OPERATOR THEORY OF NETWORKS:

SCATTERING THEORY*

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Abstract

Beginning with a set theoretical definition of a network the scattering operator is developed with emphasis upon linear passive networks. Synthesis of passive operators is outlined through cascade-load decompositions, while some emphasis is placed upon finite networks through a development using differential fields.

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1. INTRODUCTION

"This poem wants to be a description" [H1, 4th]

Among the recently developing areas of applied mathematics can be found that of the operator theory of networks. Here we try to outline the field, as scattering theory pertains, beginning with first principles and ending with the present state of the art and open problems which may be of interest to the mathematical community.

The field is one which, until recently, with one or two exceptions, has been primarily cultivated by mathematically inclined electrical engineers. It is, though, we believe, a fascinating field for both mathematicians and engineers and one where strong ties are developing and appear profitable. Thus, for example, in their artistic development of new designs for modern integrated circuits and microsystems the engineers are led to operator theoretic decompositions which can challenge creative mathematicians for generalization and vice versa. Too, as we shall see here, the techniques developed open up new areas of research in combinations of algebra and analysis, as for example in the coupling of differential field theory and Hilbert space operators.

By way of historical background, one can probably rightfully say that ideas in the field trace to those of Cauer who, in the early 1930's, set up a transformation theory of networks using electrical transformers [C3]. Later, in 1952, the mathematician B. McMillan set up a vector-space framework for networks, giving syntheses for an important class of impedance operators [M2]. Probably the first significant introduction of the scattering operator in network theory was given by Belevitch in 1951 [B1] when he pointed out a useful decomposition for synthesis. This work was extended and solidified in a beautiful paper by Oono and Yasuura in 1954 [O1], with a rigorization of the foundation material initiated by Youla, Castriota and Carlin in 1959 [Y1]. This latter showed the importance of Hilbert space concepts in treating passive scattering operators and undoubtedly motivated some of the following treatments based upon the theory of distributions of L. Schwartz [S8]. For example, using distributions Zemanian [Z2][Z4] and Beltrami & Wohlers [B3] obtained passivity conditions on time-independent operators. However, probably of even more importance has been the use of distributions, through the kernel theorem [S7], in treating time-dependent operators [A1], especially in the syntheses of Spaulding [S11] and Anderson [A3]. In the early 1960's, the field was such that it could be somewhat codified [N4], leading the Argentinian mathematician A. Gonzalez-Domínguez to be motivated to deeper studies [G3] which have recently been profitably continued by P. Dewilde [D4]. Professor Gonzalez-Domínguez has also been instrumental in calling attention to the work of M. Livschitz [L10]. One of the first uses of more abstract operator theories in network synthesis published in English was given by Saeks in 1970 [S2], this being followed by the works of Levan [L7] and Helton [H6] using the characteristic operator function, as developed by Sz.-Nagy and Foias [S16], for synthesis of passive scattering operators. Interestingly enough, these abstract theories run into difficulties when applied to the practical problems of finite networks for which an alternate development [N7], which preserves rationality and causality of operators, proceeds on an algebraic basis through differential field theory [K4].

By way of background, on the mathematical side we assume some familiarity with Hilbert space concepts, as may be obtained from a number of texts [H4][H10][R2][S13]. At points, the treatment will also be somewhat algebraic for which we will assume the availability of comprehensive texts [D9][R1][R3] for those wishing to check out details. On the engineering side, we assume less, though point out the existence of texts [N4][S4][W4] which may be helpful in clarifying some of the physical meanings. We should, however, mention that the theory, though perhaps similar in a few points to scattering theory as treated by Lax and Phillips [L1], is quite different in content from that familiar to many accustomed to flow and transportation networks based on graph theory [S10]. Too, it is worth commenting that it would be convenient to have a theory developed for which one would not constantly have to refer to the underlying functional spaces which somehow keep appearing in a manner which confuses the more basic arguments. However, as mathematicians are aware, specifications of such spaces are crucial for the validity of results, something upon which the laxity of engineers often leads them to incorrect results. In any event, we try to avoid the undue use of functional spaces in the treatment here, for which reason only passing mention of distributional results is made.

Because most practical use for the theory occurs in electrical engineering, we adhere somewhat closely to the nomenclature there. Thus $\mathbf{v}$ and $\mathbf{i}$ can conveniently be thought of as voltage and current vectors, though, of course, other interpretations are possible when thinking of other applications. By way of explanation of the notation, we will use boldface quantities to represent vectors or matrices usually of physically measurable quantities dropping the boldface for scalars and general operators.

The next section gives a definition of a network in terms of allowed pairs of port variables, this being followed by the introduction of the scattering operator in Section 3. In Section 4, passivity is introduced, this being the main constraint used for deriving results on linear networks. Synthesis, the main concern of engineering, is introduced in Section 7, this falling back upon the cascade load relationship introduced in Section 5. Section 7 also contains some clarifying examples. Since engineering designs are primarily based upon finite networks, these latter are covered in Section 6 which also contains differential field concepts of use in the synthesis of finite networks. A few generalizations are touched upon in Section 8 with some open problems mentioned in the closing discussion.

2. NETWORKS

"This forest is dense
Full of scrawny trees, and they are afraid" [H1, 4th]

We begin by defining in general, but mathematical, terms a network introducing the concepts of input power and ports; the section closes by cataloging a few important networks.

Abstractly a network N can be most primitively thought of as a set of allowed pairs of variables $\underset{\sim}{v}$ and $\underline{i}$ where a given pair is allowed if it satisfies the constraints C_N placed by the physical nature of the network. We thus write $\underset{\sim}{v}C_N\underline{i}$ if $\underset{\sim}{v}$ and $\underline{i}$ satisfy C_N and define the particular <u>network</u> so specified as

$$N = \{[\underset{\sim}{v},\underline{i}] \mid \underset{\sim}{v}C_N\underline{i}\} \tag{1}$$

Pictorially, we represent N as in Fig. 1. We comment that no preference is given to $\underset{\sim}{v}$ or $\underline{i}$; in other words, neither is considered as an input or an output since physically the same network may serve several functions. A network is then a special case of a non-oriented system [Z1, p.10].

There are two types of constraints contained within C_N, those common to all networks and those differentiating one network from another. The former are generally placed for mathematical rigor, while the latter are of most importance from an engineering viewpoint. Within the former class of constraints, we shall initially take $\underset{\sim}{v}$ and $\underline{i} \in \mathcal{D}_+(H)$ where $\mathcal{D}_+(H)$ is the space of infinitely differentiable functions of time t taking values in some Hilbert space H and which are zero until some finite time (that is, have support bounded on the left, where by the support of a function is meant the closure of the time set upon which the function is nonzero). This constraint is motivated on physical grounds since an arbitrary network may give rise to arbitrarily high derivatives, while, too, it would be put in use only after a finite time. Denoting the scalar product on H by $<\cdot,\cdot>_H$ we also subsume in the common constraints of C_N that the polarity choices on $\underline{v}$ and $\underline{i}$ are such that for the pair $[\underline{v},\underline{i}]\in N$ the total power P(t) into N at time t is given by

$$P(t) = <\underline{v}(t),\underline{i}(t)>_H \qquad (2)$$

(Note, for example, that a reversal of $\underline{i}$ would give output power instead).

Customarily, and for ease of interpretation of various results, one takes $H = R^n$ and $t\in R^1$ where R^n is real Euclidean n-space in which case N is called an n-port and $<\underline{v},\underline{i}>_H = \tilde{\underline{v}}\underline{i}$ where the tilde denotes matrix transposition. For general H the network is called a Hilbert-port [Z3]. Most normal electronic systems are n-ports while non-n-port Hilbert-ports are of practical interest for microwave structures, such as satellite communication systems.

Example 1. The constraints differentiating some important networks from others are illustrated as follows. For these we comment that the parameters, as resistance, are allowed to vary with time in which case we denote by $\mathcal{E}$ the space of infinitely differentiable (real) functions (with no constraint on the support).

a) The resistor, of resistance r, is the 1-port defined by the constraints n =1 and v = ri, $r\in\mathcal{E}$.

b) The capacitor, of capacitance $c\in\mathcal{E}$, is defined by n = 1 and i = d(cv)/dt. Similarly the inductor, of inductance $\ell\in\mathcal{E}$, has $v = d(\ell i)/dt$.

c) The gyrator, of gyration conductance $g\in\mathcal{E}$, is the 2-port defined by

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad ; \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad , \quad \underline{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

d) The <u>transformer</u> (k+m)-port, of kxm turns-ratio matrix $\underline{T} = [T_{ij}]$, $T_{ij} \in \mathcal{E}$, is defined by

$$\underline{v}_2 = \tilde{\underline{T}}\underline{v}_1 \quad , \quad \underline{v} = \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix} \quad , \quad \underline{i} = \begin{bmatrix} \underline{i}_1 \\ \underline{i}_2 \end{bmatrix}$$

$$\underline{i}_1 = -\underline{T}\underline{i}_2$$

e) The <u>nullator</u> is the 1-port having $v = i \equiv 0$ while the <u>norator</u> has v and i independently arbitrary. For the norator the network constraints C_N are essentially vacuous.

Although the networks of Example 1 are among the most useful, there are many other networks, as transmission lines and transistors, some of which, eg. transistors, can be conveniently constructed (or conceived) as interconnections of those of the Example. Too, it should be noted that the networks of Example 1 can be generalized from n-ports to Hilbert ports; for example, a Hilbert port resistor would have its resistance mapping $\mathcal{D}_+(H)$ rather than $\mathcal{D}_+(R^1)$.

3. THE SCATTERING OPERATOR

"Here, in this forest, the voice moves dripping with sweat
This is a region where trees open up, in here
The blind tree forgets that it can be seen." [H1, 4th]

Having defined a network N, we now introduce a scattering operator $\mathcal{S}$ for it, via a linear change of port variables, giving the solvability condition for the existence of $\mathcal{S}$.

If for a given N it occurs that there exist linear mappings $\mathcal{A}[\cdot]$ and $\mathcal{B}[\cdot]$ of $\mathcal{D}_+(H)$ into $\mathcal{D}_+(H)$ such that

$$\mathcal{A}[\underset{\sim}{v}] = \mathcal{B}[\underline{i}] \tag{3}$$

then we will call N <u>linear</u>. All of the networks of Example 1 are seen to be linear. However, rather than use two linear operators to describe N we would prefer to consider only one. In cases where $\mathcal{A}$ is invertable, eg. the resistor, the impedance operator $\mathcal{Z} = \mathcal{A}^{-1}\mathcal{B}$ can be used while if, as for the gyrator, $\mathcal{B}$ is invertable then the admittance operator $\mathcal{Y} = \mathcal{B}^{-1}\mathcal{A}$ exists. But, as the transformer shows, neither the impedance nor admittance operator need exist for useful networks. Consequently, it is more convenient to introduce the <u>scattering operator</u> $\mathcal{S}[\cdot]$ for which we make a change of variables to incident, $\underset{\sim}{v}^i$, and reflected, $\underset{\sim}{v}^r$, variables

$$2\underset{\sim}{v}^i = \underset{\sim}{v}+\underset{\sim}{i} \tag{4a}$$

$$2\underset{\sim}{v}^r = \underset{\sim}{v}-\underset{\sim}{i} \tag{4b}$$

to yield

$$\underset{\sim}{v}^r = \mathcal{S}[\underset{\sim}{v}^i], \quad \mathcal{S} = (\mathcal{B}+\mathcal{A})^{-1}(\mathcal{B}-\mathcal{A}) \tag{4c}$$

At this point, an orientation for N has been chosen since $\underset{\sim}{v}^i$ acts as an input with $\underset{\sim}{v}^r$ as an output. Our meaning of the word "operator" is seen to be just a synonym for "mapping of one space (inputs) into another (outputs)," the actual spaces varying with the discussion.

Physically, incident and reflected variables can be interpreted by connecting a unit resistor in series with each port of N to form an augmented network N_a, described by $[\underset{\sim}{e},\underset{\sim}{i}]$, $\underset{\sim}{e} = \underset{\sim}{v}+\underset{\sim}{i}$, as illustrated in Fig. 2. Then $\underset{\sim}{e} = 2\underset{\sim}{v}^i$, the source voltage or input, while if for some ports $\underset{\sim}{e} = \underset{\sim}{0}$ then $\underset{\sim}{v}^r = \underset{\sim}{v}$ appears across those ports' resistors as output. Consequently, the scattering operator, besides being most conveneient for the mathematical treatment of networks, is also quite useful for designing resistively terminated structures.

We comment that if N is linear so is N_a and that if N_a has an admittance operator $\mathcal{Y}_a$, $\underset{\sim}{i} = \mathcal{Y}_a[\underset{\sim}{e}]$, then $\underset{\sim}{v} = \underset{\sim}{e}-\underset{\sim}{i} = (\mathcal{I}-\mathcal{Y}_a)[\underset{\sim}{e}]$ where $\mathcal{I}$ is the identity operator, giving

$$\mathcal{Y}_a[\underset{\sim}{v}] = (\mathcal{I}-\mathcal{Y}_a)[\underset{\sim}{i}] \qquad \text{or} \qquad \mathcal{S} = \mathcal{I}-2\mathcal{Y}_a \tag{5}$$

Consequently, we are led to call N <u>solvable</u> if for every $\underset{\sim}{e}\in\mathcal{S}_+(H)$ the equation $\underset{\sim}{e} = \underset{\sim}{v}+\underset{\sim}{i}$ is satisfied by a unique $[\underset{\sim}{v},\underset{\sim}{i}]\in N$. Our result is that a solvable linear N is described by a scattering operator $\mathcal{S}$ and conversely. If the parameters r, ℓ, c, g, T are constant, the networks of Example 1 are all solvable except for the nullator, the norator, and the r = -1 resistor.

