k} \\ -n_1 n_2 T_{nk} T_{1k} & n_2^2 T_{nk}^2 \end{bmatrix}. \tag{5b}$$

Thus the residue matrices are real, symmetric, of zero determinant, and nonnegative diagonal entries (hence positive semidefinite). Since T is orthogonal, we have for (4d)

$$\sum_{k=1}^{n} K_{21}^{(k)} = -n_1 n_2 \sum_{k=1}^{n} T_{nk} T_{1k} = 0 \tag{5c}$$

while $M_{11} = M_{nn} = 0$ gives (4c) by similar reasoning using (3a).

Proof for sufficiency: We wish to construct a real symmetric $M = T\Lambda T^t$ having zero diagonal entries from $Y(P)$ satisfying the conditions of the theorem. The problem is then that of finding an appropriate orthogonal T. Since the residue matrices take the form of (5b), we first choose positive input and output transformer turns-ratios, as seen by

$$n_i^2 = \sum_{k=1}^{n} K_{ii}^{(k)}, \qquad i = 1,2. \tag{6a}$$

Next we form, in view of (5b),

$$T_{1k} = \sqrt{K_{11}^{(k)}/n_1^2}$$
$$T_{nk} = \sqrt{K_{22}^{(k)}/n_2^2}\ \mathrm{sgn}\ K_{21}^{(k)}. \tag{6b}$$

By the compact nature of $K^{(k)}$, the proper $K_{21}^{(k)}$ result with (4d) insures that the first and last row of T are orthogonal; (6a) insures these rows are properly normalized. We have on hand then the first and last row of an orthogonal matrix T, which in view of (4c), gives, using (3a), $M_{11} = M_{nn} = 0$. We next construct an orthogonal matrix T_1 having the first and last rows of T so far obtained; for example, the remainder of T_1 can be found from the first and last row by the Gram–Schmidt procedure. If $T_1 \Lambda T_1^t = M_1$ has all zero diagonal elements, we choose $T = T_1$, $M = M_1$, while if not, we proceed as follows. If some $M_{1ii} \neq 0$, then sgn $M_{1jj} = -$sgn M_{1ii} for some j, since trace $M_1 = 0$ by (4b); by a permutation of rows and columns, we can assume $i = 2$, $j = 3$. Thus the 2×2 matrix

$$\hat{M}_1 = \begin{bmatrix} M_{122} & M_{123} \\ M_{123} & M_{133} \end{bmatrix}$$

is indefinite, in which case there exists an orthogonal 2×2 matrix $\hat{T}_1$ such that $\hat{T}_1 \hat{M}_1 \hat{T}_1^t$ has a zero replacing M_{122}. Explicitly,

$$\hat{T}_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{7a}$$

with

$$\tan\theta = \frac{-M_{123} \pm \sqrt{M_{123}^2 - M_{122}M_{133}}}{M_{133}} \tag{7b}$$

where we note a solution always exists, since $M_{133} \neq 0$, $M_{122}M_{133} < 0$.

Forming the orthogonal $T_2 = (1 \dotplus \hat{T}_1 \dotplus 1_{n-3})PT_1$, where P is the permutation matrix for the preceding permutation and $\dotplus$ denotes the direct sum, $T_2 \Lambda T_2^t = M_2$ has at least one more zero on the diagonal than does M_1, while M_2 yields the same $Y(P)$ as M_1. Repeating this procedure eventually gives a suitable $M = M_i = T_i \Lambda T_i^t$ having all zero diagonal entries. Q.E.D.

Although the procedure outlined in the sufficiency proof works for any $Y(P)$ satisfying the conditions of the theorem, it is of interest to investigate an alternate construction of M when the poles occur in conjugate pairs. Thus let us assume that $Y(P)$ satisfies the conditions of the theorem, that poles occur in conjugate pairs, and that the residue matrices at conjugate poles are either equal or differ only in the sign of the (1,2), and hence (2,1), entry. We will then give an alternate construction for M.

