INT. J. ELECTRONICS, 1974, VOL. 36, NO. 5, 607–621

Scattering matrix synthesis using multiport cavities †

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[Received 2 February 1973]

The necessary and sufficient conditions are established on the scattering matrix for an n-port to be realizable as a cascade of multiport cavities having proportional ports coupled through circulators. Design equations, yielding cavity dimensions, are given including the incorporation of cavity loss; S_{21} synthesis is presented. Although the theory can be used wherever the equivalent circuit is valid, the results are seen to be practically most useful for 2-port narrow-band microwave comb filters to which the theory is applied.

1. Introduction

With the advent of satellite communication systems, the need for precise synthesis methods for microwave networks has become clear in order that equipment can be efficiently used (Atia and Williams 1971). Towards this end, we present here a synthesis method, utilizing cavity resonators, which differs considerably in philosophy from those presently in the literature (Atia and Williams 1971, 1972, Cohn 1957, Levy 1967, Rhodes 1970, Saito 1970).

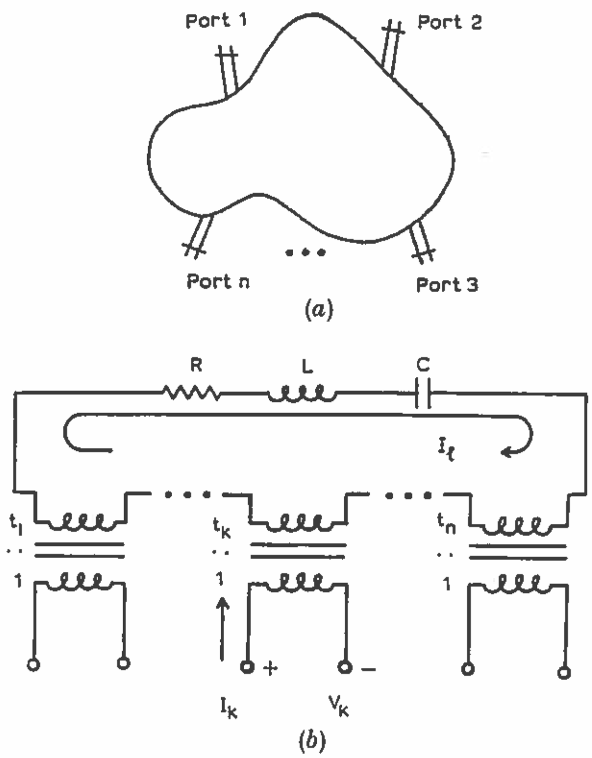
The main idea of this paper is tied to the factorization of scattering matrices (Newcomb 1966, pp. 150, 190, DeWilde *et al.* 1971). In particular, once the form of the scattering matrix for a multiport cavity is recognized, eqn. (5 *b*), the form of the scattering matrix of a cascade of such cavities coupled through circulators is known, since this overall scattering matrix is formed as a product. The problem of synthesis is then that of factorization of a given scattering matrix into appropriate realizable terms. The scattering matrix of an n-entry cavity is obtained from known equivalent circuits (Kurokawa 1969, p. 191) developable from standard electromagnetic theory and valid for a single mode of excitation (Kahan 1956, p. 63). From the physics of the situation cavity dimensions are determined in terms of scattering matrix parameters, table 1, allowing for physical constructions.

We begin in § 2 by developing the scattering matrix of a multiport cavity from its equivalent circuit. In § 3 we present the realizability conditions for a cascade synthesis of cavities whose ports are proportionally dimensioned. Since 2-port S_{21} synthesis is of most practical interest, we concisely cover this situation in § 4. In § 5 we discuss the actual dimensioning of cavities giving design equations applicable to the synthesis of the comb nature of responses for lossless structures while presenting an approximation theory. From this, the form of S_{21} is seen to be ideally suited for the design of comb

† This work was supported in part by the Air Force Office of Scientific Research under Grant AFOSR 70–1910 and the National Science Foundation under Grant GK–24036.

filters (George and Zamanakos 1954, MacFarlane 1960, Woodward 1955, p. 96, Burdic 1968, p. 75), especially in environments where large dynamic signal ranges might be expected. Finally, after the example of § 7, we discuss the results in § 8, including comments on the effects of loss and the various limitations on the theory.