4. PASSIVITY - PASSIVE OPERATORS

"A hollow region, and all in accordance
The forest burst into flower to confuse me" [H1, 4th]

In this section, we introduce passivity, the main constraint to be placed upon scattering operators. Passivity conditions in terms of $\mathcal{S}$ are derived (Thm. 1), losslessness defined, and the distributional kernel representation briefly mentioned.

To be sure, most of the electronics industry is devoted to active networks [K6][N6]; however, all such networks can be considered through the loading of passive networks in negative resistors. Hence the study of passive networks is most fundamental mathematically, and, as it turns out, quite challenging too.

We first introduce some more notation by noting that, for each $t \in R^1$, $\mathcal{D}_+(H)$ is an inner product space when supplied with the scalar product

$$<\underset{\sim}{x},\underset{\sim}{y}>_t = \int_{-\infty}^{t} <\underset{\sim}{x}(\tau),\underset{\sim}{y}(\tau)>_H d\tau \quad , \quad \underset{\sim}{x},\underset{\sim}{y} \in \mathcal{D}_+(H) \tag{6a}$$

From this, we can complete $\mathcal{D}_+(H)$ to Hilbert spaces $\mathcal{L}_2(H)_t$ [H4, p.17], one for each t, of square integrable H-valued functions associated with the scalar product $<\cdot,\cdot>_t$. A norm $\|\cdot\|_t$ is then defined for each t by

$$\|\underset{\sim}{x}\|_t = <\underset{\sim}{x},\underset{\sim}{x}>_t \quad , \quad \underset{\sim}{x} \in \mathcal{L}_2(H)_t \tag{6b}$$

We note that $\mathcal{D}_+(H) \subset \mathcal{L}_2(H)_t$ for $t \in R^1$, but not for $t = \infty$ where we will write $\mathcal{L}_2(H)$ for $\mathcal{L}_2(H)_\infty$. Finally, we will need the norm of a linear bounded transformation $T[\cdot]$ of any of these Hilbert spaces into itself for which we write, in the standard manner [H4, p.31],

$$\|T\|_t = \sup_{\|x\|_t=1} \|T[x]\|_t \tag{6c}$$

Passivity is best conceived of through the input energy E defined at time t for $[v,i]\in N$ by

$$E(t) = \int_{-\infty}^{t} P(\tau)d\tau = \langle v,i\rangle_t = \|v^i\|_t^2 - \|v^r\|_t^2 \tag{7}$$

where Eqs. (4a,b) have been used in Eq. (2). The network is then called passive if the input energy is never negative, that is $E(t)\geq 0$, $\forall t\in R^1$ and $\forall [v,i]\in N$. Note that E(t) is well defined for $\mathcal{D}_+(H)$ functions, as well as, in the passive case, for any $v^i\in\mathcal{L}_2(H)_t$, showing why Hilbert spaces are important in network theory. We also comment that all networks of Example 1, except the norator, are passive if r, ℓ, c are nondecreasing nonnegative functions of time. A network which is not passive is termed active.

A passive network, and its associated $\mathcal{S}$, is causal in the sense that $v^i(t) = 0$ for $t<t_o$ implies $v^r(t) = 0$ for $t<t_o$ (since $0\leq E(t) = -\|v^r\|_t^2$ for $t<t_o$). Physically a causal network is one for which responses do not occur before excitations. Likewise, a passive solvable network is completely solvable in the sense that if, for an incident sequence, $\lim_{j\to\infty} v_j^i = 0$ then, for the resulting reflected sequence, $\lim_{j\to\infty} v_j^r = 0$ (this is seen by noting that the limit $e = 0 = 2v^i$ yields, by solvability, a unique v^r which, by passivity, is the limit $v^r = 0$). Consequently, for each $t\in R^1\cup\{\infty\}$ $\mathcal{S}$ defines a continuous operator on $\mathcal{L}_2(H)_t$, that is, $\lim \mathcal{S}[v_j^i] = \mathcal{S}[\lim v_j^i]$ in which case the scattering operator $\mathcal{S}$ for a passive network can be extended from $\mathcal{D}_+(H)$ to the (complete) Hilbert spaces $\mathcal{L}_2(H)_t$; we will assume from now on that this extension is made. From $E(t)\geq 0$ we also see that $\|\mathcal{S}\|_t\leq 1$ while under the solvability assumption we can show that $\|\mathcal{S}\|_\infty\leq 1$ implies $\|\mathcal{S}\|_t\leq 1$ for all $t\in R^1$ [A1, p.14]. We can summarize our results.

Theorem 1. A linear solvable passive network N possesses a linear scattering operator $\mathcal{S}$ which satisfies

a) $\mathcal{S}:\mathcal{D}_+(H)\to\mathcal{D}_+(H)$

b) $\mathcal{S}:\mathcal{L}_2(H)_t\to\mathcal{L}_2(H)_t$ continuously $\forall t\in R^1\cup\{\infty\}$

c) $\|\mathcal{S}\|_\infty\leq 1$

It is convenient to call a linear $\mathcal{S}$ satisfying Thm. 1 passive and we see, by tracing back through the reasoning, that any passive $\mathcal{S}$ defines a linear solvable passive network.

Let $\mathcal{S}^a$ denote the adjoint of $\mathcal{S}$ on $\mathcal{L}_2(H)$, that is $\langle x,\mathcal{S}^a[y]\rangle_\infty = \langle\mathcal{S}[x],y\rangle_\infty$ for all x, $y\in\mathcal{L}_2(H)$, and let a Hermitian, $Q = Q^a$, operator be called nonnegative, $Q\geq 0$, if $\langle x,Qx\rangle_\infty \geq 0$ for all $x\in\mathcal{L}_2(H)$ (or sometimes, as for differential operators, for all x on a dense set, as $\mathcal{L}_2(H)\cap\mathcal{D}_+(H)$). Then from

$$E(\infty) = \langle v^i,v^i\rangle_\infty - \langle v^i,\mathcal{S}^a\mathcal{S}[v^i]\rangle_\infty = \langle v^i,(\mathcal{I}-\mathcal{S}^a\mathcal{S})[v^i]\rangle_\infty\geq 0 \qquad (8)$$

we see that a passive $\mathcal{S}$ has $\mathcal{I}-\mathcal{S}^a\mathcal{S}\geq 0$. If a scattering operator is an isometry [R2, p.280], that is if $\mathcal{S}^a\mathcal{S} = \mathcal{I}$, then a passive $\mathcal{S}$ and its corresponding network are called lossless. Physically, a lossless network excited as an augmented network with $e\in\mathcal{L}_2(H)$ returns all its energy to the augmenting resistors at $t = \infty$. We note for future reference that if $\mathcal{S}$ is unitary, that is if $\mathcal{S}^a\mathcal{S} = \mathcal{S}\mathcal{S}^a = \mathcal{I}$, then it is lossless if passive.

Although we shall not emphasize it here, it is often convenient to represent a passive scattering operator $\mathcal{S}$ by its distributional kernel $\underset{\sim}{S}$; indeed it is through such kernels that many syntheses have been achieved. Consequently, for completeness, we merely outline some of the main ideas in material which can be otherwise skipped by readers unfamiliar with the theory of (Schwartz) distributions [S8]. For this, we only consider n-ports, that is $H = R^n$, though modifications are available for general H [Z7].

To obtain the distributional kernel representation, we note, from the above, that a passive n-port $\mathcal{S}$ defines a linear continuous mapping of the space $\mathcal{D}(R^n)$, of $\mathcal{D}_+(R^n)$ functions of compact support, into $\mathcal{D}'(R^n)$, the space of n-vectors of distributions [recall that

$\mathcal{D}'(R^1)$, the space of distributions, is the topological dual of $\mathcal{D}(R^1)$; that is the distributions are linear continuous functionals on $\mathcal{D}(R^1)$]. It is known [S7][S9, p.143] that any such linear continuous mapping can be represented by an nxn matrix $\underset{\sim}{\mathcal{S}} = \underset{\sim}{\mathcal{S}}(t,\tau)$ to give

$$\underset{\sim}{v}^r(t) = \int_{-\infty}^{\infty} \underset{\sim}{\mathcal{S}}(t,\tau)\underset{\sim}{v}^i(\tau)d\tau \tag{9}$$

Rigorously the distributional kernel $\underset{\sim}{\mathcal{S}}$ is a matrix of distributions in two variables and the integral represents a scalar product taken on topologically dual spaces, this scalar product coinciding with normal integration when the latter can be performed. The conditions of Thm. 1 can be rephrased in terms of $\underset{\sim}{\mathcal{S}}$ [A1], for example $\|S\|_\infty \leq 1$ becomes

$$\delta(t-\tau)\underset{\sim}{1}_n - \int_{-\infty}^{\infty} \tilde{\underset{\sim}{\mathcal{S}}}(\lambda,t)\underset{\sim}{\mathcal{S}}(\lambda,\tau)d\tau \geq 0 \tag{10}$$

(nonnegative in the sense defined above on $\mathcal{L}_2(R^n) \cap \mathcal{D}_+(R^n)$) where $\delta(\cdot)$ is the impulse distribution (i.e., derivative of the unit step function $u(\cdot)$) and $\underset{\sim}{1}_n$ is the nxn identity matrix; causality becomes $\underset{\sim}{\mathcal{S}}(t,\tau) = \underset{\sim}{0}$ for $t<\tau$ [A1, p.12].

As an example, direct calculations show that for the capacitor

$$\mathcal{S}(t,\tau) = -\delta(t-\tau) + \frac{2}{c(t)} e^{-\int_\tau^t \frac{1}{c(\lambda)}d\lambda} u(t-\tau)$$

5. INTERCONNECTIONS - CASCADE LOADING

"But first you have to calm down the wind,
The walking wind, the dense tree-growing wind!" [H1, 5th]

Basic to the synthesis of scattering operators is the cascade-load transformation which we now define giving some examples of its use.

One of the interests of engineering is to build new constructs from basic building blocks. For this, the cascade-load connection of Fig. 3a) is of fundamental importance since any connection can be reduced to it. For example, Fig. 3b) shows how the parallel connection can so be interpreted.

For the cascade-load connection, we assume as given the scattering operators S_ℓ of the load network N_ℓ and S_c of the coupling N_c, with the latter partitioned as the ports

$$S_c = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

We say that N_ℓ <u>loads</u> N_c, and rigorously define the <u>cascade-load</u> by placing the constraints $\underset{\sim}{v}_2 = \underset{\sim}{v}_\ell$ and $\underset{\sim}{i}_2 = -\underset{\sim}{i}_\ell$ on the defining equations

$$\underset{\sim}{v}^r = S_{11}[\underset{\sim}{v}^i] + S_{12}[\underset{\sim}{v}_2^i] \tag{11a}$$

$$\underset{\sim}{v}_2^r = S_{21}[\underset{\sim}{v}^i] + S_{22}[\underset{\sim}{v}_2^i] \tag{11b}$$

$$\underset{\sim}{v}_\ell^r = S_\ell[\underset{\sim}{v}_\ell^i] \tag{11c}$$

Thus, since $\underset{\sim}{v}_\ell^i = \underset{\sim}{v}_2^r$ and $\underset{\sim}{v}_\ell^r = \underset{\sim}{v}_2^i$ in Eqs. (11), the scattering operator for N is

$$\mathcal{S} = \mathcal{S}_{11} + \mathcal{S}_{12}\mathcal{S}_\ell(\mathcal{J} - \mathcal{S}_{22}\mathcal{S}_\ell)^{-1}\mathcal{S}_{21} \tag{12}$$

In Eq. (12), we have simply indicated composition of two operators by writing them in juxtaposition. Too, we comment that even when the indicated inverse does not exist a similar expression can often be used [A2][H9], this requiring, however, detailed arguments concerning ranges and null-spaces of the operators.

<u>Example 2</u>. We show how cascade-loading can be applied to obtain the composition and real unitary transformation operations.

a) Let $\mathcal{S}_c = \begin{bmatrix} 0 & 0 & \mathcal{J} \\ \mathcal{J} & 0 & 0 \\ 0 & \mathcal{J} & 0 \end{bmatrix}$, $\mathcal{S}_\ell = \mathcal{S}_1$.

Then from Eq. (12) $\mathcal{S} = \begin{bmatrix} 0 & \mathcal{S}_1 \\ \mathcal{J} & 0 \end{bmatrix}$.

