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Tellegen's theorem and multivariable realizability theory †

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By introducing a multivariable form of Tellegen's Theorem and a generalization of energy functions, the positive-real nature of immittance matrices of finite networks composed of resistors, p_i-plane inductors and capacitors, all non-negative, transformers and gyrators, is proven, as well as the lossless constraint. Multivariable reactance functions are investigated and the relation between such functions and multivariable Hurwitz polynomials is proved.

Notations

$p=\sigma+j\omega=p_1$	the complex frequency variable,
$p_i=\sigma_i+j\omega_i$	complex variables, $1\leq i\leq n$,
$\mathbf{p}=\boldsymbol{\sigma}+j\boldsymbol{\omega}$	the n vector of variables $=(p_1, p_2, \ldots, p_n)$ of real and imaginary parts $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$,
$\text{Re}\,\mathbf{p}=\boldsymbol{\sigma}>0$	real part of $p_i>0$, for all i, $1\leqslant i\leqslant n$,
$\mathbf{p}^{(0)}$	a fixed point, generally in the open polydomain $\boldsymbol{\sigma}>0$,
$Z(\mathbf{p})$	open circuit impedance matrix of the n variables p_i,
$\tilde{Z}(\mathbf{p})$	transpose of $Z(\mathbf{p})$,
$Z^*(\mathbf{p})$	complex conjugate of $Z(\mathbf{p})$,
$Z_*(\mathbf{p})$	Hurwitz conjugation of $Z(p)$, i.e. $Z(-\mathbf{p})$ obtained by change of signs of all independent variables in $Z(\mathbf{p})$,
$Z+\tilde{Z}^*\geqslant 0$	the left-hand side is non-negative Hermitian,
$z(\mathbf{p})$	a multivariable positive real function,
$z_r(\mathbf{p})$	a multivariable reactance function,
I_m	identity matrix of order m,
$\|I_1(\mathbf{p})\|$	magnitude of multivariable function,
$\|\phi(\mathbf{p})\|\lesseqgtr 1$, for	$\|\phi(\mathbf{p})\|<1$, for $\text{Re}\,\mathbf{p}>0$, $\|\phi(\mathbf{p})\|=1$, for $\text{Re}\,\mathbf{p}$ $\text{Re}\,\mathbf{p}=0$ and $\text{Re}\,\mathbf{p}\gtrless 0$, $\|\phi(\mathbf{p})\|>1$, for $\text{Re}\,\mathbf{p}<0$.

1. Introduction

A restricted class of multivariable real rational functions, the positive-real ones, was introduced by Ozaki and Kasami (1960). Considerable progress has been made in the use of these functions. However, as several independent

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complex variables are involved, the explanations for the basic nature of these functions and the reasons for characterizing classes of networks by the accepted definitions for these functions, are either often incomplete or confusing. This is especially so as Brune's (1931, p. 196) original energy arguments cannot be justified on physical grounds as in the single variable case, and this often tends to obscure the implications of the multivariable positive-real property. Nevertheless, the 'energy functions' approach (Guillemin 1957, p. 5, Bello 1960) can be naturally extended, as we do here, and then the multivariable positive-real concept can be formulated by generalizing Tellegen's Theorem (Penfield *et al.* 1970), as we do in § 2. This is shown to provide not only insight into the multivariable positive-real property but also gives meaning and proper justification to the basic definitions for multivariable positive real matrices and functions.

It is well known that non-negative resistors, inductors and capacitors, whose impedances are functions of the complex frequency variable $p=p_1$, are necessary and sufficient to synthesize an arbitrary rational immittance driving-point function characterized by the single variable positive real property (Bott and Duffin 1949) while the addition of transformers and gyrators gives the same result on positive-real matrices (Newcomb 1966). The question next arises as to what the restrictions would be on the immittance matrices and driving-point functions of a network that is comprised of non-negative resistors, inductors and capacitors, elements with impedances of types $p_i l_i$ and $1/p_i c_i$, for $i=2, 3, \ldots, n$, where the p_i's are in general, independent complex variables (independent of each other as well as of p_1) and the l_i's and c_i's are real, non-negative constants. Any form of electric or magnetic passive reciprocal coupling between elements could be considered present, whenever appropriate, as well as gyrators.