Fig. 1



Multiport cavity and equivalent circuit.

2. Multiport cavity scattering matrix

For our purposes the scattering matrix of a multiport cavity can be developed from its equivalent circuit. Such an equivalent circuit is shown in fig. 1 (*b*) for the n-port cavity of fig. 1 (*a*). As is known (Kurokawa 1969, p. 191), this equivalent circuit approximates a cavity working in a single mode over a limited frequency range, roughly less than 10% of centre frequency. The main loop in the equivalent circuit describes the resonant behaviour of the cavity, this being characterized by the lumped series elements RLC. The various ports of the cavity are coupled to the main loop by the ideal transformers of non-negative turns ratios $t_1, t_2, \ldots, t_n$, these turns ratios being physically determined by the input hole geometries.

Assuming terminal voltages and currents, V_k and I_k at the kth port as shown in fig. 1 (*b*), we have on, straightforward analysis,

$$zI_l = \sum_{k=1}^{n} t_k V_k, \quad I_k = t_k I_l, \quad k = 1, \ldots, n \qquad (1\,a)$$

where I_l is the main loop current and z is the main loop self-impedance

$$z(p)=R+pL+\frac{1}{pC}\,; \quad C>0,\, L>0,\, R\geqslant 0 \tag{1 b}$$

Note that, for a lossless cavity, $R=0$, which is usually a good approximation for physical structures.

Substituting the first of eqns. (1 *a*) into the second, it is straightforward to see that the admittance matrix of the n-port cavity is

$$Y(p)=\frac{1}{z(p)}\,TT^t \tag{2}$$

where T is the n-vector whose components are $t_1, \ldots, t_n$; the superscript t indicates the transpose. The scattering matrix S of the cavity is given in terms of the admittance matrix Y by (Newcomb 1966, p. 52)

$$S=(1_n+Y)^{-1}(1_n-Y) \tag{3 a}$$

where 1_n is the $n\times n$ identity matrix. Direct cross-multiplication shows that

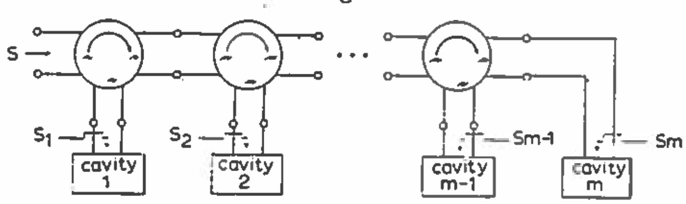
$$\left(1_n+\frac{1}{z}\,TT^t\right)^{-1}=1_n-\frac{1}{z+T^tT}\,TT^t \tag{3 b}$$

from which eqns. (2) and (3 *a*) give

$$S=1_n-\frac{2}{z+\|T\|^2}\,TT^t \tag{4}$$

where $\|T\|^2=T^tT=\sum_{k=1}^{n} t_k^2$ is the square of the norm of the turns ratio vector.

Fig. 2



Cascade of cavities coupled through circulators.