On choosing n_i as at (6a), we have two cases depending upon $n = 2m$ (even) or $n = 2m - 1$ (odd), which we bring into one. Thus, if n is odd, there is a pole at zero of residue $K^{(o)}$; we then relabel, splitting the pole at zero into two equal parts, as seen by

$$K^{(m)} = K^{(2m)} = K^{(o)}/2, \qquad \lambda_m = \lambda_{2m} = 0. \tag{8}$$

Assuming this in the following, which now allows $n = 2m$, we appropriately number the remaining λ_k such that the poles $P_k = -j\lambda_k$ are split into upper and lower (conjugate) half-plane poles to form

$$\Lambda = \left[\begin{array}{c|c} \Lambda_1 & 0_m \\ \hline 0_m & -\Lambda_1 \end{array}\right], \qquad \Lambda_1 = \mathrm{diag}\,[\lambda_1,\cdots,\lambda_m] \tag{9}$$

where $\lambda_m \geq 0$ and $\lambda_k > 0$ for $k = 1,\cdots,m-1$; 0_m is the zero $m \times m$ matrix.

To obtain T, we first find an intermediary orthogonal $m \times m$ matrix Q as follows. For the first and last rows of Q, choose, as at (6b),

$$Q_{1k} = \frac{\sqrt{K_{11}^{(k)}}}{n_1}$$
$$Q_{mk} = \frac{\sqrt{K_{22}^{(k)}}}{n_2}\ \mathrm{sgn}\ K_{21}^{(k)}, \qquad k = 1,2,\cdots,m. \tag{10}$$

For the remainder of Q, we arbitrarily construct it to be an orthogonal $m \times m$ matrix. We then choose

$$T_1 = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} Q & Q \\ \hline Q & -Q \end{array}\right] \tag{11a}$$

which is orthogonal by our choices for Q. This yields

$$M_1 = T_1 \Lambda T_1^t = \left[\begin{array}{c|c} 0_m & C \\ \hline C & 0_m \end{array}\right], \qquad C = Q\Lambda Q^t. \tag{11b}$$

If

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} K^{(k)} = K^{(k+m)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad k = 1,\cdots,m \tag{11c}$$

we choose $M = M_1$, $T = T_1$. If, however, conjugate poles all have equal residues, then we permute rows and columns one and $m + 1$ to obtain the corresponding M.

In general, the matrix C, constructed as in the preceding, will have all nonzero entries. There are several procedures to generate equivalent networks which possess less coupling elements. These procedures apply similarity transformations which reduce certain nonzero elements in C (or M) to zero. One such method which can be applied in the case of symmetrical networks to yield a "canonical form" is to use the Givern's procedure [see [6], p. 256] to reduce C to a tridiagonal form. For the general (nonsymmetrical network) case, the following procedure can be applied.

Let D be the $(m-2) \times (m-2)$ matrix obtained from deleting the first and last rows and columns of C. Since D is real-symmetric, it can be diagonalized by an orthogonal $(m-2) \times (m-2)$ matrix V, as seen by

$$D = V^t \mu V \tag{12a}$$

where μ is a diagonal matrix. Then construct the orthogonal $m \times m$ matrix S as

$$S = 1 \dotplus V \dotplus 1. \tag{12b}$$

Finally, the matrix C^1 given by

$$C^1 = SCS^t \tag{12c}$$

will be zero everywhere except for the first and last rows and columns and the main diagonal. In general, this will yield the smallest number of coupling elements.

IV. Transfer-Function Synthesis

Here we present conditions and outline the methods for which singly terminated and doubly terminated networks will result using the synchronously tuned coupled cavities of Fig. 1.

First we treat the case where the 2-port of Fig. 1 is terminated by a unit resistor at the output, for which the voltage transfer ratio of the network is

$$\frac{V_2}{V_1}(P) = \frac{-y_{21}}{1 + y_{22}}. \quad (13)$$

Sufficient conditions for realizability are given as follows.

Theorem 2

A voltage transfer ratio $V_2/V_1(P) = M(P)/N(P)$ is realizable by the singly terminated network of Fig. 1 if a) $M(P)$ and $N(P)$ are real polynomials in P with N strictly Hurwitz, b) degree $M \leq$ (degree N) $- 2$, and c) M is even if degree N is odd and odd if degree N is even.