In § 3 the multivariable positive-real matrix concept is formulated for an m-port comprised of elements of the types just mentioned, after introducing a multivariable form of Tellegen's Theorem in § 2. In § 4, the lossless constraint for multivariable m-ports is also proved using Tellegen's Theorem. Multivariable reactance functions are investigated and the multivariable counterpart of Foster's reactance theorem is proved using the formulation arrived at earlier. In § 5 the relation between multivariable reactance functions and strictly Hurwitz polynomials is proved using the maximum modulus theorem for analytic functions of several variables.

2. Tellegen's theorem

Consider an m-port network which, when closed on sources, contains a finite number of branches, the branch variables being assumed to be functions of the n complex variables $\mathbf{p}=(p_1, p_2, \ldots, p_n)$. Then, tie-set equations can be written as (Newcomb 1968, pp. 28–29)

$$I(\mathbf{p})=\tilde{T}J(\mathbf{p}), \quad TV(\mathbf{p})=0 \tag{1}$$

where $\tilde{T}$ is the transpose of T the tie-set matrix (with constant elements), $J(\mathbf{p})$ is the matrix of loop currents and $I(\mathbf{p})$, $V(\mathbf{p})$ are, respectively, the column matrices of branch currents and voltages. If $\mathscr{C}[I(\mathbf{p})]$ is a linear operator, operating on the components of $I(\mathbf{p})$, then operating on $\tilde{I}(\mathbf{p})$, multiplying by

$V(\mathbf{p})$ and using (1) gives *Tellegen's Theorem* (Penfield *et al.* 1970, p. 14) generalized to the multivariable case

$$\mathscr{O}[\tilde{I}(\mathbf{p})]\,V(\mathbf{p})=0. \tag{2}$$

Partitioning $I(\mathbf{p})$ and $V(\mathbf{p})$ into source and internal loop components denoted by subscripts s and l, one can rewrite (2), using proper polarity conventions, as

$$\mathscr{O}[\tilde{I}_s(\mathbf{p})]\,V_s(\mathbf{p})=\mathscr{O}[\tilde{I}_l(\mathbf{p})]\,V_l(\mathbf{p}) \tag{3}$$

Next consider an m-port of this type whose internal branches have impedances giving a loop impedance matrix

$$Z_l(\mathbf{p})=R+\sum_{i=1}^{n} p_iL_i+\frac{1}{p_i}S_i \tag{4}$$

where L_i and C_i are real, constant, symmetric non-negative definite matrices, and the symmetric part of the real constant matrix R is also non-negative definite. That is, the m-port is constructed from a finite number of resistors, p_i-plane inductors and capacitors, all non-negative, and transformers and gyrators. Such a network we will call *positive*.

If $Z(\mathbf{p})$ denotes the open-circuit impedance matrix of the m-port, then (3) gives

$$\mathscr{O}[\tilde{I}_s(\mathbf{p})]Z(\mathbf{p})I_s(\mathbf{p})=\mathscr{O}[\tilde{I}_l(\mathbf{p})]Z_l(\mathbf{p})I_l(\mathbf{p}) \tag{5}$$

where $\tilde{I}_s(\mathbf{p})$ can be arbitrarily chosen.

3. Positive reality

We first extend the definition of positive-reality (Newcomb 1966, p. 96) in the obvious way, where $\operatorname{Re}\mathbf{p}>0$ means $\sigma_i>0$ for all i, and the upper asterisk denotes complex conjugation.

Definition 1

An $m\times m$ matrix $Z(\mathbf{p})$ of the multi-variable $\mathbf{p}$ is called *positive-real* if,

(*a*) $Z(\mathbf{p})$ is holomorphic in $\operatorname{Re}\mathbf{p}>0$,

(*b*) $\tilde{Z}^*(\mathbf{p})=Z(\mathbf{p}^*)$ in $\operatorname{Re}\mathbf{p}>0$,

(*c*) $Z(\mathbf{p})+\tilde{Z}^*(\mathbf{p})$ is non-negative definite in $\operatorname{Re}\mathbf{p}>0$.

Note that in the 1-port case, $Z(\mathbf{p})=z(\mathbf{p})$, condition (*c*) is the standard condition $\operatorname{Re}z(\mathbf{p})\geqslant 0$ in $\operatorname{Re}\mathbf{p}>0$.