This $\mathcal{S}_c$ satisfies Thm. 1 with $\|\mathcal{S}_c\|_t = 1$ and hence defines a passive network (which is lossless), called the <u>circulator</u>. The circulator for $H = R^{3n}$ can be obtained further as a cascade-load of n gyrators having $g = -1$ with these acting as a load on wires [N4, pp. 54,150]. Too, wires, which are direct connections, are a special case of the transformer, all turns ratios being ± 1.

b) Let, as from a), $\mathcal{S}_c = \begin{bmatrix} 0 & \mathcal{S}_1 \\ \mathcal{J} & 0 \end{bmatrix}$, $\mathcal{S}_\ell = \mathcal{S}_2$.

Then $\mathcal{S} = \mathcal{S}_1\mathcal{S}_2$.

c) Alternatively, let $\mathcal{S}_c = \begin{bmatrix} 0 & 0 & \mathcal{J} \\ \mathcal{J} & 0 & 0 \\ 0 & \mathcal{J} & 0 \end{bmatrix}$, $\mathcal{S}_\ell = \mathcal{S}_2 \dotplus \mathcal{S}_1$.

where $\dotplus$ denotes the direct sum. Then $\mathcal{S} = \mathcal{S}_1\mathcal{S}_2$.

d) Let the coupling network be an ideal transformer with an orthogonal turns-ratio matrix. Then $\mathcal{S}_c = \begin{bmatrix} 0 & \mathcal{T} \\ \mathcal{T}^a & 0 \end{bmatrix}$ where $\mathcal{T}$ is the operator representation of $\underline{T}$ in $\mathcal{L}_2(H)$ and $\mathcal{T}^a$ its adjoint. We obtain $\mathcal{S} = \mathcal{T}\mathcal{S}_\ell\mathcal{T}^a$.

It is true that if $\mathcal{S}_\ell$ and $\mathcal{S}_c$ are passive then $\mathcal{S}$ obtained by cascade-loading is passive (since the total energy is the sum of the energies into the coupling and load subnetworks). However, a similar result only holds in special cases for the lossless property (since the coupling network, though lossless, may trap energy in the load [A1, p.31]); the cascade load of finite, time-invariant, lossless networks is again lossless [A2].

6. FINITE NETWORKS - DIFFERENTIAL FIELDS

"Make me this poem, make it warm enough for winter
and I will inhabit this line for a long long time" [H1, 7th]

Next we turn to a more algebraic approach, that based upon differential field theory [K4][R3], the results being primarily developed for finite networks. First, we introduce the differential operator from which a scattering matrix representation, and passivity conditions upon it, are obtained. The latter part of the section outlines details and results, particularly factorizations, using differential field theory, as needed for the following section on synthesis (where Examples 3 and 4 illustrate various calculations).

By a finite network we shall mean any network conceived as an interconnection of a finite number of those of Example 1. Practically the vast majority of constructed linear networks are finite and it is from them that the impetus for a theory of networks has stemmed. It is to be noted that for finite networks the dynamics can be considered to enter only through the derivative in which case we are led to introduce the derivative operator

$$p = \frac{d[\cdot]}{dt} \tag{13}$$

In terms of p, we will consider rational descriptions of networks.

Consider as given a differential field F, that is a field in which a (derivative) mapping into itself is defined taking $a \in F$ into $a' \in F$ satisfying $(ab)' = a'b + ab'$, $(b^{-1})' = -b'/b^2$ where also $b \in F$; a' is called the derivative of a. We shall assume F to be formally real, that is $-1 \neq \Sigma a_i^2$ with $a_i \in F$, and interpret the elements of F as functions of time, examples being R^1 for which all elements are constants, $a' = 0$, and the set of rational functions in t with real coefficients. With F on hand we set up polynomials in p with coefficients in F, called <u>differential polynomials</u>,

$$P(p,t) = a_k(t)p^k + \ldots + a_1(t)p + a_o(t) \quad , \quad a_i \epsilon F \tag{14a}$$

for which, in order to perform polynomial multiplication, we take our operator p to act algebraically over F according to the rule

$$pa = ap + a' \qquad a,a'\epsilon F \tag{14b}$$

We call k the degree, writing $k = \delta[P]$, and introduce the adjoint of P of Eq. (14a) as

$$P^a(p,t) = (-1)^k p^k a_k + \ldots + p^2 a_2 - pa_1 + a_o \tag{14c}$$

At this point, we can specify a domain upon which the differential polynomials can operate, initially this being chosen as F. Thus we introduce a multiplication • such that $P(p,t) \bullet a \epsilon F$ for $a\epsilon F$, this being naturally defined through $p \bullet a = a'$. Next, by proper choices of F, as for $F = R^1$, we extend the domain of definition for • and consider matrices of differential polynomials such that they map $\mathcal{D}_+(R^n)$ into $\mathcal{D}_+(R^n)$. Indeed most finite networks can be described by [C2]

$$\underset{\sim}{A}(p,t) \bullet \underset{\sim}{v} = \underset{\sim}{B}(p,t) \bullet \underset{\sim}{i} \tag{15}$$

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are matrices of differential polynomials.

On using Eq. (14b) in defining multiplication, the set $\mathcal{P}$ of differential polynomials over F forms a ring which in fact satisfies conditions allowing it to be imbedded in its skew-field, $\mathcal{Q}$, of quotients [W1, p.137](that is, $\mathcal{P}$ has no divisors of zero and for any $P_1, P_2 \epsilon \mathcal{P}$ there exist $\overline{P}_1$, $\overline{P}_2$, $\hat{P}_1$, $\hat{P}_2 \epsilon \mathcal{P}$ for which $\overline{P}_1 P_1 = \overline{P}_2 P_2$ and $P_1 \hat{P}_1 = P_2 \hat{P}_2$). In other words, $\mathcal{Q}$ is the differential quotient ring of rational differential operators. With division now available we can, as at Eq. (4c), form the scattering (matrix) operator $\underset{\sim}{s}(p,t)$ with

$$\underset{\sim}{s} = (\underset{\sim}{B}+\underset{\sim}{A})^{-1}(\underset{\sim}{B}-\underset{\sim}{A}) \tag{16}$$

Besides having the advantage of yielding a concrete representation for $\mathcal{S}$ the composition product is here seen to correspond to simple matrix multiplication. The material of all of the previous sections carries through and, for example, we have (with $\underset{\sim}{1}_n$ the nxn identity matrix, since $H = R^n$ now)

$$\underset{\sim}{z} = \underset{\sim}{A}^{-1}\underset{\sim}{B} = (\underset{\sim}{1}_n + \underset{\sim}{s})(\underset{\sim}{1}_n - \underset{\sim}{s})^{-1}, \quad \underset{\sim}{y} = \underset{\sim}{B}^{-1}\underset{\sim}{A} = (\underset{\sim}{1}_n - \underset{\sim}{s})(\underset{\sim}{1}_n + \underset{\sim}{s})^{-1} \quad (17a)$$

$$\underset{\sim}{s} = \underset{\sim}{1}_n - 2\underset{\sim}{y}_a = (\underset{\sim}{1}_n + \underset{\sim}{y})^{-1}(\underset{\sim}{1}_n - \underset{\sim}{y}) = (\underset{\sim}{z} + \underset{\sim}{1}_n)^{-1}(\underset{\sim}{z} - \underset{\sim}{1}_n) \quad (17b)$$

$$= \underset{\sim}{s}_{11} + \underset{\sim}{s}_{12}\underset{\sim}{s}_{\ell}(\underset{\sim}{1}_m - \underset{\sim}{s}_{22}\underset{\sim}{s}_{\ell})^{-1}\underset{\sim}{s}_{21} \quad (17c)$$

Since $\underset{\sim}{s}$ operates on $\mathcal{L}_2(R^n)$, the main passivity constraint, $\|\mathcal{S}\|_\infty \le 1$, takes the form of $\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s} \ge 0$ where the adjoint $\underset{\sim}{s}^a$ is formed by transposing $\underset{\sim}{s}$ and taking the adjoint of each entry, using Eq. (14c) on the differential polynomials. Although this method of forming the adjoint works on $\mathcal{L}_2(R^n)$ it is worth pointing out that it does not generally work on $\mathcal{L}_2(R^n)_t$. In any event, the <u>resistivity</u> matrix $\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s}$ is a nonnegative Hermitian operator on $\mathcal{L}_2(R^n)$ whose rank, as we shall see in section 7, is related to the minimum number of resistors in N. Theorem 1 can now be reformulated.

Theorem 2: If $\underset{\sim}{s}(p,t)$ describes a linear passive finite N then necessarily

a) $\underset{\sim}{s} : \mathcal{D}_+(R^n) \to \mathcal{S}_+(R^n)$
b) $\underset{\sim}{s} : \mathcal{L}_2(R^n) \to \mathcal{L}_2(R^n)$ causally
c) $\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s} \ge 0$

Conversely, given an nxn matrix $\underset{\sim}{s}$ of rational differential operators it defines a linear passive N if the conditions a), b) and c) of Theorem 2 are satisfied [$\mathcal{S}$ for Thm. 1 appears continuous by obtaining the distributional kernel $\underset{\sim}{\mathcal{S}}(t,\tau) = \underset{\sim}{s}(\delta'(t-\tau),\tau)$ where $\delta' = d\delta(t)/dt$ is the derivative of the impulse; N is solvable by the existence of $\underset{\sim}{s}$]. However, finiteness is most conveniently obtained through a synthesis.

In order to obtain the factorizations necessary for synthesis, and hence as background preparation for synthesis, we present some more detailed results on differential quotient rings. This, however, is a digression from the main chain of thought and may, perhaps, be skipped on an initial reading. As preliminary to the factorizations we first turn to Galois type extensions. For this, we recall that

$$P(p,t)\cdot u(t) = 0 \quad , \quad P\epsilon\mathcal{P} \tag{18}$$

represents a differential equation in which case, under suitable conditions, there exists [K4, p.21] a Picard-Vessiot extension field $\mathfrak{F}$ of F such that a) $P\cdot u = 0$ has solutions $u_i \epsilon \mathfrak{F}$ of which there are $\delta[P]$ linearly independent (over constants) u_i and b) F and $\mathfrak{F}$ have the same field of constants. Suitable conditions are that 1) the base field F has characteristic zero, that is there are an infinite number of distinct multiples of unity (as for R^1) and 2) the field of constants is algebraically closed when $j = \sqrt{-1}$ is adjoined, that is every polynomial with constant coefficients in F has a root $c = a + jb$ with $a,b\epsilon F$ and $a' = b' = 0$. These suitable conditions are consistent with any physical theory, hence we assume from now on that F has been replaced by $\mathfrak{F}$, for which we write F, whenever needed.

Given $u\epsilon F$ direct calculation shows that $p\frac{1}{u}\cdot u = \frac{1}{u}(p - \frac{u'}{u})\cdot u = 0$ so that, following the ideas of Frobenius [F2,p.190], if u also satisfies $P\cdot u = 0$ then $p\frac{1}{u}$ is a right factor of P, $\mathcal{P}$ being, as we shall see, a ring with irreducible factor decomposition [R1, p.310] $\{$note that if $(p-\alpha)\cdot u = 0$ then $u' - \alpha u = 0$ or $u(t) = u(t_1)\exp(-\int_{t_1}^{t}\alpha(\lambda)d\lambda)\}$. Cancelling this factor and repeating gives a factorization of P into irreducible factors of degree one (with formally real coefficients). If P is self-adjoint then $\frac{1}{u}(-p)$ can be cancelled from the left (assuming otherwise gives a contradiction) and we obtain (as an extension of the factorization of Frobenius [F2,p.193])

$$P = P^a = (-p)^{\ell} b_k p^{\ell} + (-p)^{\ell-1} b_{k-2} p^{\ell-1} + \ldots + (-p) b_2 p + b_o, \quad k = 2\ell \tag{19a}$$

$$= [c_1(-p)c_2(-p)\ldots c_{\ell}(-p)]c_o[pc_{\ell}p\ldots pc_2 pc_1] \tag{19b}$$

$$= (-p-\alpha_1)\ldots(-p-\alpha_{\ell})b_k(p-\alpha_{\ell})\ldots(p-\alpha_1) = P_1^a b_k P_1 \tag{19c}$$

Now, if this self-adjoint P is nonnegative, $\langle x,Px\rangle_\infty = \langle P_1x,b_kP_1x\rangle_\infty = \langle y,b_ky\rangle_\infty \geq 0$ shows that $b_k \geq 0$ in which case it can be factored into its Hermitian nonnegative square-root giving the result:

$$\text{If } P = P^a \geq 0 \;, \quad P\epsilon\mathcal{P}, \text{ then } P = P_o^aP_o \;, \quad P_o\epsilon\mathcal{P}$$

where $P_o = \sqrt{b_k}P_1$.