Proof: Following standard techniques [see [7], p. 417], write

$$\frac{V_2}{V_1} = \frac{M/\text{Ev } N}{1 + (\text{Od } N/\text{Ev } N)} \text{ if deg } N \text{ is even} \quad (14a)$$

$$= \frac{M/\text{Od } N}{1 + (\text{Ev } N/\text{Od } N)} \text{ if deg } N \text{ is odd} \quad (14b)$$

where Ev N is the even part and Od N is the odd part of N. From (13) and (14a) and (14b), we can identify the short-circuit admittance parameters as

$$y_{21}(P) = M/\text{Ev } N$$

$$y_{22}(P) = \text{Od } N/\text{Ev } N \text{ if deg } M \text{ is even} \quad (14c)$$

$$y_{21}(P) = -M/\text{Od } N,$$

$$y_{22}(P) = \text{Ev } N/\text{Od } N \text{ if deg } M \text{ is odd.} \quad (14d)$$

We can now obtain the partial fraction expansions of y_{12} and y_{22} as

$$y_{22}(P) = \sum_{k=1}^{n} \frac{K_{21}^{(k)}}{P - P_k}$$

$$y_{22}(P) = \sum_{k=1}^{n} \frac{K_{22}^{(k)}}{P - P_k}. \quad (15)$$

Note that y_{22} has no poles at ∞, while condition c) guarantees that the $K_{21}^{(k)}$ are real. Since $N(P)$ is Hurwitz, it follows that y_{22} is a reactance function of P and, therefore, all the poles P_k are simple, purely imaginary, and occur in conjugate pairs also insuring the first of (4c). It also follows that $K_{22}^{(k)} > 0$ for $k = 1,2,\cdots,n$ and the residues at conjugate poles will be equal. Furthermore, it is seen by rationalizing and observing the numerator coefficients of p^{n-1} that condition b) insures that $\sum_{k=1}^{n} K_{21}^{(k)} = 0$.

Finally, we let

$$y_{11}(P) = \sum_{k=1}^{n} \frac{K_{11}^{(k)}}{P - P_k} \quad (16a)$$

where

$$K_{11}^{(k)} = \frac{K_{21}^{(k)2}}{K_{22}^{(k)}} \geq 0, \qquad k = 1,2,\cdots,n. \quad (16b)$$

This also insures the second of (4c).

Therefore, all conditions of Theorem 1 are satisfied, and the synthesis can be made following in fact the construction of (11).

A similar result holds for the more practical case of doubly terminated networks. If the network of Fig. 1 is loaded at port two by a conductance G_2 and fed by a source at port one of internal conductance G_1, then

$$\frac{V_2}{V_s} = -\frac{G_1}{G_2}\frac{I_2}{I_s} = \frac{-G_1 y_{21}}{(y_{11} + G_1)(y_{22} + G_2) - y_{21}^2}. \quad (17)$$

We then have the following.

Theorem 3

A transfer voltage ratio $(V_2/V_s)(P) = M(P)/N(P)$ is realizable by the doubly terminated network of Fig. 1, with G_1 as source and G_2 as load conductances, if a) M and N are real polynomials with N strictly Hurwitz, b) degree $M \leq$ (degree N) $- 2$, and c) $1 - 4(G_2/G_1)|(V_2/V_s)(j\lambda)|^2 \geq 0$.

Proof: Following the standard Darlington procedure [see [7], pp. 424, 402], we create the 2×2 lossless admittance matrix from $I_2/I_s = -(G_2/G_1)V_2/V_s$. This Y has compact poles [see [7], p. 408] and the reasoning of the proof of Theorem 2 applies to show that Theorem 1 is satisfied. The only point which could be in doubt is (4d) which follows from (17), where the numerator degree is two less than the denominator degree. Q.E.D.