We will show that positive networks, those discussed in § 2, have positive-real impedance matrices. First we recall the *Theorem of Osgood* (Gunning and Rossi 1965, p. 2):

> A function, $f(\mathbf{p})$, of n complex variables, continuous in an open polydomain $\operatorname{Re}\mathbf{p}>0$, is holomorphic in this polydomain if and only if it is holomorphic in each variable separately.

It now follows that for the class of networks under consideration $Z(\mathbf{p})$ is holomorphic in $\operatorname{Re}\mathbf{p}>0$, that is, condition (*a*) of the positive-real definition is necessary. This is a consequence of taking $\mathbf{p}$ to have all components fixed but one, in which case $Z(\mathbf{p})$ becomes a positive (but not necessarily real) matrix in that one variable, by virtue of (4) (Belevitch 1968, p. 71), for which condition (*a*) holds for that variable. If $I_s(\mathbf{p})$ is chosen holomorphic in $\text{Re}\,\mathbf{p}>0$ then one can also assume $I_l(\mathbf{p})$ holomorphic in $\text{Re}\,\mathbf{p}>0$, as will be needed in the next paragraph.

To prove condition (*b*) we choose $\mathscr{O}[\cdot]$ in (5) to be the identity operator, then it is seen that for $\mathbf{p}$ real, $Z(\mathbf{p})$ is real in $\text{Re}\,\mathbf{p}>0$ (by using (4) and choosing $I_s(\mathbf{p})$ real, which then generates real $I_l(\mathbf{p})$). Finally, to prove condition (*c*) $\mathscr{O}[\cdot]$ is chosen to give the complex conjugate, i.e. $\mathscr{O}[I(\mathbf{p})]=I^*(\mathbf{p})$, then

$$\tilde{I}_s^*(\mathbf{p})[\tilde{Z}^*(\mathbf{p})+Z(\mathbf{p})]I_s(\mathbf{p})=\tilde{I}_l^*(\mathbf{p})[\tilde{Z}_l^*(\mathbf{p})+Z_l(\mathbf{p})]I_l(\mathbf{p}) \qquad (6)$$

from which, it follows, using (4) and the foregoing holomorphicity condition on $I_l(\mathbf{p})$, $I_s(\mathbf{p})$ and $Z(\mathbf{p})$ that $\tilde{Z}^*+Z\geqslant 0$ in $\text{Re}\,\mathbf{p}=\boldsymbol{\sigma}>0$. Thus, using the generalized version of Tellegen's Theorem we have proven :

Theorem 1

The impedance matrix of a positive network is necessarily positive-real (and rational).

The converse occurs in Koga (1968).

We comment that the class of networks under discussion does give rational $Z(\mathbf{p})$. However, the technique used also applies to the same class of networks but with possibly an infinite number of branches, in which case condition (*b*) cannot be relaxed to read for any $\mathbf{p}$, since $I_l(\mathbf{p})$ at (6) need no longer be rational in $\mathbf{p}$, and condition (*a*) must be postulated since it does not follow from condition (*c*). In contrast, as with the single variable situation, when $Z(\mathbf{p})$ is rational, condition (*c*) can be shown to yield holomorphicity, condition (*a*). Although this analyticity is often accepted (Koga 1968) its proof depends on some nice properties of functions of several variables. Towards the proof we first note that clearly, for any m vector $I(\mathbf{p})$ at a point of its analyticity

$$\tilde{I}^*[\tilde{Z}^*(\mathbf{p})+Z(\mathbf{p})]I(\mathbf{p})=2\,\text{Re}\,[\tilde{I}^*(\mathbf{p})Z(\mathbf{p})I(\mathbf{p})]. \qquad (7)$$