We next consider an interconnection of m such cavities as shown in fig. 2, that is, a cascade of m cavities coupled through $3n$-port circulators. As is known, the scattering matrix of this structure is equal to the product of the individual cavity scattering matrices (Newcomb 1966, p. 150)

$$S=S_1S_2\ldots S_m \tag{5 a}$$

where the scattering matrix of the ith cavity is given by, as prescribed by eqn. (4),

$$S_i(p)=1_n-\frac{2}{z_i(p)+\|T_i\|^2}\,T_iT_i^t, \quad z_i(p)=R_i+pL_i+\frac{1}{pC_i} \tag{5 b}$$

3. Cavity realizability theorem

The most important case of interest is in the design of structures whose input ports are compatible in the sense that the port geometries are set out in a similar manner. By this, we mean precisely that the turns ratio vectors of the cavities are proportional. We make this assumption from now on for which the connection of fig. 2 has

$$T_i = k_i T, \quad i = 1, \ldots, m \tag{6}$$

where the positive k_i are scalar constants and T is a fixed n-vector of non-negative entries. A set of m cavities satisfying eqn. (6) will be said to have *proportional ports.*

For proportional cavities, we have the following realizability theorem for structures of the form of fig. 2.

Theorem of realizability

The necessary and sufficient conditions for a scattering matrix $S(p)$ to be realizable as a cascade of m n-port cavities with proportional ports coupled through circulators (as in fig. 2) are that S be expressible in terms of real parameters as

$$S(p) = 1_n - \frac{TT^t}{\|T\|^2}\left[1 - \prod_{i=1}^{m} \frac{p^2+\beta_i p+\gamma_i}{p^2+\alpha_i p+\gamma_i}\right] \tag{7 a}$$

where T is a non-zero n-vector of non-negative entries and

$$\alpha_i > 0, \quad \alpha_i - \beta_i \geqslant 0, \quad \gamma_i > 0, \quad i = 1, \ldots, m \tag{7 b}$$

Proof

(*a*) Necessity. From eqns. (5 *b*) and (6) we can express the n-port cavity scattering matrix as

$$S_i = 1_n - K\left[1 - \frac{z_i - k_i^2\|T\|^2}{z_i + k_i^2\|T\|^2}\right] = 1_n - K\left[1 - \frac{p^2+\beta_i p+\gamma_i}{p^2+\alpha_i p+\gamma_i}\right],$$

$$K = K^2 = \frac{TT^t}{\|T\|^2} \tag{8 a}$$

where

$$\alpha_i = \frac{R_i + k_i^2\|T\|^2}{L_i}, \quad \beta_i = \frac{R_i - k_i^2\|T\|^2}{L_i}, \quad \gamma_i = \frac{1}{L_i C_i} \tag{8 b}$$

Solving these latter yields

$$2\frac{R_i}{L_i} = \alpha_i + \beta_i, \quad 2\frac{k_i^2}{L_i} = \frac{\alpha_i - \beta_i}{\|T\|^2}, \quad C_i L_i = \frac{1}{\gamma_i} \tag{8 c}$$

which, with $L_i > 0$, shows the necessity of eqn. (7 *b*). Next we note, using the form of eqn. (8 *a*), that all scattering matrices commute for $S = \prod_{i=1}^{m} S_i$ of eqn. (5 *a*). Direct multiplication then yields eqn. (7 *a*).

(*b*) Sufficiency. We exhibit a synthesis procedure by extracting a term from the product by multiplication through its inverse. We have, again by direct multiplication with eqn. (8 *a*),

$$S_i^{-1} = 1_n - K\left[1 - \frac{p^2+\alpha_i p+\gamma_i}{p^2+\beta_i p+\gamma_i}\right] \tag{8 d}$$

Given a realizable S we first put it into the form of eqn. (7 *a*). Then, applying the inverse of S_i to S, using eqn. (8 *d*), shows that the extraction of S_i leaves a remainder $S_i^{-1}S$ which satisfies the Theorem of Realizability. Continuing, fig. 2 results as the degree lowers by 2 at each extraction.

Q.E.D.

By way of comment, we see that $S(p)$ is bounded real, being the scattering matrix of a passive n-port (Newcomb 1966, p. 98). Thus $S(p)$ is analytic in the right half p-plane, which is also clear from eqn. (7 *b*). Also, if $R_i = 0$ then $\beta_i = -\alpha_i$ and S_i is paraunitary.