At this point, we are in a position to give a (matrix) Gauss factorization, this resting upon the scalar factorization just obtained. For this, consider as given a self-adjoint nonnegative matrix $Q(p,t)$ of differential polynomials. On partitioning $Q = Q^a$ with q_{11} a scalar differential polynomial, we write by inspection

$$Q(p,t) = \begin{bmatrix} q_{11} & q_{12} \\ q_{12}^a & q_{22} \end{bmatrix} = \begin{bmatrix} q_{11} & 0 \\ q_{12}^a & 1_{n-1} \end{bmatrix} \begin{bmatrix} q_{11}^{-1} & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ 0 & 1_{n-1} \end{bmatrix} \quad \text{(20a)}$$

with

$$Q_2 = q_{22} - q_{12}^a\, q_{11}^{-1}\, q_{12} \geq 0 \qquad \text{(20b)}$$

where $q_{11} \neq 0$ is assumed, perhaps by applying permutations of rows and columns. Since $q_{11} = q_{11}^a > 0$ we can apply our factorization of Eq. (19) to get $q_{11} = q_1^aq_1$. Too, since we are working in the quotient ring $\mathcal{Q}$, and in order to recover a polynomial matrix from Q_2, we can find a differential polynomial Q_1 and a differential polynomial matrix Q_{12} such that $q_1^{-a}q_{12} = Q_{12}Q_1^{-1}$ (where the superscript -a denotes the adjoint's inverse). From Eq. (20b) we have

$$Q_2 = Q_1^{-a}[Q_1^aq_{22}Q_1 - Q_{12}^aQ_{12}]Q_1^{-1} \geq 0 \qquad \text{(20c)}$$

Continuing the iteration, after interchanging factors with Q_1^{-1} and absorbing any permutations in the outer matrices finally gives a <u>Gauss factorization</u>

$$Q = [M^a\tilde{1}_{\rho,n}L^{-a}][L^{-1}1_{\rho,n}M] = q^aq \qquad \text{(21)}$$

where $\underset{\sim}{1}_{\rho,n}$ is the $\rho \times n$ zero matrix with its first ρ columns replaced by $\underset{\sim}{1}_{\rho}$; ρ is the rank of $\underset{\sim}{Q}$, and $\underset{\sim}{M}$ and $\underset{\sim}{L}$ are polynomial with $\underset{\sim}{L}$ diagonal. We comment that because of the nonuniqueness of irreducible factors in $\mathcal{P}$ various properties for $\underset{\sim}{L}$ are possible. For example, if $F = R^1$, $\underset{\sim}{L}$ can consist of Hurwitz polynomials with real constant coefficients [N4, p.168].

We have used the fact that $\mathcal{P}$ has irreducible factor decomposition, this following [R1, p.325] from $\mathcal{P}$ being a Euclidean ring. That is, a Euclidean algorithm exists [O2, p.483] based upon either left or right division. Specifically given $P_1, P_2 \epsilon \mathcal{P}$ with degrees satisfying $\delta[P_1] \geq \delta[P_2] > 0$ there exist $\overline{P}_1, P_3 \epsilon \mathcal{P}$ such that $P_1 = \overline{P}_1 P_2 + P_3$ and having $\delta[\overline{P}_1] = \delta[P_1] - \delta[P_2]$ and $\delta[P_3] \leq \delta[P_2] - 1$ from which we obtain the sequence $P_2 = \overline{P}_2 P_3 + P_4, \ldots, P_{k-2} = \overline{P}_{k-2} P_{k-1} + P_k$, $P_{k-1} = \overline{P}_{k-1} P_k$ for some k. Using similar operations, elementary matrices $\underset{\sim}{P}$, $\underset{\sim}{Q}$ (having entries, and inverses with entries, in $\mathcal{P}$) can be found such that for any (square) matrix $\underset{\sim}{A}$ with entries in $\mathcal{P}$ a <u>Smith form</u> holds [W1, p.139]

$$\underset{\sim}{P}(p,t)\underset{\sim}{A}(p,t)\underset{\sim}{Q}(p,t) = \text{diag}\,[a_1(p,t),\ldots,a_r(p,t),\underset{\sim}{0}] \qquad (22)$$

where a_i is a left and right factor of a_j, and $\delta[a_i] < \delta[a_j]$ for $i < j$; the rank r is independent of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$. The result extends easily to nonsquare $\underset{\sim}{A}$ while another extension to rational matrices gives a Smith-McMillan form [K1][N4, P.178] as follows. We remove a left denominator polynomial b_ℓ to write a given rational matrix $\underset{\sim}{B}$ as $\underset{\sim}{B} = \frac{1}{b_\ell}\underset{\sim}{A}$ with $\underset{\sim}{A}$ a differential polynomial matrix. Expanding $\underset{\sim}{A}$ in a Smith form as above we next find a differential polynomial b and (elementary) $\underset{\sim}{P}_\ell$ such that $b\underset{\sim}{P}_\ell = \underset{\sim}{P}b_\ell$ to give the Smith-McMillan form

$$\underset{\sim}{P}_\ell\left(\frac{1}{b_\ell}\underset{\sim}{A}\right)\underset{\sim}{Q} = \frac{1}{b}\,\text{diag}[a_1,\ldots,a_r,0] = \text{diag}\left[\frac{1}{\beta_1}\alpha_1,\frac{1}{\beta_2}\alpha_2,\ldots,\frac{1}{\beta_r}\alpha_r,\underset{\sim}{0}\right] \qquad (23a)$$

where, by virtue of possible cancellations, α_i divides α_{i+1} and β_i divides β_{i-1}. Similar results hold if a right denominator is extracted. The Smith-McMillan form allows us to define the degree, $\delta[\cdot]$, of a rational matrix $\underset{\sim}{B} = \frac{1}{b_\ell}\underset{\sim}{A}$ as

$$\delta[\underset{\sim}{B}] = \sum_{i=1}^{r} \delta[\frac{1}{\beta_i}\alpha_i] \tag{23b}$$

where $\delta[\frac{1}{\beta_i}\alpha_i] = \max\{\delta[\alpha_i], \delta[\beta_i]\}$ is the largest of the numerator or denominator degrees after all possible cancellations have been made. With appropriate interpretations the customary properties of the degree hold [N4, p.176], for example $\delta[\underset{\sim}{B}] = 0$ if and only if $\underset{\sim}{B}$ is independent of p.

7. SYNTHESIS - OPERATOR DECOMPOSITIONS

"Here everything is as usual
Like a river into the sea,
I bring full darkness to night;
Everlasting.
You may walk by, but the night does not halt" [H1, 6th]

This section culminates the previous ones by outlining several synthesis methods. The first embeds a passive scattering operator in a unitary operator to physically yield a lossless coupling having a resistive load, while the second uses the characteristic operator function for a lossless coupling having a lossless (capacitive) load. Then the differential field concepts are applied toward synthesis of finite networks through rational operator decompositions. Unfortunately most syntheses require the development of considerable background material, so much of what we present will merely outline what is involved.

In engineering terms, a <u>synthesis</u> can be considered as the formation of a network from a given scattering operator through the interconnection of (supposedly simpler) subnetworks. For a given $\mathcal{S}$, there are consequently various degrees of synthesis depending upon how simple the subnetworks are required to be. Of most practical interest is the synthesis of rational $\underline{s}(p,t)$, for which we will mean by a <u>complete synthesis</u> that all subnetworks are either resistors, capacitors, inductors, gyrators or transformers.

Mathematically, any synthesis can be conceived of as the generation from $\mathcal{S}$ of appropriate coupling, $\mathcal{S}_c$, and load, $\mathcal{S}_\ell$, scattering operators for the cascade-load connection, Eq. (12).

For the first synthesis to be considered, we will place all loss in the load in the form of unit resistors, as illustrated in Fig. 4a); thus $\mathcal{Z}_\ell = \mathcal{I}$ or $\mathcal{S}_\ell = 0[=(\mathcal{Z}_\ell + \mathcal{I})^{-1}(\mathcal{Z}_\ell - \mathcal{I})]$ giving via

Eq. (12), $\mathcal{S} = \mathcal{S}_{11}$. In this case, a lossless $\mathcal{S}_c$ is desired, that is $\mathcal{S}_c^a\mathcal{S}_c = \mathcal{J}$. One possibility is the unitary transformation [S2]

$$\mathcal{S}_c = \begin{bmatrix} \mathcal{S} & (\mathcal{J}-\mathcal{S}\mathcal{S}^a)^{1/2} \\ (\mathcal{J}-\mathcal{S}^a\mathcal{S})^{1/2} & -\mathcal{S}^a \end{bmatrix} \tag{24}$$

where passivity guarantees the existence of the square roots since $\mathcal{J} - \mathcal{S}^a\mathcal{S} \geq 0$ and with it $\mathcal{J} - \mathcal{S}\mathcal{S}^a \geq 0$ (since $\|\mathcal{S}^a\mathcal{S}\|_\infty = \|\mathcal{S}\|_\infty^2 = \|\mathcal{S}\mathcal{S}^a\|_\infty$). However, the $\mathcal{S}_c$ so formed may be noncausal, hence not passive, unless further restrictions are placed (for efforts in this latter direction see [S11, p.56][A1, p.27] for the introduction of an antecedal adjoint and [L6][L9] for the use of certain projection operators). Noncausality of $\mathcal{S}_c$ can of course stem from that of $\mathcal{S}^a$ as is seen, for example, from the kernel representation for the capacitor as given at the end of Section 4 for which

$$\mathcal{S}^a(t,\tau) = -\delta(t - \tau) + \frac{2}{c(\tau)} e^{-\int_t^\tau \frac{1}{c(\lambda)} d\lambda} u(\tau-t)$$

which is nonzero for $\tau > t$, hence non-causal; see also [L7] and [S2, p.929] for other examples. Even when this $\mathcal{S}_c$ is passive, it is not generally attractive in an engineering sense since a) it usually yields nonfinite subnetworks when finite ones can be obtained and b) more resistors than necessary may be required for the synthesis of $\mathcal{S}$.

For a second, and similarly mathematical, synthesis we summarize the use of the characteristic operator function [F1] [S17][D8] as outlined by Helton [H6]. In essence, this method consists of extracting capacitors (or inductors if preferred) into the load. In fact, if inductors or capacitors are present, we note that $\mathcal{S}$ is a function of our operator p, $\mathcal{S} = \mathcal{S}(p)$. First, $\mathcal{S}(p)$, a passive operator on $\mathcal{L}_2(H)$, is written as a direct sum of a unitary p-independent operator $\mathcal{S}_u$ and a <u>purely contractive</u> operator $\mathcal{S}_k(p)$, (that is, $\mathcal{S}_k$ satisfies $\|\mathcal{S}_k(p)\|_\infty \leq 1$ with $\|\mathcal{S}_k(1)\|_\infty < 1$), through the use of a unitary p-independent operator W [H6][S18, Prop.V(2.1)]

$$\mathcal{S}(p) = W^a[\mathcal{S}_u \dotplus \mathcal{S}_k(p)]W \tag{25a}$$

Next $\mathcal{S}_k$ is further decomposed by finding an appropriate Hilbert space $\underline{H}$ and a p-independent contraction operator T on $\underline{H}$ such that [H6][S17, pp.38,51]

$$\mathcal{S}_k(p) = [-UP_*TV] + [U(\mathcal{J}-TT^a)^{1/2}][\frac{p+1}{p-1}\mathcal{J}-T^a]^{-1}[(\mathcal{J}-T^aT)^{1/2}V] \tag{25b}$$

where U and V are onto isometric p-independent maps of $(\mathcal{J}-TT^a)^{1/2}\underline{H}$ into $\mathcal{L}_2(H)$ and H into $(\mathcal{J}-T^aT)^{1/2}\underline{H}$ respectively; P_* is the orthogonal projection of $\underline{H}$ onto $(\mathcal{J}-TT^a)^{1/2}\underline{H}$. Here $\theta(\frac{p-1}{p+1}) = \mathcal{S}_k(p)$ is called the <u>characteristic operator function</u>, in (p-1)/(p+1), for T. The decomposition of Eq. (25b) is well-determined for a purely contractive operator [S17, pp.51-53], though the construction of T is not especially simple. However, given the characteristic operator function decomposition of Eq. (25b), if we form the p-independent unitary map, on $\mathcal{L}_2(H) \dotplus \underline{H}$,

$$\mathcal{S}_{c2} = \begin{bmatrix} -UP_*TV & U(\mathcal{J}-TT^a)^{1/2} \\ (\mathcal{J}-T^aT)^{1/2}V & T^a \end{bmatrix} \tag{25c}$$

then $\mathcal{S}_k(p)$ results by loading in $\mathcal{S}_\ell = \frac{1-p}{1+p}\mathcal{J}$, that is (a Hilbert port of) unit capacitors. The final structure is that of Fig. 4b) where it is seen that only lossless subnetworks are used. Consequently, besides having the shortcomings of the first synthesis an infinite number of capacitors is needed to simulate a resistor. As a further investigation shows [H6], this method allows the synthesis of any bounded-real H-valued matrix function of p where p is treated as a complex variable; by definition (and with * denoting the complex conjugate) $\mathcal{S}(p)$ is bounded-<u>real</u> if

1) $\mathcal{S}(p)$ is analytic in Re $p > 0$
2) $\mathcal{S}^*(p) = \mathcal{S}(p^*)$ in Re $p > 0$
3) $\langle \underline{x}, (\mathcal{J} - \tilde{\mathcal{S}}^*(p)\mathcal{S}(p))\underline{x} \rangle_H \geq 0$ in Re $p > 0$ $\forall \underline{x} \in H$

Classical syntheses of rational bounded-real scattering matrices having real constant coefficients are well-documented [O1][N4][B2] and extensive.

