On Network Realizability Conditions*

Recently Hazony and Nain[1] stated a set of realizability constraints on the Z matrix of a passive n-port. As pointed out by Slepian[2] these conditions are not sufficient for realization. Since the revised conditions stated by Hazony and Nain[3] are also not sufficient, it seems that further clarification is necessary. This is especially true since other somewhat different, but admittedly incomplete, conditions also have been given.[4]

Here we give the necessary and sufficient conditions for an impedance matrix Z to correspond to a finite, passive n-port. Because of space limitations we refer elsewhere for the proof and don't even attempt a definition of terms such as linear, finite, passive and n-port.

Consider a linear, time-invariant, passive n-port N described by an $n \times nZ$ matrix. Then by exciting with increasing exponentials we can see that Z is necessarily positive real,[5] *i.e.*,

1) $Z(p)$ is analytic in Re $p>0$
2) $Z^*(p)=Z(p^*)$ in Re $p>0$
3) $Z_H(p)$ is positive semi-definite in Re $p>0$.

Here $p=\sigma+j\omega$, a superscript asterisk denotes complex conjugation, and Z_H is the Hermitian part of Z.

If N is finite then Z must be rational and condition 2) implies real coefficients; we then call Z real-rational. In the real-rational case, that considered by Hazony and Nain,[3] 3) implies 1). In a form closer to that given given by Hazony and Nain[1] the positive real conditions in the real-rational case can be stated as follows:

Theorem: The necessary and sufficient conditions that an $n \times n$ matrix $Z(p)$ be the impedance matrix of a finite, passive n-port N are

1) Z is real-rational and
2) Z is analytic in Re $p>0$ and
3) Poles of Z on Re $p=0$ are simple (including infinity) and
4) The residue matrix of Z for each pole on Re $p=0$ (including infinity) is Hermitian with every principal minor non-negative and
5) All principal minors of Z_H are non-negative for each p on Re $p=0$ for which they are defined.

Conditions 4) and 5) are equivalent to the respective statements that the residue matrices (on Re $p=0$) and $Z_H(j\omega)$ are positive semi-definite. Each of these conditions can be given a physical interpretation. Thus 1) states that N can be built with a finite number of real valued elements (including gyrators and transformers). 2) and 3) state that N is stable but perhaps not asymptotically stable. 4) indicates that poles on $p=j\omega$ are due to lossless subnetworks. Finally 5) shows that the average power input in the sinusoidal steady state is non-negative (recall that the steady state can't be defined for open circuit natural frequencies). Several synthesis methods prove the sufficiency; such are those of Oono and Yasuura,[6] Belevitch[7] and Newcomb.[8] The necessity proof relates the conditions of the theorem to the positive-real definition.[5] This follows the standard pr test[9] and is available in notes for lectures given at Stanford.

To some extent we can compare this with the statements in the previous notes. We can write

$$Z = R_{SY} + R_{SS} + jX_{SY} + jX_{SS}$$

where the R's and X's are real matrices and the subscripts SY and SS stand for symmetric and skew-symmetric, respectively. In Hazony and Nain[3] the condition det (Re Z) = det $(R_{SY}+R_{SS}) \geq 0$ for $p=j\omega$ is stated. Note that this doesn't agree with condition 5) of the above theorem which requires det $(R_{SY}+jX_{SS}) \geq 0$. Of course the requirement of det (Re $Z(j\omega)) \geq 0$ is a wrong condition, as seen by the example

$$Z(p) = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

which has det $Z(p)=3>0$ but can't be realized by a passive N. In Hazony[4] the condition det $R_{SY} \geq 0$ for $p=j\omega$ is given. This is seen to be a necessary condition, but much more is required, since det $Z_H(j\omega) \geq 0$ must hold. The necessity of det $R_{SY} \geq 0$, $p=j\omega$, can be seen by connecting transformers to N in the manner attributed to Brune.[10] However, this interpretation fails when looking at $Z_H(j\omega)$ where the nonphysical complex transformer would have to be used.

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[1] D. Hazony and H. J. Nain, "A synthesis procedure for an n-port network," Proc. IRE, vol. 49, pp. 1431–1432; September, 1961.

[2] P. Slepian, "Comments on a synthesis procedure for an n-port network," Proc. IRE (*Correspondence*), vol. 50, p. 81; January, 1962.

[3] D. Hazony and H. J. Nain, "Author's comments," Proc. IRE (*Correspondence*), vol. 50, p. 81; January, 1962.

[4] D. Hazony, "Two extensions of the Darlington synthesis procedure," IRE Trans. on Circuit Theory, vol. CT-9, pp. 284–288; September, 1961, See p. 287.

[5] D. C. Youla, L. J. Castriota, and H. J. Carlin, "Bounded real scattering matrices and the foundations of linear passive network theory," IRE Trans. on Circuit Theory, vol. CT-6, pp. 102–124; March, 1959. See p. 122 (Def. 21).

[6] Y. Oono and K. Yasuura, "Synthesis of finite passive $2n$-terminal networks with prescribed scattering matrices," *Memoirs of the Faculty of Engineering*, Kyushu University, vol. 14, pp. 125–177; May 1954. See pp. 153–158 and 163–167.

[7] V. Belevitch, "On the Brune process for n-ports," IRE Trans. on Circuit Theory, vol. CT-7, pp. 280–296; September, 1960.

[8] R. Newcomb, "Synthesis of Non-Reciprocal and Reciprocal Finite Passive $2N$-Poles," Ph.D. Thesis, University of California, Berkeley; 1960.

[9] D. F. Tuttle, Jr., "Network Synthesis," vol. 1, John Wiley and Sons, Inc., New York, N. Y., p. 182; 1958.

[10] E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, Inc., New York, N. Y., pp. 7-9; 1957.

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