V. Synthesis Example

An example which illustrates the synthesis technique developed in this paper is the design of a sixth order elliptic function waveguide filter. Since this type of filter is realizable by symmetrical networks, the even (or odd) mode coupling matrix can be used to simplify the computations. The general form of the coupling matrix of a symmetrical network and the definitions of the even and odd mode coupling matrices and admittances are shown in Fig. 2. A typical sixth order voltage transfer ratio is

$$\frac{V_2}{V_s} = \frac{M(P)}{N(P)} \quad (18a)$$

where

$$M(P) = P^4 + 5.8000187\,P^2 + 7.7931287$$

$$N(P) = 8.4976927\,(P^6 + 2.4998811\,P^5 + 4.6159331\,P^4 + 5.5088462\,P^3 + 4.7520099\,P^2 + 2.7320542\,P + 0.9170876). \quad (18b)$$

Applying Darlington's procedure [see [7], pp. 424, 402] and

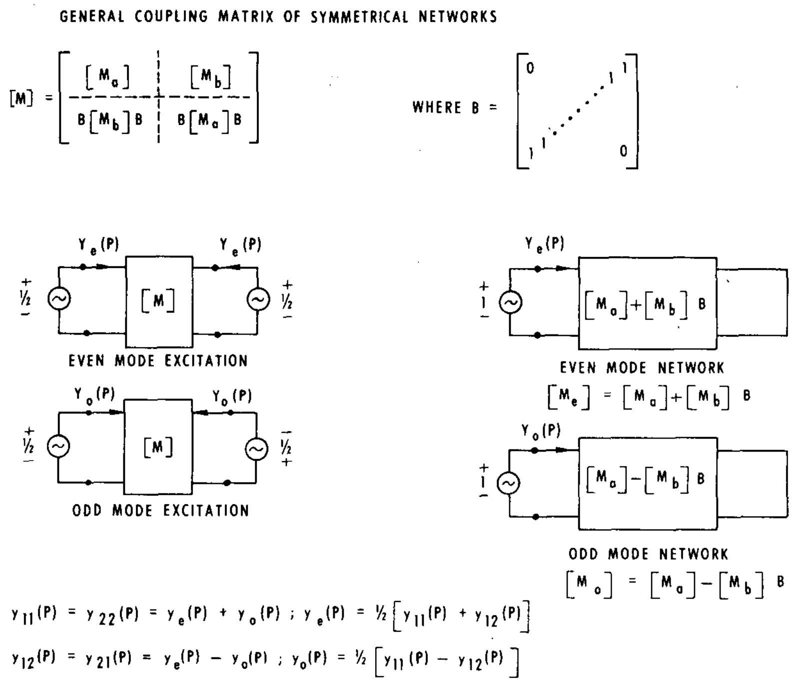

Fig. 2. Even and odd mode coupling matrices for a symmetrical network.

the definitions in Fig. 2 to yield a symmetric network, for which $2y_e = y_{11} + y_{21}$, $y_{11} = y_{22}$, the even mode short circuit admittance $y_e(P)$ can be obtained as

$$y_e(P) = \frac{K_e^{(1)}}{P - P_{1e}} + \frac{K_e^{(2)}}{P - P_{2e}} + \frac{K_e^{(3)}}{P - P_{3e}} \tag{19a}$$

where the even mode residues and eigenvalues are given by

$$\begin{aligned} K_e^{(1)} &= 0.576152 & P_{1e} &= j0.4812277 = j\lambda_{1e} \\ K_e^{(2)} &= 0.441353 & P_{2e} &= -j1.1431542 = j\lambda_{2e} \\ K_e^{(3)} &= 0.232436 & P_{3e} &= j1.2309347 = j\lambda_{3e}. \end{aligned} \tag{19b}$$

The transformer turns ratios $n_1 = n_2 = 1.18007$, follow from (6a). Dividing the residues by n_1^2, we get the square of the first row of an even mode 3×3 orthogonal matrix T_e, the rest of which can be constructed by the Gram–Schmidt procedure. For example, if the two vectors (1,0,0) and (0,1,0) are used, the resulting T_e matrix would be

$$T_e = \begin{bmatrix} 0.678928 & 0.594221 & 0.431228 \\ 0.736205 & -0.549482 & -0.398762 \\ 0.000000 & 0.587342 & -0.809336 \end{bmatrix}. \tag{20a}$$