We comment that the two syntheses so far presented can be tied together by noting that $\mathcal{S}_c$ of Eq. (24) can be considered as resulting from the characteristic operator function $\theta(\lambda)$ for $T = -\mathcal{S}$, using $U = V = P_* = \mathcal{J}$, through

$$\theta(\lambda) = \mathcal{S} + (\mathcal{J}-\mathcal{S}\mathcal{S}^a)^{1/2}(\tfrac{1}{\lambda}\mathcal{J} - \mathcal{S}^a)^{-1}(\mathcal{J}-\mathcal{S}^a\mathcal{S})^{1/2}$$

and choosing $\lambda = 0$ when loading $\mathcal{S}_c$ in $\mathcal{S}_\ell(\lambda) = \lambda\mathcal{J}$. Since $\mathcal{S} = \mathcal{S}(p)$, the introduction of the variable λ also opens the possibility of considering various parameters for other purposes, such as gain for sensitivity analyses [N6, p.202].

As stated above, the two syntheses outlined to this point, though mathematically elegant, have engineering shortcomings. Consequently, for a synthesis which is more realistic in engineering terms, we turn to that of a given rational scattering matrix operator $\underline{s}(p,t)$. Assuming $\underline{s}$ is passive, for which both $\underline{1}_n - \underline{s}^a\underline{s}$ and $\underline{1}_n - \underline{s}\underline{s}^a$ are nonnegative, we form a unitary $\underline{s}_c$ to extract a resistive load following a philosophy similar to that of the first synthesis. Letting ρ_c denote the number of resistors used, the unitary constraint is

$$\underline{s}_c^a\underline{s}_c = \underline{s}_c\underline{s}_c^a = \underline{1}_{n+\rho_c} \quad ; \quad \underline{s}_c = \begin{bmatrix} \underline{s} & \underline{s}_{12} \\ \underline{s}_{21} & \underline{s}_{22} \end{bmatrix} \begin{matrix} \}n \\ \}\rho_c \end{matrix} \quad \text{(columns: } \underbrace{\ }_{n}\ \underbrace{\ }_{\rho_c}) \tag{26a}$$

which upon expansion yields (as the (1,1), (1,1) and (2,1) terms)

$$\underline{1}_n - \underline{s}^a\underline{s} = \underline{s}_{21}^a\underline{s}_{21}\ , \quad \underline{1}_n - \underline{s}\underline{s}^a = \underline{s}_{12}\underline{s}_{12}^a, \quad \underline{s}_{22}\underline{s}_{12}^a = -\underline{s}_{21}\underline{s}^a \tag{26b}$$

Since the rank of a product is no bigger than the number of rows in a factor, the first equality of Eq. (26b) shows that

$$\rho_c \geq \rho = \text{rank}\ (\underline{1}_n - \underline{s}^a\underline{s}) \tag{26c}$$

which fixes a lower bound, which we shall obtain, on the number of resistors needed to synthesize $\underset{\sim}{s}$ (we observe that the ranks of $\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s}$ and $\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a$ are equal since, when the impedance $\underset{\sim}{z}$ exists, by Eq. (17), $\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a = 2(\underset{\sim}{z} + \underset{\sim}{1}_n)^{-1}(\underset{\sim}{z} + \underset{\sim}{z}^a)(\underset{\sim}{z}^a + \underset{\sim}{1}_n)^{-1}$ and $\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s} = 2(\underset{\sim}{z}^a + \underset{\sim}{1}_n)^{-1}(\underset{\sim}{z} + \underset{\sim}{z}^a)(\underset{\sim}{z} + \underset{\sim}{1}_n)^{-1}$; when $\underset{\sim}{z}$ does not exist $\underset{\sim}{1}_n - \underset{\sim}{s}$ is singular and there exists vectors $\underset{\sim}{x}$ such that $\underset{\sim}{x} = \underset{\sim}{s}\underset{\sim}{x}$ in which case $\underset{\sim}{x} = \underset{\sim}{s}^a\underset{\sim}{x}$ [S18, p.8] giving $(\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s})\underset{\sim}{x} = (\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a)\underset{\sim}{x} = \underset{\sim}{0}$ while on the space complementary to such $\underset{\sim}{x}$'s an impedance exists; in the time-independent case the result follows also from $\underset{\sim}{s}\underset{\sim}{s}^a$ and $\underset{\sim}{s}^a\underset{\sim}{s}$ having equal eigenvalues [M1, p.23]). If we succeed in finding $\underset{\sim}{s}_{21}, \underset{\sim}{s}_{12}$ and $\underset{\sim}{s}_{22}$ to satisfy Eqs. (26b) then $\underset{\sim}{s}_c$ is unitary, as is easily checked. To find $\underset{\sim}{s}_{21}$ and $\underset{\sim}{s}_{12}$ we factor $\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s}$ and $\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a$ by the Gauss factorization developed in the last section (any other factorization which insures rational factors could be used). Introducing a right pseudo-inverse $\underset{\sim}{s}_{12}^{(-a)}$ for $\underset{\sim}{s}_{12}^a$, that is $\underset{\sim}{s}_{12}^a\underset{\sim}{s}_{12}^{(-a)} = \underset{\sim}{1}_{\rho_c}$, the third term of Eq.(26b) gives $\underset{\sim}{s}_{22} = -\underset{\sim}{s}_{21}\underset{\sim}{s}^a\underset{\sim}{s}_{12}^{(-a)}$. Explicitly, we first write all quantities as differential polynomials through

$$\underset{\sim}{s} = \underset{\sim}{K}g^{-1} = g_2^{-1}\underset{\sim}{K}_2 \tag{27a}$$

where $g, g_2, K_{ij}, K_{2ij} \in \mathcal{P}$ with K_{ij} and K_{2ij} entries in the matrices $\underset{\sim}{K}$ and $\underset{\sim}{K}_2$. Using Eq. (21) we have

$$\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s} = g^{-a}[g^a g\underset{\sim}{1}_n - \underset{\sim}{K}^a\underset{\sim}{K}]g^{-1} = g^{-a}\underset{\sim}{M}^a\underset{\sim}{\tilde{1}}_{\rho,n}\underset{\sim}{L}^{-a}\underset{\sim}{L}^{-1}\underset{\sim}{1}_{\rho,n}\underset{\sim}{M}g^{-1} \tag{27b}$$

$$\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a = g_2^{-1}[g_2 g_2^a\underset{\sim}{1}_n - \underset{\sim}{K}_2\underset{\sim}{K}_2^a]g_2^{-a} = g_2^{-1}\underset{\sim}{M}_2\underset{\sim}{\tilde{1}}_{\rho,n}\underset{\sim}{L}_2^{-1}\underset{\sim}{L}_2^{-a}\underset{\sim}{1}_{\rho,n}\underset{\sim}{M}_2^a g_2^{-a} \tag{27c}$$

or finally, with $\underset{\sim}{s}_{12}^{(-1)} = \underset{\sim}{L}_2\underset{\sim}{1}_{\rho,n}\underset{\sim}{M}_2^{-1}g_2$ and using Eq. (26b)

$$\underset{\sim}{s}_c = \begin{bmatrix} \underset{\sim}{s} & g_2^{-1}\underset{\sim}{M}_2\underset{\sim}{\tilde{1}}_{\rho,n}\underset{\sim}{L}_2^{-1} \\ \underset{\sim}{L}^{-1}\underset{\sim}{1}_{\rho,n}\underset{\sim}{M}g^{-1} & -\underset{\sim}{L}^{-1}\underset{\sim}{1}_{\rho,n}\underset{\sim}{M}g^{-1}\underset{\sim}{K}_2^a\underset{\sim}{M}_2^{-a}\underset{\sim}{\tilde{1}}_{\rho,n}\underset{\sim}{L}_2^a \end{bmatrix} \tag{27d}$$

Consequently, a unitary coupling scattering matrix operator $\underset{\sim}{s}_c(p,t)$ has been constructed which is rational with coefficients in a formally real field (when $\underset{\sim}{s}$ is passive). There is actually considerable freedom in choosing the diagonal polynomial matrices $\underset{\sim}{L}$ and $\underset{\sim}{L}_2$; it appears that these can be chosen such that for any fixed time $\underset{\sim}{s}_c$ maps $\mathcal{L}_2(R^n)_t$ into $\mathcal{L}_2(R^n)_t$ with nonnegative energy $E(t) \geq 0$ [N7]. However, only when $F = R^1$, that is when the base field of the differential algebra is the reals, is it yet proven that a passive $\underset{\sim}{s}_c$ can be obtained by this procedure [N4, p.288]. The case $F = R^1$ is that of time-invariant networks, $\underset{\sim}{s}(p,t) = \underset{\sim}{S}(p)$, for which complete synthesis can be given for (rational) passive $\underset{\sim}{S}(p)$, these scattering operators being of necessity bounded-real [Y1].

The process of forming $\underset{\sim}{s}_c$ is conveniently illustrated by example.

Example 3. Let it be desired to find a lossless $\underset{\sim}{s}_c$ for the passive

$$\underset{\sim}{s}(p,t) = \underset{\sim}{1}_n - \frac{1}{p+\alpha(t)} 2\alpha(t)\underset{\sim}{u}\underset{\sim}{\tilde{u}} \quad ; \quad \underset{\sim}{u}' = \underset{\sim}{0} \ , \quad \underset{\sim}{\tilde{u}}\underset{\sim}{u} = 1 \ , \ \alpha\epsilon F$$

$$= \underset{\sim}{1}_n - 2\alpha\frac{1}{p+\alpha+\frac{\alpha'}{\alpha}} \underset{\sim}{u}\underset{\sim}{\tilde{u}}$$

where $\underset{\sim}{u}$ is an n-vector, constant and normalized as shown, and the second expression results from the first since $\frac{1}{p+\alpha}\alpha = a\frac{1}{p+b}$ yields possible a and b from $\alpha(p+b) = (p+\alpha)a$ which is $\alpha p + \alpha b = ap + a' + \alpha a$ implying $a = \alpha$, $b = (a' + \alpha a)/\alpha$. For Eq. (27a), we have $g = p + \alpha + \alpha'/\alpha$, $g_2 = p + \alpha$, $\underset{\sim}{K} = (p+\alpha+\alpha'/\alpha)\underset{\sim}{1}_n - 2\alpha\underset{\sim}{u}\underset{\sim}{\tilde{u}}$, $\underset{\sim}{K} = (p + \alpha)\underset{\sim}{1}_n - 2\alpha\underset{\sim}{u}\underset{\sim}{\tilde{u}}$. On direct multiplication we have

$$\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a = \frac{1}{p+\alpha} 2\alpha' \frac{1}{-p+\alpha} \underset{\sim}{u}\underset{\sim}{\tilde{u}} \qquad (=\underset{\sim}{s}_{12}\underset{\sim}{s}^a_{12})$$

$$\underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s} = \frac{1}{-p+\alpha+\frac{\alpha'}{\alpha}} 2\alpha' \frac{1}{p+\alpha+\frac{\alpha'}{\alpha}} \underset{\sim}{u}\underset{\sim}{\tilde{u}} \qquad (=\underset{\sim}{s}^a_{21}\underset{\sim}{s}_{21})$$

For the Gauss factorization at Eq. (27), we have the rank $\rho = 1$, hence $\underset{\sim}{1}_{\rho,n} = \underset{\sim}{1}_{1,n} = [1 \mid \underset{\sim}{0}]$ with $\underset{\sim}{1}_{1,n}\underset{\sim}{M} = 2\alpha'\underset{\sim}{\tilde{u}}$. Noting that $\alpha' \geq 0$ is a condition for factorization and hence passivity we have

$$\underset{\sim}{s}_{12} = \frac{1}{p+\alpha} (2\alpha'\underset{\sim}{u}) \frac{1}{\sqrt{2\alpha'}} = \frac{1}{p+\alpha} \sqrt{2\alpha'}\ \underset{\sim}{u}$$

$$\underset{\sim}{s}_{21} = \frac{1}{\sqrt{2\alpha'}} (2\alpha'\underset{\sim}{\tilde{u}}) \frac{1}{p+\alpha+\frac{\alpha'}{\alpha}} = \frac{\sqrt{2\alpha'}}{\alpha} \underset{\sim}{\tilde{u}} \frac{1}{p+\alpha} \alpha$$

Then $\underset{\sim}{s}_{12}^{a} = \tilde{u}\sqrt{2\alpha'}\,\frac{1}{-p+\alpha}$ and $\underset{\sim}{s}_{12}^{(-a)}$ for $\underset{\sim}{s}_{12}^{a}\underset{\sim}{s}_{12}^{(-a)} = 1$ is $\underset{\sim}{s}_{12}^{(-a)} = (-p+\alpha)\,\frac{1}{\sqrt{2\alpha'}}\underset{\sim}{u}$ for which we have

$$\underset{\sim}{s}_{22} = -\underset{\sim}{s}_{21}\underset{\sim}{s}^{a}\underset{\sim}{s}_{12}^{(-a)} = -\frac{\sqrt{2\alpha'}}{\alpha}\,\tilde{u}\,\frac{1}{p+\alpha}\alpha[\underset{\sim}{1}_n - 2\alpha\frac{1}{-p+\alpha}\underset{\sim}{u}\tilde{u}](-p+\alpha)\frac{1}{\sqrt{2\alpha'}}\,\underset{\sim}{u}$$

$$= \frac{\sqrt{2\alpha'}}{\alpha}\,\frac{1}{p+\alpha}[\alpha(p-\alpha)+2\alpha^2]\frac{1}{\sqrt{2\alpha'}} = \frac{\sqrt{2\alpha'}}{\alpha}\,\frac{1}{p+\alpha}\,(p+\alpha-\frac{\alpha'}{\alpha})\,\frac{\alpha}{\sqrt{2\alpha'}}$$

Finally, using these entries we have the (n+1)x(n+1) unitary matrix

$$\underset{\sim}{s}_c = \begin{bmatrix} \underset{\sim}{1}_n - \frac{1}{p+\alpha}\,2\alpha\underset{\sim}{u}\tilde{u} & \underset{\sim}{u}\frac{1}{p+\alpha}\sqrt{2\alpha'} \\ \frac{\sqrt{2\alpha'}}{\alpha}\,\frac{1}{p+\alpha}\,\alpha\tilde{u} & \frac{\sqrt{2\alpha'}}{\alpha}\,\frac{1}{p+\alpha}(p+\alpha-\frac{\alpha'}{\alpha})\,\frac{\alpha}{\sqrt{2\alpha'}} \end{bmatrix}$$

We comment that the $\mathcal{L}_2$ mapping constraint, b) of Thm. 1, seems to also require $\alpha > 0$ for passivity.