Then assume the presence of a singularity of $f(\mathbf{p})=\tilde{I}^*(\mathbf{p})Z(\mathbf{p})I(\mathbf{p})$ at $\mathbf{p}=\mathbf{p}^{(0)}$ in $\text{Re}\,\mathbf{p}>0$, where, since $Z(\mathbf{p})$ is taken rational, $\mathbf{p}^{(0)}$ will be non-essential, though of the first or second kind (Kaplan 1963, p. 67). First assume that the singularity is of the first kind and expand the single variable function

$$g(p_1)=\tilde{I}^*(\mathbf{p})Z(\mathbf{p})I(\mathbf{p})\big|_{p_2=p_2^{(0)},\,\ldots,\,p_n=p_n^{(0)}}$$

about the point $p_1=p_1^{(0)}$ in a Laurent series. Then, of course, $\text{Re}\,g(p_1)$ changes sign in a suitable neighbourhood of $p_1=p_1^{(0)}$. If $\mathbf{p}^{(0)}$ is a non-essential singularity of the second kind then it is known that the locus of the non-essential singularities of $f(\mathbf{p})$ is of dimension $(2n-4)$ in a space of dimension $2n$; therefore these cannot disconnect the space (Kalplan 1963, pp. 65–67). In fact, if $f(\mathbf{p})$ were a function of two complex variables, i.e. if $n=2$, then the non-essential singularities of the second kind would form a set of isolated points. As the domain of the rational multivariable meromorphic function $f(\mathbf{p})$ including singularities of the first kind and excluding those of the second kind is connected (Kaplan 1963, p. 66) it follows that in an arbitrarily close neighbourhood of $\mathbf{p}^{(0)}$, there must exist a connected space containing singularities of the first kind. Any such singularity at $\mathbf{p}=\mathbf{p}^{(1)}$ with $\operatorname{Re}\mathbf{p}^{(1)}>0$ in the neighbourhood of $\mathbf{p}=\mathbf{p}^{(0)}$ is considered, and by the argument advanced previously from a Laurent series expansion of a single variable, it results that $\operatorname{Re}\hat{g}(p_1)$ changes sign in a suitable neighbourhood of $p_1=p_1^{(1)}$ where again

$$\hat{g}(p_1)=\tilde{I}^*(\mathbf{p})Z(\mathbf{p})I(\mathbf{p})\big|_{p_2=p_2^{(1)},\ \ldots,\ p_n=p_n^{(1)}}$$

is a function of a single complex variable.

The definition of positive-reality of course holds for 1-ports. Thus, if in (6) all current sources except the one at the ith port are deactivated then (6) reduces to

$$z_{ii}(\mathbf{p})=\frac{1}{|I_{si}|^2}\tilde{I}_1^*(\mathbf{p})[\tilde{Z}_1^*(\mathbf{p})+Z_1(\mathbf{p})]I_1(\mathbf{p}) \tag{8}$$

where $z_{ii}(\mathbf{p})$ is the driving-point impedance at the ith port. Equation (8) is then a useful representation of a driving-point function from which the positive-real nature can be deduced.

4. Reactance matrices and reactance functions

We now consider that in (4) the matrix R is skew-symmetric, implying that resistors are absent in the m-port. In this case a positive network is called *lossless*. Now let $\mathcal{O}[\,\cdot\,]$ be such that it yields the Hurwitz conjugate (replacement of $\mathbf{p}$ by $-\mathbf{p}$), i.e. $\mathcal{O}[I(\mathbf{p})]=I_*(\mathbf{p})=I(-\mathbf{p})$. Then (5) yields

$$\tilde{I}_{s*}(\mathbf{p})[\tilde{Z}_*(\mathbf{p})+Z(\mathbf{p})]I_s(\mathbf{p})=0 \tag{9}$$

for all $\mathbf{p}$ since the right-hand side vanishes using (4). Since (9) holds for all $I_s(\mathbf{p})$ it follows that besides being positive-real, the impedance matrix of a lossless m-port satisfies

$$Z(\mathbf{p})=-\tilde{Z}(-\mathbf{p}) \tag{10}$$

A positive-real matrix satisfying (10) is called a *reactance* matrix. Of course, by duality the admittance matrix satisfies identical constraints while the transformation

$$S(\mathbf{p})=(Z+I_m)^{-1}(Z-I_m) \tag{11}$$

yields the bounded-real constraint (Newcomb 1966, p. 94, Koga 1968, p. 4) on the scattering matrix with $S(\mathbf{p})=[\tilde{S}_*(\mathbf{p})]^{-1}$ in the lossless case (Newcomb 1966, p. 101, Koga 1968, p. 5).

