A THEOREM IN LUMPED-DISTRIBUTED NETWORK SYNTHESIS*

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Summary

One of the areas in which the theory of multivariable positive real functions has found applications is in the synthesis of driving-point functions of certain classes of lumped-distributed networks. Thus, a single variable transcendental function problem is converted into a multivariable rational function problem. In this paper, a necessary and sufficient condition is obtained for a polynomial of n complex variables, and a class of single variable transcendental functions generated therefrom, to be multivariable Hurwitz, and single variable Hurwitz, respectively. Several consequences of the theorem are discussed.

1. Introduction

Large-scale use of integrated circuits, along with demands for microminiaturization, have necessitated the introduction of disciplines for systematic analysis and synthesis of lumped-distributed networks composed of a finite number of commensurate and/or incommensurate transmission lines along with lumped elements. The multivariable approach has been popularized for this purpose, as the study of single variable transcendental functions in one complex variable is converted to the study of multivariable rational functions in several independent complex variables. Such study necessarily rests on the subject of analytic functions of several variables, a topic which has been reasonably well explored by mathematicians. With reference to particular classes of

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lumped-distributed networks (the distributed network elements are assumed to be uniform, commensurate or incommensurate transmission lines, with prescribed nonnegative real characteristic impedances and one-way time delays), Koga [1] gave his proof of the multivariable bounded-reality criteria. This result has recently been discussed by Youla [2] and Csurgay [3]. The proof, originally given, dwells heavily on the properties of algebraic and algebroidal functions. The complexity, along with attendant lack of clarity, makes it desirable to find a simple proof suitable for presentation of the material in a regular class, where multivariable theory is expected to play an important future role, especially because of its equally important applications in areas outside network theory. The present communication is the result of an attempt to establish the aforementioned result by a clear and constructive proof. So far, we have been unable to establish the complete validity of the result using the chosen approach. However, the main theorem to be established here appears to have, over and above the simplicity of proof, useful information that might validate (or invalidate) Koga's result.

2. Main Result

The following theorem is established here.

Theorem: The multivariable polynomial $g(\lambda_o, \lambda_1, \ldots, \lambda_n)$ has no zeros in the open polydomain $\mathrm{Re}\lambda_i > 0 (i=0,1,\ldots,n)$ if and only if the function $g\big(\alpha_o p+\beta_o, \tanh(\alpha_1 p+\beta_1), \ldots, \tanh(\alpha_n p+\beta_n)\big)$ has no zeros in the open region $\mathrm{Re}p > 0$ for all nonnegative α_i and β_i which are not simultaneously zero in like indexed pairs, $0 \le i \le n$.

Proof: a) If part. We wish to show $g(\lambda_o, \lambda_1, \ldots, \lambda_n) \neq 0$ in $\mathrm{Re}\lambda_i > 0$ if $g\big(\alpha_o p+\beta_o, \tanh(\alpha_1 p+\beta_1), \ldots, \tanh(\alpha_n p+\beta_n)\big) \neq 0$ in $\mathrm{Re}p > 0$ for arbitrary α_i, β_i satisfying the nonnegative constraints. Thus assume the contrary, that is for $g\big(\alpha_o p+\beta_o, \tanh(\alpha_1 p+\beta_1), \ldots, \tanh(\alpha_n p+\beta_n)\big) \neq 0$ in $\mathrm{Re}p > 0$ for all suitable nonnegative α_i, β_o there is a set λ_{io} with $\mathrm{Re}\lambda_{io} > 0$ such that $g(\lambda_{oo}, \lambda_{1o}, \ldots, \lambda_{no}) = 0$. In this case, set $\alpha_o = \lambda_{oo}, \alpha_i = \tanh^{-1}\lambda_{io}, i=1,\ldots,n$ and choose $p_o = \sigma_o + j\omega_o$ such that

$$\measuredangle p_o = \tan^{-1}(\omega_o/\sigma_o) \ge \tan^{-1}(\mathrm{Im}\gamma_i/\mathrm{Re}\gamma_i) = \measuredangle\gamma_i \text{ for all } i=0,1,\ldots,n$$

Here we use the periodicity and positive-real properties of tanh to insure $\mathrm{Im}\gamma_i \ge 0, \mathrm{Re}\gamma_i > 0$. This choice of p_o guarantees that we can find nonnegative α_i, β_i such that

$$\gamma_i=\alpha_i p_o+\beta_i \qquad i=0,1,\ldots,n$$

as a graphical comparison of γ_i with p_o shows. But this choice of p_o and α_i,β_i gives $g\big(\alpha_o p_o+\beta_o, \tanh(\alpha_1 p_o+\beta_1),\ldots,\tanh(\alpha_n p_o+\beta_n)\big)=0$ contrary to assumption.