Then the even mode coupling matrix M_e can be obtained from (3a) as

$$M_e = \begin{bmatrix} -0.04707 & 0.87935 & -0.27290 \\ 0.87935 & 0.49343 & 0.20162 \\ -0.27290 & 0.20162 & -1.01542 \end{bmatrix}. \tag{20b}$$

From this even mode coupling matrix, both the matrices M_a and M_b (see Figs. 2 and 3(a)) can be constructed using $M_e = M_a + M_b B$ as

$$M_a = \begin{bmatrix} 0 & 0.87935 & 0 \\ 0.87935 & 0 & 0.20162 \\ 0 & 0.20162 & 0 \end{bmatrix} \tag{20c}$$

$$M_b = \begin{bmatrix} -0.27290 & 0 & -0.06707 \\ 0 & 0.49343 & 0 \\ -1.01542 & 0 & -0.27290 \end{bmatrix}. \tag{20d}$$

The coupled cavity network resulting from this matrix M_e shown in Fig. 3(a), is obviously difficult to construct and tune, since it contains the maximum number of possible couplings. A structure which contains the minimum number of coupling elements can be obtained by reducing the matrix M_e into a tridiagonal form such as

$$M_e' = \begin{bmatrix} -0.04707 & 0.920718 & 0 \\ 0.920718 & 0.24678 & 0.593338 \\ 0 & 0.593338 & 0.768716 \end{bmatrix} \tag{20e}$$

which can be realized by the more practical structure of Fig. 3(b). A waveguide filter having this coupling matrix was constructed, and its calculated and measured responses are shown in Fig. 4.

It is interesting to note that the procedure illustrated in this example can be applied in general to any transfer function realizable by a symmetrical network, such as for elliptic function filters, to obtain a canonical form as that shown in Fig. 3(b). Similarly, the same ideas can be used to obtain the corresponding antimetrical networks.

VI. Discussion and Conclusion

In Theorem 3, we have given the practically significant conditions for realizability of a given voltage transfer function in terms of doubly terminated synchronously tuned multiple-coupled cavities. It is interesting to note

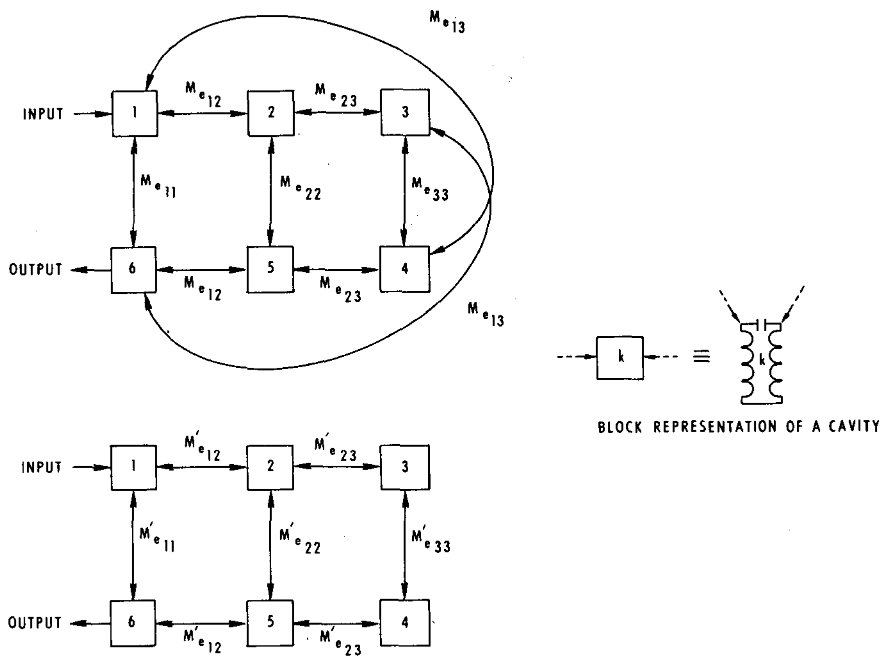

Fig. 3. (a) Coupled cavity structure resulting from M_e of (20a). (b) Coupled cavity structure resulting from M_e of (20b).