Factorization of K yields the turns ratio vector $T_i = k_i T$. Thus, besides the n turns ratio entries of T itself, there are four other parameters (R_i, L_i, C_i, k_i) determined by the physical cavity, whereas eqn. (8 *c*) only specifies three, we see that there is freedom in specifying L_i, say. Thus we wish to choose the normalization parameter, L_i say, such that the cavities can be conveniently dimensioned.

4. $S_{21}(p)$ conditions

In the last section we discussed realizability conditions placed upon the full scattering matrix. However, most often it is one or more off-diagonal elements which are specified, perhaps only in magnitude for real frequencies. Here we discuss the situation with regard to $S_{21}(p)$ for 2-ports from which results for n-ports can be readily inferred; discussion of $|S_{21}(j\omega)|$ occurs in § 6.

From eqn. (7 *a*)

$$S_{21}(p) = K_{21}\left[1 - \prod_{i=1}^{m} \frac{p^2+\beta_i p+\gamma_i}{p^2+\alpha_i p+\gamma_i}\right] \tag{9}$$

Since the conditions on α_i, β_i, γ_i are known from eqn. (7 *b*), the only unknown realizability question concerns K_{21}. We can write

$$-K_{21} = \frac{t_2 t_1}{\|T\|^2} = \frac{t_2/t_1}{1+(t_2/t_1)^2}, \quad t_2 t_1 \geqslant 0 \tag{10}$$

which, as a function of t_2/t_1, is seen to have a maximum of $+\frac{1}{2}$ at $t_1 = t_2$ (and a minimum of $-\frac{1}{2}$ at $t_1 = -t_2$). Thus, if $0 \leqslant -K_{21} \leqslant \frac{1}{2}$, we can solve for real non-negative t_2/t_1 and hence obtain a structure for each section in the form of fig. 1 (*b*) (on arbitrarily specifying one of t_1, t_2 and multiplying each by k_i of eqn. (8 *c*) to get the actual turns ratios for the section as seen by eqn. (6)). Thus we have the realizability conditions on a 2-port S_{21} as

$$0 \leqslant -K_{21} \leqslant \tfrac{1}{2}, \quad \alpha_i > 0, \quad \alpha_i - |\beta_i| \geqslant 0, \quad \gamma_i > 0, \quad i = 1, \ldots, m \tag{11}$$

Note that if more than two ports are present, the denominator in eqn. (10) is increased, in which case eqns. (11) still hold, thus giving the realizability conditions on an n-port S_{21}. In other words, given S_{21} of the form of eqn. (9) satisfying eqn. (11) a scattering matrix satisfying the Theorem of Realizability can be created, for any $n \geqslant 2$.

5. Cavity and aperture dimensioning

In this section we present basic equations to determine waveguide cavity and coupling aperture dimensions for the realization of a basic 2-port scattering matrix

$$S(p) = 1_2 - \frac{1}{t_1^2 + t_2^2}\begin{bmatrix} t_1^2 & t_1 t_2 \\ t_2 t_1 & t_2^2 \end{bmatrix}\left[1 - \frac{p^2 + \beta p + \gamma}{p^2 + \alpha p + \gamma}\right] \tag{12}$$