As multivariable reactance functions form a very important subclass of multivariable positive-real functions, their properties will be derived separately. In fact for some purposes practically useful networks consisting of lumped reactances and lossless commensurate transmission lines can be characterized by two variable reactance functions (Ansell 1964). Again, if in (4) the symmetric part of R is zero, and if all current sources except the one at the ith port is deactivated, (5) reduces to

$$z_{iir}(\mathbf{p}) = \frac{1}{|I_{gi}|^2}\left[\sum_{i=1}^{n}\left(p_i T_i(\mathbf{p}) + \frac{V_i(\mathbf{p})}{p_i}\right) + j V_g(\mathbf{p})\right] \tag{12 a}$$

where

$$T_i(\mathbf{p}) = \tilde{I}_1^*(\mathbf{p}) L_i \tilde{I}_1(\mathbf{p}), \quad V_i(\mathbf{p}) = \tilde{I}_1^*(\mathbf{p}) S_i \tilde{I}_1(\mathbf{p}) \tag{12 b}$$

are non-negative semidefinite Hermitian forms for $i = 1, 2, \ldots, n$, and $j V_g(\mathbf{p})$ is a skew Hermitian form resulting from the non-zero skew symmetric part of R. Now the left side of (12 a) is independent of $I_g(\mathbf{p})$ and consequently so is the right for which we are free to choose any $I_g(\mathbf{p})$. Thus we choose $I_g(\mathbf{p})$ real so that $j V_g(\mathbf{p}) = 0$, as this will not alter the properties of $z_{iir}(\mathbf{p})$, the function under study. Similar results as those to be obtained below can also be reached by maintaining $j V_g(\mathbf{p})$ non-zero as has been done in the single variable case by Su (1965, p. 80). However, choosing $\tilde{I}_g(\mathbf{p})$ such that $j V_g(\mathbf{p}) = 0$, and $|I_{gi}(\mathbf{p})| = 1$, yields the more compact and manageable form

$$z_{iir}(\mathbf{p}) = z_r(\mathbf{p}) = \sum_{i=1}^{n}\left(p_i T_i(\mathbf{p}) + \frac{V_i(\mathbf{p})}{p_i}\right). \tag{13}$$

From simple arguments using (13), (10) and the positive-real nature of a typical reactance function we conclude the necessity of the following multi-variable extension of the single variable results (Balabanian 1958, p. 66).

Theorem 2

A non-zero multivariable rational function $z_r(\mathbf{p})$ is a reactance function if and only if

(1) $z_r(\mathbf{p})$ is real for $\mathbf{p}$ real,

(2) $\mathrm{Re}\,[z_r(\mathbf{p})] \gtreqless 0$ for $\mathrm{Re}\,\mathbf{p} \gtreqless 0$.

Here the symbolism means, for example, that $\mathrm{Re}\, z_r < 0$ in $\mathrm{Re}\,\mathbf{p} < 0$; it should be noted that the behaviour for all $\mathbf{p}$ is not covered, for example, when $\mathrm{Re}\, p_1 > 0$ and $\mathrm{Re}\, p_2 < 0$. We see that these two conditions completely characterize a rational $z_r(\mathbf{p})$ as being a reactance function since such a z_r is clearly positive-real and (10) follows from its validity for $\mathbf{p} = j\boldsymbol{\omega}$ since $z_r(j\boldsymbol{\omega})$ is a rational form in $\boldsymbol{\omega} = \mathbf{p}/j$.

Some further useful results will be arrived at by using the representation of a reactance function as given in (9). Consider any point $\mathbf{p} = \mathbf{p}^{(1)}$, having $n-1$ components fixed such that $\mathrm{Re}\,(p_i) = \mathrm{Re}\,(p_i^{(1)}) = 0$, for $i = 2, 3, \ldots, n$. Then

$$z_r(\mathbf{p}^{(1)}) = \sum_{i=2}^{n}\left(j\omega_i^{(1)} T_i + \frac{V_i}{j\omega_i^{(1)}}\right) + (\sigma_1 + j\omega_1) T_1 + \frac{V_1}{(\sigma_1 + j\omega_1)} \tag{14}$$

shows that if T_1 and V_1 are not identically zero $z_r(\mathbf{p}^{(1)}) = 0$ cannot be satisfied for $\sigma_1 \neq 0$. Furthermore, under these conditions $z_r(\mathbf{p}^{(1)}) = 0$ will be satisfied if $\sigma_1 = 0$ by some choice of ω_1 since the right of (14) represents a structure of complex resistors and single variable, $j\omega_1$, inductors and capacitors (Belevitch 1968, p. 185). Therefore, zeros (and by the dual representation, singularities) of a reactance function of n variables will always show up in Re $\mathbf{p}=0$ and none will be in either Re $\mathbf{p}<0$ or Re $\mathbf{p}>0$.