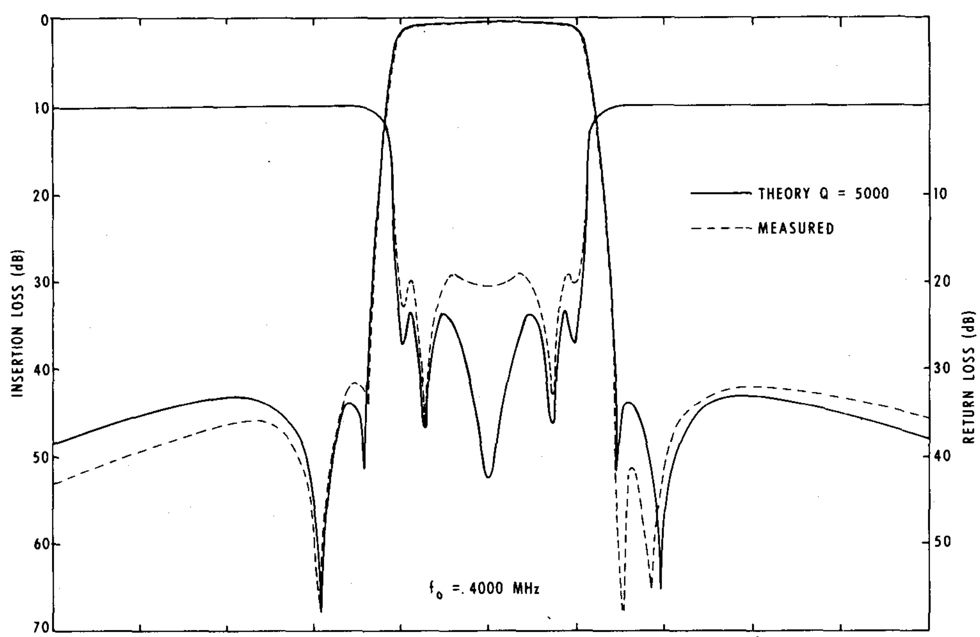

Fig. 4. Measured and calculated response of a sixth order filter.

that the only condition added over that for lumped RLC networks is the second order vanishing at infinity of the transfer function, condition b) of Theorem 3. However, practically, the results are only valid over the narrow band of frequencies for which the couplings behave as the constant matrix jM of (1b). As shown by previously reported experiments [1], [5], and by the example of Section V, the theory can be accurately applied in practice.

At (12c), we have mentioned a transformation on a synthesized M which yields structures often of more interest than the general ones obtained. In the important practical case of realizing filter functions whose insertion loss ratios are expressible as

$$\left|\frac{V_2}{V_s}\right|^2 = \frac{1}{1+\Phi^2}$$

where Φ is any odd rational function of $\lambda = P/j$, it is known that a symmetrical network always exists [see [8], p. 306].

Hence, the procedure illustrated by the example in Section V can be used to obtain a canonical form as that shown in Fig. 3(b).

REFERENCES

[1] A. E. Atia and A. E. Williams, "New types of waveguide bandpass filters for satellite transponders," *COMSAT Tech. Rev.*, vol. 1, no. 1, pp. 21–43, 1971.

[2] S. B. Cohn, "Direct-coupled resonator filters," *Proc. IRE*, vol. 45, no. 2, pp. 187–196, Feb. 1957.

[3] M. Saito, "Synthesis for general-coupled resonator transmission networks," *Electron. Commun. Japan*, vol. 53-A, no. 5, pp. 26–33, 1970.

[4] J. D. Rhodes, "The theory of generalized interdigital networks," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 280–288, Aug. 1969.

[5] A. E. Atia and A. E. Williams, "Narrow-bandpass waveguide filters," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 258–265, Apr. 1972.