There are several ways to obtain a complete synthesis of a passive unitary $\underset{\sim}{s}(p,t)$. One method is to convert to an admittance, while another is to factor $\underset{\sim}{s}$. Both of these methods we only outline to catch the main features to give a general feeling, since the details presently involve extensive calculations using distributions [S11][A3].

For the first method, one forms $\underset{\sim}{y}(p,t) = (\underset{\sim}{1}_n-\underset{\sim}{s})^{-1}(\underset{\sim}{1}_n+\underset{\sim}{s})$ after perhaps an orthogonal transformation, as at Example 2d), used to guarantee $\underset{\sim}{1}_n - \underset{\sim}{s}$ nonsingular [N4,p.131]. The unitary condition of $\underset{\sim}{s}$ gives $\underset{\sim}{y} + \underset{\sim}{y}^{a} = \underset{\sim}{0}_n$ which, in conjunction with the passivity constraint, guarantees the existence of the expansion [S11, p.53]

$$\underset{\sim}{y}(p,t) = \underset{\sim}{T}_{\infty}(t)p\tilde{\underset{\sim}{T}}_{\infty}(t) + \underset{\sim}{Y}_0(t) + \underset{\sim}{T}_0(t)\frac{1}{p}\tilde{\underset{\sim}{T}}_0(t) \quad ; \quad \underset{\sim}{Y}_o = -\tilde{\underset{\sim}{Y}}_o \tag{28}$$

Synthesis readily occurs from Eq. (28) which yields a parallel connection of three subnetworks: a transformer of turns ratio matrix $\underset{\sim}{T}_{\infty}$ cascade-loaded in unit capacitors, gyrators for $\underset{\sim}{Y}_o$, and a transformer of turns ratio $\underset{\sim}{T}_o$ cascade-loaded in unit inductors [for this note that a transformer, by Example 1d), having $\underset{\sim}{v}_2 = \tilde{\underset{\sim}{T}}\underset{\sim}{v}_1$, $\underset{\sim}{i}_1 = -\underset{\sim}{T}\underset{\sim}{i}_2$, cascade-loaded in a network with $-\underset{\sim}{i}_2 = \underset{\sim}{i}_\ell = \underset{\sim}{y}_\ell(p,t)\cdot\underset{\sim}{v}_2$ gives $\underset{\sim}{i}_1 = \underset{\sim}{T}\underset{\sim}{y}_\ell\tilde{\underset{\sim}{T}}\cdot\underset{\sim}{v}_1$ or $\underset{\sim}{y}(p,t) = \underset{\sim}{T}(t)\underset{\sim}{y}_\ell(p,t)\tilde{\underset{\sim}{T}}(t)$]. Figure 5 shows the end result (when $\underset{\sim}{1}_n - \underset{\sim}{s}$ is nonsingular; otherwise a transformer is inserted on the left). When $\underset{\sim}{s}$ is independent of t, the matrices on the right of Eq. (28) will still normally be time dependent. However, terms can either be combined [S12] or a direct partial-fraction expansion made [N4, p.202] to insure that no time dependent subnetworks are used. Again, an example is in order.

<u>Example 4.</u> Consider

$$\underset{\sim}{s} = \underset{\sim}{1}_n - \frac{1}{p+\alpha}\beta\underset{\sim}{u}\tilde{\underset{\sim}{u}} = \underset{\sim}{1}_n - \beta\frac{1}{p+\alpha+\beta'/\beta}\underset{\sim}{u}\tilde{\underset{\sim}{u}} \quad ; \quad \underset{\sim}{u}' = \underset{\sim}{0},\ \tilde{\underset{\sim}{u}}\underset{\sim}{u} = 1$$

This is lossless if $0 < \alpha\epsilon F$ and $2\alpha = \beta - \frac{\beta'}{\beta}$, as calculation of $\underset{\sim}{1}_n - \underset{\sim}{s}\underset{\sim}{s}^a = \underset{\sim}{1}_n - \underset{\sim}{s}^a\underset{\sim}{s} = \underset{\sim}{0}_n$ shows. We then form

$$\underset{\sim}{y} = (\underset{\sim}{1}_n - \underset{\sim}{s})(\underset{\sim}{1}_n + \underset{\sim}{s})^{-1}$$

$$= \left(\frac{1}{p+\alpha}\beta\underset{\sim}{u}\tilde{\underset{\sim}{u}}\right)\frac{1}{2}\left(\underset{\sim}{1}_n + \frac{1}{p+\alpha-\beta/2}\,\frac{\beta}{2}\,\underset{\sim}{u}\tilde{\underset{\sim}{u}}\right)$$

Here the inverse is found by solving for x in the equation $(\underset{\sim}{1}_n + \underset{\sim}{s})[\frac{1}{2}(\underset{\sim}{1}_n + x\underset{\sim}{u}\tilde{\underset{\sim}{u}})] = \underset{\sim}{1}_n$. This equation for x also shows that $\frac{1}{p+\alpha}\,\frac{\beta}{2}\,\frac{1}{p+\alpha-\beta/2} = -\frac{1}{p+\alpha}\,\beta/2 + \frac{1}{p+\alpha-\beta/2}\,\beta/2$ which, when inserted in the expression for $\underset{\sim}{y}$, gives, since $2\alpha - \beta = -\beta'/\beta$,

$$\underset{\sim}{y} = \frac{1}{p+\alpha-\beta/2}\,\frac{\beta}{2}\,\underset{\sim}{u}\tilde{\underset{\sim}{u}} = \frac{1}{p-\beta'/2\beta}\,\frac{\beta}{2}\,\underset{\sim}{u}\tilde{\underset{\sim}{u}}$$

$$= (\sqrt{\beta/2}\,\underset{\sim}{u})\,\frac{1}{p}\,(\sqrt{\beta/2}\,\tilde{\underset{\sim}{u}})$$

which is Eq. (28) [the last equality results by solving for x in $x\frac{1}{p}x = \frac{1}{p-\beta'/2\beta}\beta/2$ which is $(p-\beta'/2\beta)x = \frac{\beta}{2x}p$]. In terms of Fig. 5 the synthesis uses one unit time-invariant inductor and a transformer of turns-ratio matrix $\underset{\sim}{T}_o = \sqrt{\beta/2}\,\underset{\sim}{u}$.

Another method of obtaining a complete synthesis of a rational unitary passive $\underline{s}(p,t)$ is to factor $\underline{s}$ into degree one factors

$$\underline{s} = \underline{a}_o\underline{s}_1\underline{s}_2 \cdots \underline{s}_d \quad ; \quad \delta[\underline{a}_o] = 0,\ \delta[\underline{s}_i] = 1,\ \delta[\underline{s}] = d \qquad (29a)$$

where details show [A3, p.43]

$$\underline{a}_o = \underline{a}_o(t) \text{ is orthogonal and} \qquad (29b)$$

$$\underline{s}_i(p,t)=[\underline{1}_n+\tilde{\underline{T}}_i(t)\frac{1}{p}\underline{T}_i(t)]^{-1}[\underline{1}_n-\tilde{\underline{T}}_i(t)\frac{1}{p}\underline{T}_i(t)]\ , \quad i=1,\ldots,d=\delta[s] \qquad (29c)$$

with the rank one turns ratio (1xn) matrices $\underline{T}_i$ determined as follows: Find the distributional kernel representation of $\underline{s}$, this takes the form [S11, p.44]

$$\underline{\mathcal{S}}(t,\tau) = \underline{a}_o(t)[-\delta(t-\tau)\underline{1}_n + \underline{\Phi}(t)\tilde{\underline{\Psi}}(\tau)u(t-\tau)] \qquad (30)$$

where, as before, $u(\cdot)$ is the unit step function, $\delta(\cdot) = u'(\cdot)$ is the impulse distribution, and $\underline{\Phi}$ and $\underline{\Psi}$ are nxd matrices. The number of nonzero columns, d, in $\underline{\Phi}$ is the degree of $\underline{s}$ from which the synthesis philosophy is to reduce the columns to zero by removing, for each $\underline{s}_i$, one column at a time, each removal coinciding with a factor of $\underline{s}$ with each $\underline{s}_i$ factor coming from a network containing one inductor and a transformer. In actual fact, the removal of a column of $\underline{\Phi}$ is accompanied by a removal of a row of $\tilde{\underline{\Psi}}$ since the lossless constraints on $\underline{s}$ guarantee [A3, p.36] that there exists a positive semidefinite constant matrix $\underline{C}$ to satisfy $\underline{s}^a \cdot \underline{\Phi} = -\underline{\Psi}\underline{C}$. Taking $\underline{\varphi}(t)$ to be the jth column of $\underline{\Phi}$ and c the (j,j) entry of $\underline{C}$ one then chooses

$$\underline{T}_1(t) = [2(c + \int_t^\infty \tilde{\underline{\varphi}}(\tau)\underline{\varphi}(\tau)d\tau)]^{-1/2}\underline{\varphi}(t) \qquad (31)$$

This particular choice of $\underline{T}_1$, which is physically a turns-ratio matrix, guarantees that the degree decreases, that is, $\delta[\underline{s}] = \delta[\underline{s}_1^a\underline{a}_o^a\underline{s}] + 1$. On extracting $\underline{a}_o\underline{s}_1$ one then repeats on the remainder, $\underset{\sim}{s}_1^a \underset{\sim}{a}_o \underset{\sim}{s} = (\underset{\sim}{a}_o \underset{\sim}{s}_1)^{-1} \underset{\sim}{s}$, to get $\underset{\sim}{T}_2$, continuing the process d times. Given the expansion of E . (29a), we iteratively apply the cascade-load result of Example 2c) for realizing the products. The orthogonal scattering matrix $\underset{\sim}{a}_o$ comes from transformers and gyrators while each $\underset{\sim}{s}_i$, $i = 1, \ldots, d$, is synthesized by converting to $\underset{\sim}{y}_i = \tilde{\underset{\sim}{T}}_i \frac{1}{p} \underset{\sim}{T}_i$ and using the synthesis already given at Eq. (28). As with previous synthesis, time-invariant syntheses exist [D2] [D3].

Finally, we give some active syntheses for which we observe that if $\mathcal{S}$ is a bounded linear scattering operator with $1 < \|\mathcal{S}\|_\infty \le k$ then $\frac{1}{k}\mathcal{S}$ has $\|\frac{1}{k}\mathcal{S}\|_\infty \le 1$. Then $\mathcal{S}$ results [L4][S2] from the passive

$$\mathcal{S}_c = \begin{bmatrix} 0 & \frac{1}{k}\mathcal{S} \\ \mathcal{I} & 0 \end{bmatrix} \tag{32a}$$

by cascade-loading with (negative) resistors having $\mathcal{S}_\ell = k\mathcal{I}$ (which is $\mathcal{Z}_\ell = \frac{1+k}{1-k}\mathcal{I}$). This $\mathcal{S}_c$ in turn results from cascade-loading the circulator of Example 2a) by $\mathcal{S}_\ell = \frac{1}{k}\mathcal{I}$. A similar result using $\mathcal{Z}_\ell = -\mathcal{I}$ can be found in [S2] while another method of active synthesis, also due to Levan [L3][L5], stems from forming [compare Eq. (24)]

$$\mathcal{S}_J = \begin{bmatrix} \mathcal{S} & -(\mathcal{S}\mathcal{S}^a - \mathcal{I})^{1/2} \\ (\mathcal{S}^a\mathcal{S} - \mathcal{I})^{1/2} & \mathcal{S}^a \end{bmatrix} \tag{32b}$$

which satisfies $\mathcal{S}_J^a J \mathcal{S}_J = \mathcal{S}_J J \mathcal{S}_J^a = J$ with $J = \mathcal{I} \dotplus (-\mathcal{I})$ (hence $\mathcal{S}_J$ is called the <u>J-lossless</u> extension of $\mathcal{S}$). By terminating the coupling network defined by $\mathcal{S}_J$ in (positive) unit resistors $\mathcal{S}$ results. $\mathcal{S}_J$, which is itself active, can be synthesized by using Eq. (32a), in which $\mathcal{S}$ is replaced by $\mathcal{S}_J$, though other methods are presently under investigation.