Next an extension of the well-known Foster's Theorem will be arrived at using (13). The results to be arrived at have been given by Ozaki and Kasami 1960, p. 255) but the approach is different. Our technique makes use of the Cauchy–Riemann equations for analytic functions of several variables (Gunning and Rossi 1965, p. 4). From (13),

$$z_r(\mathbf{p})=r(\boldsymbol{\sigma}, \boldsymbol{\omega})+jx(\boldsymbol{\sigma}, \boldsymbol{\omega}) \tag{15}$$

where

$$r(\boldsymbol{\sigma}, \boldsymbol{\omega})=\sum_{i=1}^{n}\left(\sigma_i T_i+\frac{\omega_i V_i}{\sigma_i^2+\omega_i^2}\right) \tag{16 a}$$

$$x(\boldsymbol{\sigma}, \boldsymbol{\omega})=\sum_{i=1}^{n}\left(\omega_i T_i-\frac{\omega_i V_i}{\sigma_i^2+\omega_i^2}\right) \tag{16 b}$$

From (16 a)

$$\left.\frac{\partial r}{\partial \sigma_i}\right|_{\boldsymbol{\sigma}=0}=T_i+\frac{V_i}{\omega_i^2}\geqslant 0, \quad i=1, 2, \ldots, n. \tag{17}$$

Applying the Cauchy–Riemann equations in several variables, it follows that

$$\left.\frac{\partial x}{\partial \omega_i}\right|_{\boldsymbol{\sigma}=0}\geqslant 0, \quad i=1, 2, \ldots, n \tag{18}$$

except at points where $z_r(\mathbf{p})$ has singularities in Re $\mathbf{p}=0$ (as the Cauchy–Riemann equations do not hold here). Equation (18), which is the multivariable counterpart of Foster's Reactance Theorem, is included in the next theorem.

Theorem 3

A rational multivariable reactance function $z_r(\mathbf{p})=r(\boldsymbol{\sigma}, \boldsymbol{\omega})+jx(\boldsymbol{\sigma}, \boldsymbol{\omega})$ has the following properties :

(a) $z_r(\mathbf{p})$ is holomorphic in Re $\mathbf{p}>0$ and Re $\mathbf{p}<0$,

(b) $\left.\dfrac{\partial x}{\partial \omega_i}\right|_{\boldsymbol{\sigma}=0}\geqslant 0, \quad i=1, 2, \ldots, n.$

5. Multivariable reactance functions and Hurwitz polynomials

In this section the characterization for multivariable reactance functions given in (13) is used to prove a useful relation between multivariable reactance functions and strictly Hurwitz polynomials. Such a relation was proved in a lengthy manner for the two-variable case by Ansell (1962, pp. 11–20).

First we define a *real* polynomial as one with real coefficients and then turn to multivariable Hurwitz polynomials (Saito 1966, p. 354).

Definition 2

A real polynomial $f(\mathbf{p})$ of n independent complex variables is called a *strictly Hurwitz* polynomial if it has no zeros in the closed polydomain Re $\mathbf{p}\geqslant 0$.

Then the following generalization from the single variable case is valid.

Theorem 4

A real non-constant polynomial $f(\mathbf{p})=f_e(\mathbf{p})+f_o(\mathbf{p})$, with f_e and f_o relatively prime even and odd parts, is strictly Hurwitz if and only if $f_o(\mathbf{p})/f_e(\mathbf{p})$ is a non-zero n-variable reactance function.

Proof

Whether $2f_e=f+f_*$ and $2f_o=f-f_*$ are relatively prime or not can be tested by using standard procedures (Bose 1971, Walker 1972, pp. 25–27).