[6] F. R. Gantmacher, *The Theory of Matrices*, vol. 1. New York: Chelsea Publishing Co., 1959.

[7] M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*. New York: Wiley, 1960.

[8] S. Darlington, "Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics," *J. Math. Phys.*, vol. 30, no. 9, pp. 257–353, Sept. 1939.

Constant Argument Immittance Realization by a Distributed *RC* Network

SUHASH C. DUTTA ROY, SENIOR MEMBER, IEEE

Abstract—The driving point properties of a ladder type distributed RC network, characterized by the distributions $r(x) = r_o \exp(-ax)$ and $c(x) = c_o \exp(-bx)$, are investigated. It is shown that the infinite length network exhibits, at one of the ports, a constant argument impedance of the form $s^{-a/(a+b)}$, in an asymptotic manner. With a finite length, the approximation is found to be valid over a finite frequency band only, while the asymptotic behavior is that of an impedance $s^{-1/2}$, irrespective of the values of a and b.

I. INTRODUCTION

THE problem of realizing a constant argument immittance of the form s^ν, $0 < |\nu| < 1$, has attracted considerable attention in the recent past, see [1] and the references cited there, and [2]–[5]. Such immittances are useful in a number of practical situations [1], and their realizability raises a number of fundamental questions, which have been dealt with in [4]. One of the early works on this problem is due to Morrison [6], who suggested two Foster and two Cauer approximations in lumped RC form. A distributed RC version of one Foster form has been studied in [5] and [7]. The Cauer forms of [6] were studied by the present author in both lumped and distributed forms, and it was commented in [1] (see footnote 1) that these structures approximate $s^{-1/2}$ only and not $s^{-\nu}$, $0 < \nu < 1$. In this paper, we present a detailed investigation of the distributed version of Morrison's Cauer I structure; the Cauer II structure has not been considered as it cannot be fabricated in the distributed form with present-day technology. The results reveal the following interesting correction to footnote 1 in [1]: the network, viewed as a two-port, realizes an impedance $s^{-1/2}$ at one port but is capable of realizing $s^{-\nu}$, $0 < \nu < 1$ at the other port. The erroneous conclusion in [1] stems from the fact that attention was concentrated on the port realizing $s^{-1/2}$ only.

Manuscript received May 29, 1973; revised November 14, 1973.

The author was with the Department of Electrical and Electronic Engineering, University of Leeds, Leeds LS2 9JT, England, and is now with the Department of Electrical Engineering, Indian Institute of Technology, New Delhi 110029, India.

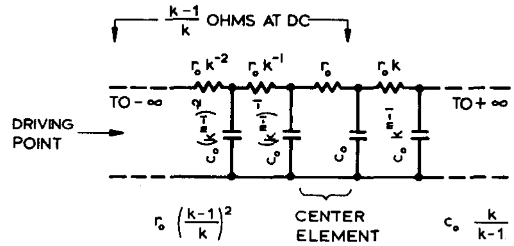

Fig. 1. Morrison's Cauer I structure for realizing an admittance s^ν, $\nu = 1/m$, and $0 < \nu < 1$. An equality sign should be inserted between r_0 and $[(k-1)/k]^2$, and also between c_0 and $k/(k-1)$.

The properties of both finite and infinite lengths of the distributed network have been examined. In the latter case, it is found that the realization of $s^{-\nu}$ is valid in an asymptotic sense. For a finite length, the approximation to $s^{-\nu}$ is shown to hold only for a finite range of frequencies, while the asymptotic behavior is always that of an $s^{-1/2}$ impedance.

II. THE DISTRIBUTED *RC* NETWORK AND ITS SOLUTION

Morrison's Cauer I structure [6] for realizing an admittance s^ν, $\nu = 1/m$, and $0 < \nu < 1$, is reproduced in Fig. 1, with slight change in notations. The corresponding distributed network will have a resistance and capacitance per unit length, at distance x from the center, given by

$$r(x) = r_o k^x \text{ and } c(x) = c_o (k^{m-1})^x \tag{1}$$

where $-\infty < x < +\infty$. We shall consider, without loss