Table 1. Design parameter calculations given α, β, γ, kt_i $(i = 1, \ldots, n)$

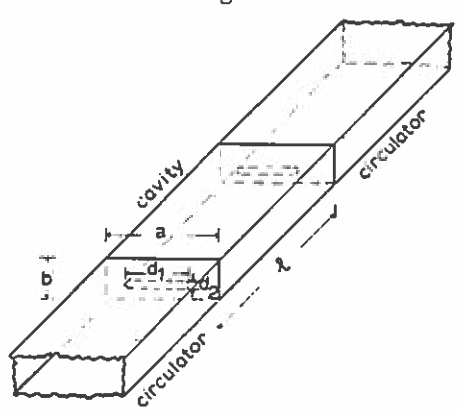
Design parameter	Calculation	Comments
a	Choose	Use standard waveguide a=depth, b=height, in m
b	Choose	
	$\frac{1}{2}\lambda = \dfrac{3\pi \times 10^8}{\sqrt{\gamma}}$	λ in m/sec
	$\frac{1}{2}\lambda_g = \dfrac{\lambda/2}{\sqrt{\left[1 - \left(\frac{(\lambda/2)}{a}\right)^2\right]}}$	a in m
l	$l = \frac{1}{2}\lambda_g$	l=cavity length in m
	$Q_{ex} = \dfrac{2\\|kT\\|^2}{(kt_i)^2}\dfrac{\sqrt{\gamma}}{(\alpha - \beta)}$	$\\|kT\\|^2 = 2(kt_i)$ if $n=2$ and $-K_{21} = \frac{1}{2}$
	$M_i = \dfrac{lab}{2}\sqrt{\left(\dfrac{l\lambda_g}{\lambda^2} \cdot \dfrac{1}{\pi Q_{ex}}\right)}$	For apertures in transverse field, as fig. 3; others use M_i as in Matthaei *et al.* (1964, p. 461)
d_i	$d_i = {}^3\sqrt{(6M_i)}$	d_i=ith aperture diameter in m If $\lambda/d_i \gg 1$ use elliptical hole (Cohn 1957, Matthaei *et al.* 1964, Kretzschmar 1970)
	$R_S = \dfrac{377\pi(\alpha + \beta)}{{}^8\sqrt{\gamma}} \times \left[\dfrac{2b(a^2 + l^2)^{3/2}}{al(a^2 + l^2) + 2b(a^3 + l^3)}\right]$	R_S in Ω
σ	$\sigma = \dfrac{4\pi^2 \times 10^{-7} f}{R_S^2}$	σ=cavity wall conductivity in ℧/m Choose appropriate lossy material

satisfying the realizability conditions $\alpha > 0$, $\alpha - |\beta| \geqslant 0$, $\gamma > 0$. For the sake of clarity, we restrict our attention to 2-port cavities in rectangular waveguides operated in the fundamental TE_{10} mode, but the same principles apply to other multiport waveguide shapes and modes. Table 1 summarizes the results for a low-loss rectangular waveguide realization with apertures as shown in fig. 3. We assume that the parameters α, β, γ and the turns ratios kt_1, kt_2 (with k as in eqn. (8)) have been determined from S as in eqn. (12), and that the waveguide dimensions a and b are specified. Then the cavity length l for the TE_{101} mode is half a guide wavelength (Ramo and Whinnery 1953, p. 421)

$$l = \frac{\lambda_g}{2} = \frac{\lambda}{2\sqrt{[1-(\lambda/2a)^2]}}, \quad \lambda = \frac{2\pi v}{\sqrt{\gamma}} \tag{13}$$

where λ is the free-space wavelength at the resonant frequency and v is the free-space propagation velocity (3×10^8 m/sec).

Fig. 3



Waveguide cavity resonator with elliptical aperture couplings.

The dimensions of a coupling iris of a given shape is determined from the turns ratio, using Bethe's small aperture theory (Marcuvitz 1951, p. 241, Cohn 1957). The magnetic polarizability M of the iris is obtained by equating the two expressions for the external Q of the cavity due to the transformer coupling (Kurokawa 1969, p. 193, Matthaei *et al.* 1964, p. 461)

$$Q_{ex} = \frac{\omega_0 L}{(Kt)^2} = \frac{2\|T\|^2\sqrt{\gamma}}{(\alpha-\beta)t^2} = \frac{l^3a^2b^2\lambda_g}{4\pi M^2\lambda^2} \tag{14}$$

The diameter d of a circular coupling aperture can be determined from the relation (Marcuvitz 1951, p. 241)

$$M = d^3/6 \tag{15}$$

However, should d turn out to be relatively large with respect to the wavelength dimension, the circular aperture may not be practical. One can revert to other shapes of apertures (e.g. elliptic, rectangular) whose dimensions can be determined from graphs and equations given by Matthaei *et al.* (1964, p. 234), Cohn (1952) and Kretzschmar (1970).