For clarity and later purposes, we comment that, if γ_k is the γ_i with the largest angle, we can choose $p_o=\gamma_k$, in which case $\alpha_k=1$, $\beta_k=0$.

b) Only if part. For this we wish to show that $g(\lambda_o,\lambda_1,\ldots,\lambda_n)\neq 0$ in $Re\lambda_i>0$ implies $g\big(\alpha_o p+\beta_o, \tanh(\alpha_1 p+\beta_1),\ldots,\tanh(\alpha_n p+\beta_n)\big)\neq 0$. Thus assume $g\big(\alpha_o p_o+\beta_o, \tanh(\alpha_1 p_o+\beta_1),\ldots,\tanh(\alpha_n p_o+\beta_n)\big)=0$ for some p_o with $Rep_o>0$ and nonnegative α_i,β_i which are not simultaneously zero in like indices. Choosing $\lambda_o=\alpha_o p_o+\beta_o$, $\lambda_i=\tanh(\alpha_i p_o+\beta_i)$ shows, however, the impossibility since $g(\lambda_o,\lambda_1,\ldots,\lambda_n)$ would then be zero with $Re\lambda_i>0$ by the positive reality of tanh.

Q.E.D.

The Theorem covers $Re\lambda_i>0$, but it is clear that the same result holds for $Re\lambda_i\geq 0$ by considering $Rep\geq 0$ for all nonnegative α_i,β_i which may now be simultaneously zero.

3. Some Special Results

In this section, some special results that follow from the theorem, stated and proved above, are given. Rational functions and polynomials of two independent complex variables play very significant roles in network theory [4] as well as in other areas, such as bidimensional recursive filtering [5]. The following assertion can be justified.

Assertion The two variable polynomial, $g(p_1,p_2)$, has no zeros in the open polydomain, $Rep_1>0$, $Rep_2>0$, if and only if both of the following conditions are satisfied;

(i) $g\big(p, \tanh(\alpha_1 p+\beta_1)\big)$ has no zeros in $Rep>0$ for all $\alpha_1\geq 0, \beta_1\geq 0$ (not both simultaneously zero)

and

(ii) $g(\alpha_o p+\beta_o, \tanh p)$ has no zero in $Rep>0$, for all $\alpha_o\geq 0$, $\beta_o\geq 0$ (not both simultaneously zero)

Outline of proof: Observing the comment to part a) of the proof of the Theorem, we see that condition (i) holds for those p_1,p_2 with $\measuredangle p_1\geq\measuredangle p_2$

while condition (ii) holds for $\measuredangle p_1 \leq \measuredangle p_2$. In order to test for all p_1 and p_2 in $\text{Re}p_1 > 0$, $\text{Re}p_2 > 0$, we need to simultaneously consider both cases.

Q.E.D.

The result stated in the above assertion enables one to convert the test for location of zeros of a transcendental function of one complex variable and four real variables to a test for location of zeros of two transcendental functions of one complex variable and two real variables, each. Similar results as stated in the assertion can be written down for n-variable polynomials, when $n > 2$.

As shown in [6], the results of [7] are in error. The results of [7] in turn appear to rest upon the validity of the following statement. Consequently, it appears important to obtain a clear and concise verification, or a counter-example, for the statement, perhaps by using ideas of the main Theorem.

Statement (True or False?)

The (n+1) variable polynomial $g(p, \lambda_1, \lambda_2, \ldots, \lambda_n)$ has no zeros in the region $\text{Re}p > 0$, $\text{Re}\lambda_i > 0, 1 \leq i \leq n$, if and only if the function $g(p, \tanh \alpha_1 p, \tanh \alpha_2 p, \ldots, \tanh \alpha_n p)$ has no zeros in the region $\text{Re}p > 0, \alpha_i > 0, 1 \leq i \leq n$.

4. Discussion

A theorem containing a necessary and sufficient condition is obtained for a polynomial of (n+1) complex variables to be multivariable Hurwitz. An open question is then posed. The justification for posing such a question arises from the fact that the number of degrees of freedom in the multivariable polynomial, $g(p, \lambda_1, \lambda_2, \ldots, \lambda_n)$, for $n \geq 1$, is 2n and thus larger than the number of degrees of freedom, n+2, in the generated function, $g(p, \tanh \alpha_1 p, \tanh \alpha_2 p, \ldots, \tanh \alpha_n p)$. By way of comparison, when physically representing transmission lines the form $g(p, z_{o1} \tanh \alpha_1 p, \ldots, z_{on} \tanh \alpha_n p)$ results which has 2n degrees of freedom when the real characteristic impedances z_{oi} are taken into account. The validity of the statement can be interpreted in terms of a normalization of characteristic impedances to unity. The freedom to perform such a multiple normalization is not as yet completely clear. A favorable answer to the question on the statement will lead to a constructive, simple proof of the multivariable bounded reality criteria.

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