8. GENERALIZATIONS

"No time, no time to see the green before it burst open."[H1,1st]

Here we discuss several extensions of the previous ideas which are of especial interest for future developments.

A. Complex Spaces

Throughout this work, we have assumed real physical elements since practically these are the only ones available. However, there is essentially no step we have carried out which will not mathematically work with complex quantities. In most cases, the necessary modifications to incorporate complex elements will be clear; for example condition 2) of the bounded-real definition gets deleted with the corresponding deletion of the word "real" while a possible definition of energy is [Y1][N3]

$$E(t) = \mathrm{Re} \int_{-\infty}^{t} < \underline{v}(\tau), \underline{i}(\tau) >_H d\tau \qquad (33)$$

B. Complex Normalization

In defining incident and reflected quantities, we have also made a particular choice by assuming unit resistors in the augmentation of Fig. 2. A common choice of interest for cascade synthesis is

$$2R_o^{1/2}\underline{v}^i = \underline{v} + Z_o\underline{i} \quad , \quad 2R_o^{1/2}\underline{v}^r = \underline{v} - Z_o^a\underline{i} \qquad (34)$$

where Z_o is a given operator with $2R_o = Z_o + Z_o^a > 0$. Since $\underline{v}$ and $\underline{i}$ are the primary quantities defining a network and Z_o is free to be chosen, Eqs. (34) are said to define a scattering operator, $\underline{v}^r = \mathcal{S}[\underline{v}^i]$, normalized with respect to Z_o. A physical interpretation similar to that of Fig. 2 can still be given [N4, p.286].

The scattering matrix used throughout the previous sections is seen to be normalized with respect to the identity, that is $Z_o = \mathcal{J}$.

C. State-Space Methods

If the cascade-load formula can be written as

$$\mathcal{S}(p) = \mathcal{S}_{11} + \mathcal{S}_{12}(p\mathcal{J} - \mathcal{S}_{22})^{-1}\mathcal{S}_{21} \tag{35a}$$

with the $\mathcal{S}_{ij}$ independent of p, then we call this a state-space decomposition, the nomenclature following from the incident variable on the load $\mathcal{S}_{\ell}(p) = \frac{1}{p}\mathcal{J}$ acting as a state. For rational time-independent matrices state-space techniques are well-developed [Y2][N5] while for nonrational matrices a theory based upon distributions has been set up by Kamen [K2][K3]. We point out that similar ideas apply if one can write

$$\mathcal{S}(p) = \mathcal{S}_{11} + \mathcal{S}_{12}\ (\mathcal{S}_{\ell}^{-1}(p) - \mathcal{S}_{22})^{-1}\ \mathcal{S}_{21} \tag{35b}$$

with the $\mathcal{S}_{ij}$ p-independent for which the case of $\mathcal{S}_{\ell}(p)$ bounded-real is of most significance; a particular instance occurs at Eq. (25b).

D. Factorization Syntheses

In the rational time-independent nxn matrix case factorization synthesis, as at Eq. (29a), can occur [B2][D3] through the use of complex normalizations and degree one factors which take the form [B2, p.329]

$$\underset{\sim}{S}_i(p) = \underset{\sim}{1}_n - \frac{2\alpha_i \underset{\sim}{u}_i \underset{\sim}{\tilde{u}}_i^*}{p+p_i^*} \tag{36}$$

where the n-vector $\underset{\sim}{u}_i$ is normalized to $\underset{\sim}{\tilde{u}}_i^* \underset{\sim}{u}_i = 1$ and $p_i = \alpha_i + j\omega_i$, with $\alpha_i > 0$, $j = \sqrt{-1}$, are appropriately chosen to decrease the degree [D3]. Factorization can occur even when the given scattering operator is not lossless, but in that case necessitates a change of normalization after the removal of a given factor. Consequently, a new type of multiplication is introduced which is nonassociative, being dependent upon the normalizations used [D2, p.124].

In the case where an nxn time-independent scattering matrix $\underset{\sim}{S}(p)$ is not rational (and hence of "infinite" degree), though still bounded-real on $H = R^n$, a factorization synthesis can occur if and only if $\underset{\sim}{S}(p)$ is roomy [D4]. By definition such a bounded-real $\underset{\sim}{S}(p)$ is roomy if $M = M(\omega_o)$ spans R_n for some real ω_o where $M(\omega)$ is the space of analytic n-vectors $\underset{\sim}{A}(j\omega)$ such that $\underset{\sim}{S}(-j\omega)\underset{\sim}{A}(j\omega)$ is analytic in ω. Physically, M represents the orthogonal complement of the state-space of $\underset{\sim}{S}(p)$. Since factorization of a roomy scattering matrix into degree one factors in general requires an infinite number of factors, each of which individually appears as in Eq. (36), the product integral representation of Potapov [P2][G1] becomes of most interest [G3], as well as the theory of invariant subspaces [H4] and transfer scattering matrices [D2]. Toward further research in this area possible factorizations of the characteristic operator function [F1, p.68] should be of interest.

E. Equivalence Theory

Two syntheses for the same $\mathcal{S}$, but using different $\mathcal{S}_c$ and $\mathcal{S}_\ell$ in the cascade-load, are called equivalent. Equivalent syntheses are of considerable engineering interest since they allow for various choices and freedom in physical construction.

Equivalence theory has been extensively developed for bounded-real matrices on $H = R^n$ in the two cases of $\mathcal{S}_\ell = 0$ [O1][N4, p.31] and $\mathcal{S}_\ell = \frac{p-1}{p+1}\mathcal{J}$ [A4][N5, p.71], The latter through state-space methods. In the former case, it is a question of investigating unitary $\mathcal{S}_c$ of the form

$$\mathcal{S}_c = \left[\begin{array}{c|c} \mathcal{J} & 0 \\ \hline 0 & U_\ell \end{array}\right] \left[\begin{array}{c|cc} \mathcal{S} & \mathcal{S} & 0 \\ \hline \mathcal{S}_{21} & \mathcal{S}_{22} & 0 \\ 0 & 0 & \mathcal{S}_{33} \end{array}\right] \left[\begin{array}{c|c} \mathcal{J} & 0 \\ \hline 0 & U_r \end{array}\right] \tag{37a}$$

where U_ℓ, U_r and $\mathcal{S}_{33}$ are arbitrary unitary matrices and an initial coupling scattering operator is given, constructed from $\mathcal{S}$, $\mathcal{S}_{12}$, $\mathcal{S}_{21}$ and $\mathcal{S}_{22}$. Of interest for future research in this direction are the unitary $\mathcal{S}_c$ which are unitary dilations [D8, Section 1], that is $\mathcal{S}_c$ for which $\mathcal{S}_c^n$ has $\mathcal{S}^n$ in the (1,1) position for any positive integer n, the unitary dilation S_c being uniquely determined by S when S_c is minimal [D8].

In the second (state-space type) case it is a question of investigating p-independent S_c of the form

$$S_c = \left[\begin{array}{c|c} \mathcal{I} & 0 \\ \hline 0 & \mathcal{T}^{-1} \end{array}\right] \left[\begin{array}{c|ccc} S_{11} & 0 & S_{12}^{b} & S_{12}^{c} \\ \hline S_{21}^{a} & S_{22}^{aa} & S_{22}^{ab} & S_{22}^{ac} \\ S_{21}^{b} & 0 & S_{22}^{bb} & S_{22}^{bc} \\ 0 & 0 & 0 & S_{22}^{cc} \end{array}\right] \left[\begin{array}{c|c} \mathcal{I} & 0 \\ \hline 0 & \mathcal{T} \end{array}\right] \tag{37b}$$

for a given (minimal realization) S_{11}, S_{12}^{b}, S_{21}^{b}, S_{22}^{bb} initially satisfying $S = S_{11} + S_{12}^{b}(\frac{p+1}{p-1}\mathcal{I}-S_{22}^{bb})^{-1} S_{21}^{b}$, all other indicated entries in S_c being arbitrary.

9. DISCUSSION

"Perhaps it is time
For some conversation?
Why not?
As follows:" [H1, 1st]

In a relatively short space we have given most of the key ideas within the operator theory of networks, at least as they apply to engineering designs in scattering operator terms. Consequently, portions of the treatment have been sketchy and some topics, such as sensitivity, stability, and multivariable synthesis have been almost completely omitted. Too, in an effort to be concise while complete we have subsumed some classical results in a more general framework, such that, for example, one familiar with network synthesis as normally taught [V1] will find the results rather hidden, though present.

The synthesis techniques given here rely quite heavily upon the cascade-load decomposition which, because of its generality, finds other uses besides. For example, it can be advantageously used in computer-aided design [B6] as well as to show that all finite networks can be generated using interconnections of two basic elements, the capacitor and the 3-port differential voltage controlled current source [B5]. Too, it may be worthwhile commenting that the scattering operator is not just a theoreticians' tool; common transistors now come specified in terms of their scattering parameters.

There is one aspect of particular importance to engineering design which we have as yet not commented upon. This is the theory of specification from which given scattering operators are obtained and for which syntheses are performed. Classically, specifications have resulted through approximation theories [H11] which, however, only recently have been considered within an operator theory framework [D1] and then only in the sense of modelling of existing systems. In order that the theory may become practically useful, methods of specification of scattering operators are needed so that meaningful systems will be synthesized.

Finally, we comment that open problems can be found throughout by a careful study. Among these will be seen a need for the development of an algebraic theory for nonrational operators in p through a differential field type treatment. Such operators include such things as transmission-lines for which engineers have long proceeded with various forms of calculii. Too, the abstract theory of differential fields does not seem to have been tied into Hilbert space techniques beyond the treatment given here. Along this line, the various properties of the Picard-Vessiot extensions seem to need investigation in particular cases of practical importance. Concerning Hilbert ports, very little has been done by way of investigating the behavior of networks where the ports shift around in time. Similarly the theory of network equivalence has scarcely been touched upon beyond that available for rational time-independent operators.

In closing, we believe it could be said that the operator theory of networks has a history of development stemming from practical needs which brought it to the mathematical maturity we have been able to present here. We feel its future is bright and may well be worth an investment by the mathematical community in view of recent discussions [W2].

"The sun and the moon
The tree in the clouds, and man, in woman's power...
And it fell silent, the greedy poem." [H1, 8th]

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"Keep yourself warm
When the pools are freezing" [H1, 9th]

FIGURE TITLES

1. Pictorial Representation of Network N

2. The Augmented Network N_a

3. Cascade-Load Connection
    a) General Connection
    b) Parallel Connection as Special Case

4. Synthesis Configurations
    a) Resistive Extraction
    b) Capacitor Extraction

5. Synthesis of Unitary Passive $\underset{\sim}{s}(p,t)$

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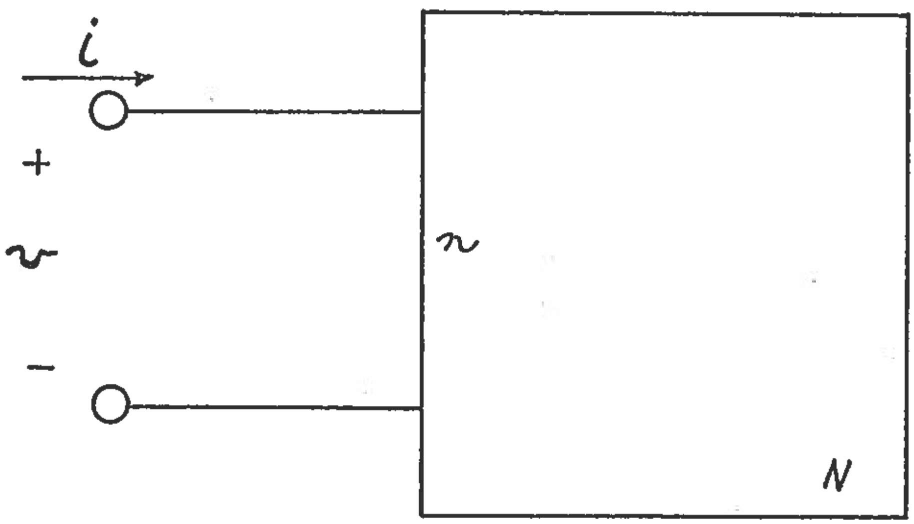
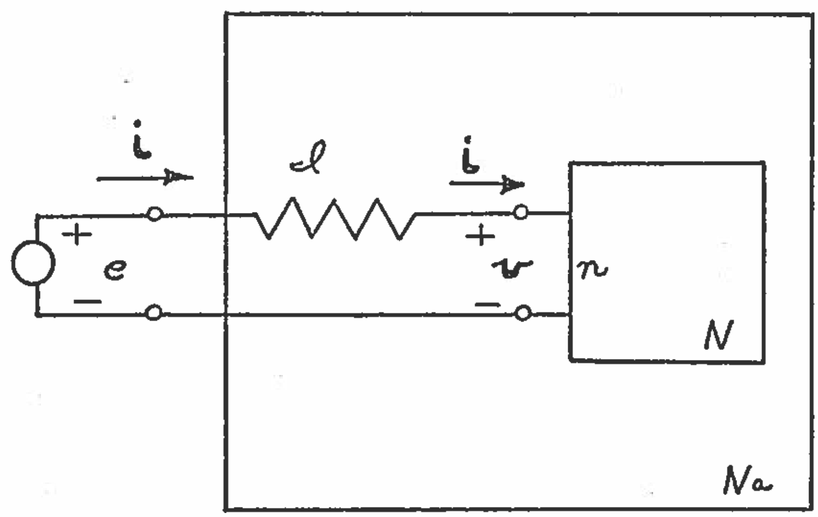