If Part : Let f_o/f_e be a non-zero n-variable reactance function. Then $g(\mathbf{p})=f_o/f_e+1=(f_o+f_e)/f_e$ is an n-variable positive-real function, as is then $1/g(\mathbf{p})$ which must have no zeros of f_o+f_e in Re $\mathbf{p}>0$ by condition (*a*) of positive-reality. By Theorem 2, $f_o/f_e\neq -1$ on Re $\mathbf{p}=0$, therefore f_o+f_e has no zeros in Re $\mathbf{p}=0$. Whence f_o+f_e is strictly Hurwitz.

Only If Part : Suppose $f(\mathbf{p})$ is a strictly Hurwitz non-constant polynomial. Then $f_o\neq 0$ as otherwise $f=f_e$ has zeros on Re $\mathbf{p}=0$ by the arguments at (14). Form

$$\phi(\mathbf{p})=\frac{f_*}{f}=\frac{f_e-f_o}{f_e+f_o}=\frac{1-\dfrac{f_o}{f_e}(\mathbf{p})}{1+\dfrac{f_o}{f_e}(\mathbf{p})}. \tag{19}$$

As f_o/f_e is an odd rational function, Re $(f_o/f_e)(j\boldsymbol{\omega})=0$. Therefore

$$|\phi(\mathbf{p})|=1 \quad \text{for} \quad \text{Re } \mathbf{p}=0. \tag{20}$$

As $\phi(\mathbf{p})$ is holomorphic in Re $\mathbf{p}\geqslant 0$ (note that $\phi(\mathbf{p})$ is continuous on the boundary Re $\mathbf{p}=0$), applying the maximum modulus theorem for analytic functions of several variables (Bochner and Martino 1948, pp. 107–109), it follows that in any arbitrarily large domain where $\phi(\mathbf{p})$ is analytic with the proviso that the domain is bounded, $|\phi(\mathbf{p})|<1$, as $f_o\neq 0$. In other words,

$$|\phi(\mathbf{p})|<1, \quad \text{for bounded non-zero } \mathbf{p} \text{ in Re } \mathbf{p}>0. \tag{21}$$

It is noted that the result is generally not valid if the domain is unbounded, unless $|\phi(\mathbf{p})|$ approaches zero as $|p_1|^2+|p_2|^2+\ldots+|p_n|^2$ approaches infinity (Bochner and Martino 1948, p. 108). Also, as $1/\phi(\mathbf{p})$ is analytic in Re $\mathbf{p}\leqslant 0$, it follows similarly that $|1/\phi(\mathbf{p})|<1$ in the any bounded portion of the domain Re $\mathbf{p}<0$. Therefore

$$|\phi(\mathbf{p})|>1, \quad \text{for bounded non-zero } \mathbf{p} \text{ in Re } \mathbf{p}<0. \tag{22}$$

Now $\phi(\mathbf{p})$ is continuous in $\mathbf{p}$ for sufficiently large p_i, hence, by (20), (21), (22),

$$|\phi(\mathbf{p})|\gtreqless 1 \quad \text{for} \quad \text{Re } \mathbf{p}\lesseqgtr 0. \tag{23}$$

Using (19) and (23) it follows that

$$\text{Re}\left(\frac{f_o}{f_e}(\mathbf{p})\right)\gtreqless 0 \quad \text{for} \quad \text{Re } \mathbf{p}\gtreqless 0 \tag{24}$$

in which case Theorem 2 applies to show that f_0/f_e is a reactance function when $f_e \neq 0$, as assumed.

6. Discussion

Here, among other things, we have proven the necessity of the positive-real condition for positive multivariable networks. Previously there appears to have been no complete proof and presently we are unable to proceed through a direct energy consideration, as in the single variable case. Consequently we have introduced a multivariable version of Tellegen's Theorem which allows the straightforward proof to be given. Actually a close look at the procedure shows that if at (4) we had $Z_1(\mathbf{p}) = \sum Z_i(p_i)$ with the Z_i positive-real, the same results can be obtained, that is, the $Z(\mathbf{p})$ at the ports will be positive-real; further generalizations of the classes of networks allowed for the validity of Theorem 1 can be thought up almost at will.

In Theorem 4 an extension of the single variable test is given for multivariable Hurwitz polynomials, the proof falling back upon the rather interest ing Theorem 2. A method then of testing a multivariable Hurwitz polynomial is to attempt a lossless synthesis of the odd over even part through a positive network. As a consequence one would hope in the future for simpler such syntheses than those presently available (Koga 1968).

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