Figure 1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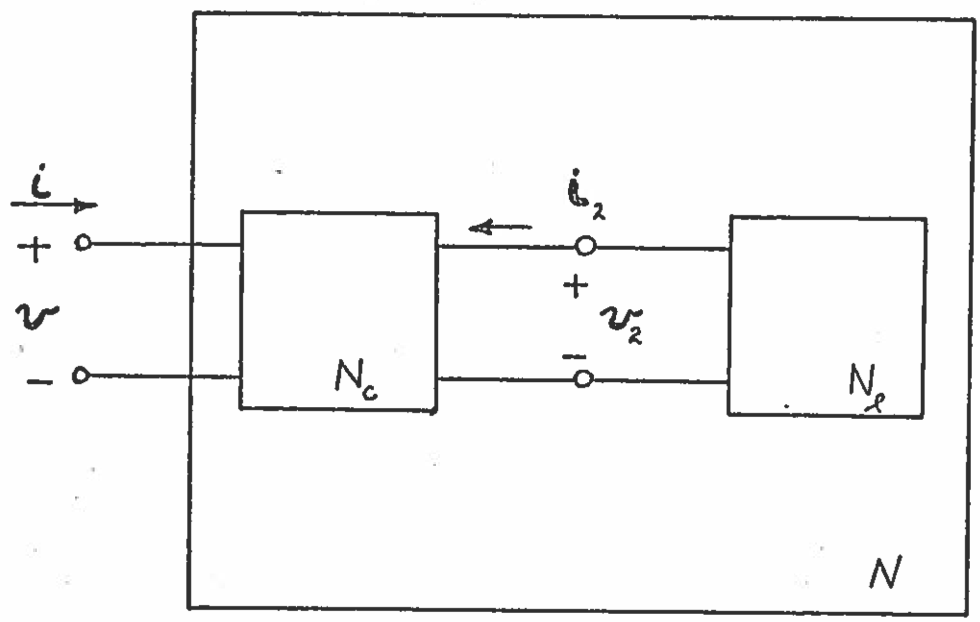
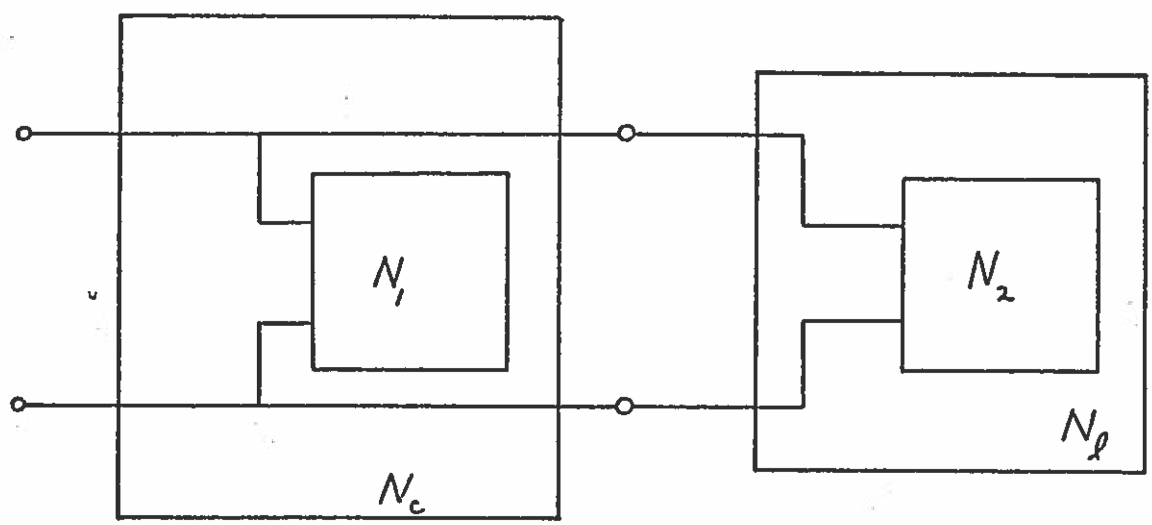


a)

Figure 3 a).



b)

Figure 3b)

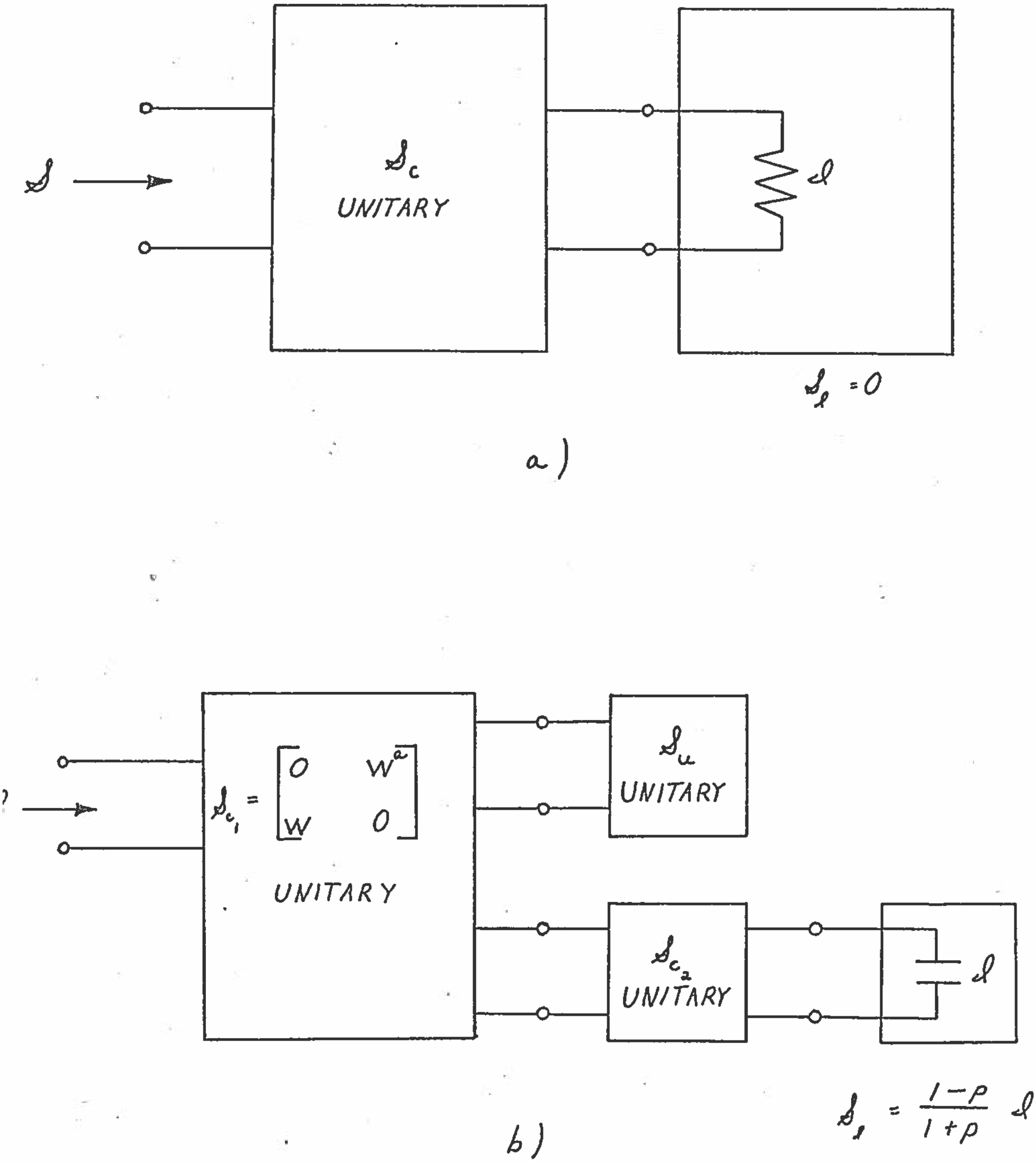
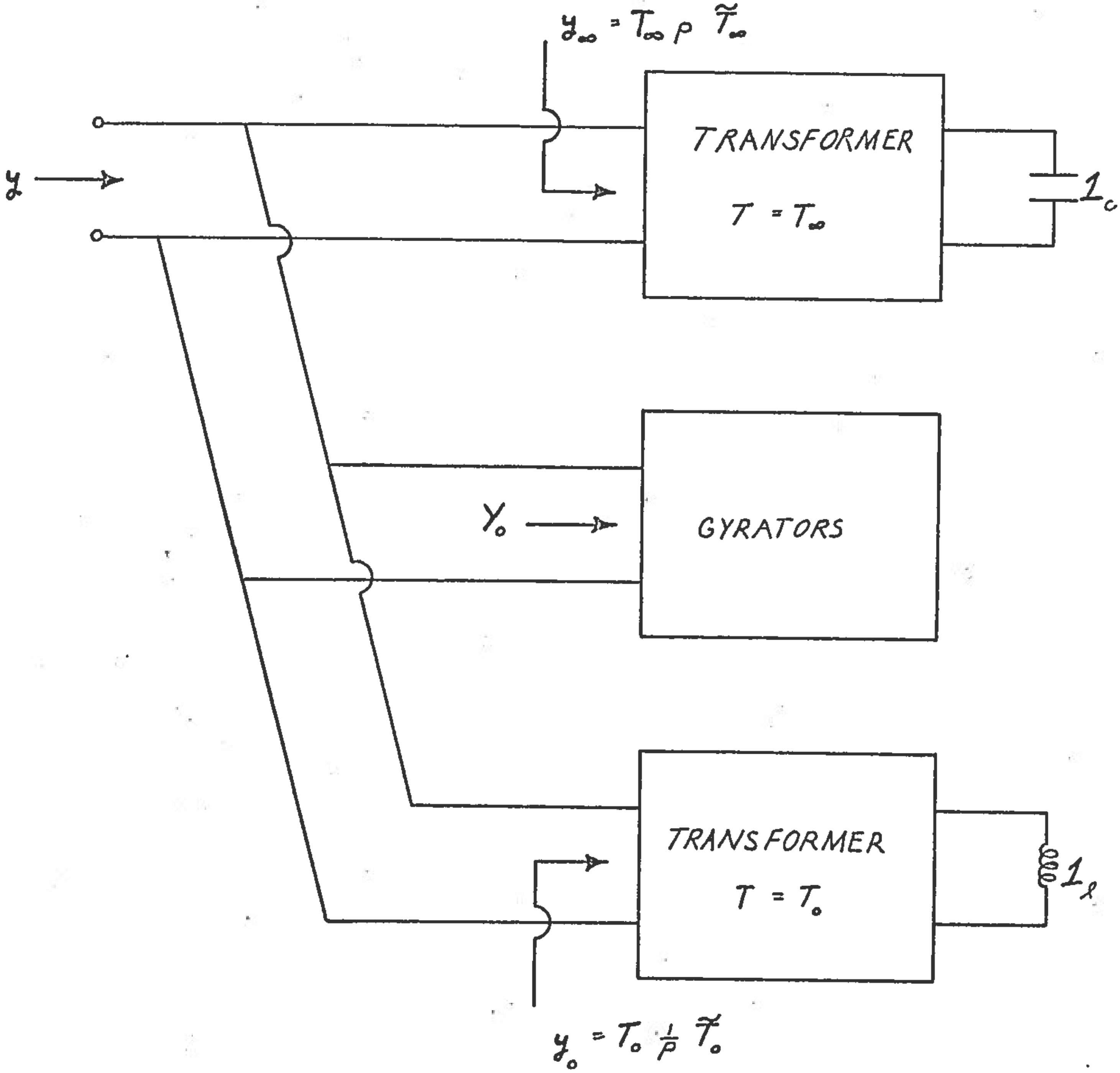

S
S_c
UNITARY
l
$S_l = 0$
a)
$S_{c_1} = \begin{bmatrix} 0 & W^a \\ W & 0 \end{bmatrix}$
UNITARY
S_u
UNITARY
S_{c_2}
UNITARY
l
$S_l = \frac{1-p}{1+p}\, l$
b)

Figure 4



$y_\infty = T_\infty \, p \, \tilde{T}_\infty$
y
TRANSFORMER
$T = T_\infty$
1_c
Y_0
GYRATORS
TRANSFORMER
$T = T_0$
1_ℓ
$y_0 = T_0 \frac{1}{p} \tilde{T}_0$

Figure 5