Small cavity losses can be considered by calculating the unloaded quality factor, Q_u, of the resonator ; this is given by (Ramo and Whinnery 1953, p. 424)

$$Q_u = \frac{\omega_0 L}{R} = \frac{2\sqrt{\gamma}}{\alpha+\beta} = \frac{\pi}{4}\frac{\eta}{R_S}\left[\frac{2b(a^2+l^2)^{3/2}}{al(a^2+l^2)+2b(a^3+l^3)}\right] \tag{16}$$

where R_S is the surface resistivity and η ($\approx 377\ \Omega$ in air) is the free space intrinsic impedance. R_S is related to conductivity σ by the relation (Ramo and Whinnery, 1953, p. 239) $R_S = \sqrt{(\pi f \mu/\sigma)}$, where μ is the material permeability ($4\pi \times 10^{-7}$ Hy/m for waveguide materials) and f is the operating frequency. For standard waveguide materials σ can be found in a standard reference (Westman 1957, p. 45) or published curves (Ramo and Whinnery 1953, p. 238).

Some comments on normalization may be worth while. We see from eqn. (7 *a*) that if the frequency is scaled by $p = \Omega_n p_n$ for which

$$\beta_{i_n} = \beta_i/\Omega_n, \quad \alpha_{i_n} = \alpha_i/\Omega_n, \quad \gamma_{i_n} = \gamma_i/\Omega_n^2 \tag{17 a}$$

then, if the impedance level is raised, these parameters are left invariant. A change in the impedance level, however, is effected by a change in the dimension a, since at a fixed frequency $f = 1/\lambda$, the characteristic impedance of the guide is (Ramo and Whinnery 1953, p. 367)

$$Z_0 = Z_{TE} = \frac{\eta}{\sqrt{\left[1 - \left(\frac{\lambda}{2a}\right)^2\right]}} \tag{17 b}$$

From the beginning we have used the scattering matrix which can be considered as a normalized one with all normalizing port impedances equal to Z_0 (Newcomb 1966, p. 75). Thus, in raising the impedance level, say from $1\ \Omega$ to $Z_0\ \Omega$, we divide the turns ratios t_i by Z_0 (Newcomb 1966, p. 75) which, on using eqns. (7) and (8), leaves S invariant, as expected (Newcomb 1966, p. 74). Indeed, cavity dimensions can be expressed in terms of units of λ, as table 1 shows, with denormalization to the operating wavelength occurring for constructions, the latter being accompanied automatically by an impedance level shift via eqn. (17 *b*).

6. Comb filter approximation

In this section a method of prescribing any off-diagonal element of a realizable n-port scattering matrix in terms of real frequency data is considered. As will be seen, the practical characteristics are those of comb filters (George and Zamanakos 1954, MacFarlane 1960). Without loss of generality, the element we shall deal with is taken as S_{21}, but it is clear that the same arguments apply for any S_{lm}, $l \neq m$.

We recall the form of S_{21} as

$$S_{21}(p) = K_{21}\left[1 - \prod_{i=1}^{m} \frac{p^2+\beta_i p+\gamma_i}{p^2+\alpha_i p+\gamma_i}\right] \tag{9}$$

which is subject to the realizability constraints of eqn. (11). Since practical cavities have very small losses we carry out the approximation for the lossless case in which $R_i = 0$ for all i, or $\alpha_i = -\beta_i$ from eqn. (8 *c*). Then the factors $[p^2-\alpha_i p+\gamma_i]/[p^2+\alpha_i p+\gamma_i]$ are all-pass which for real frequencies have unit magnitude and hence can be expressed in terms of the phase angles θ_i as

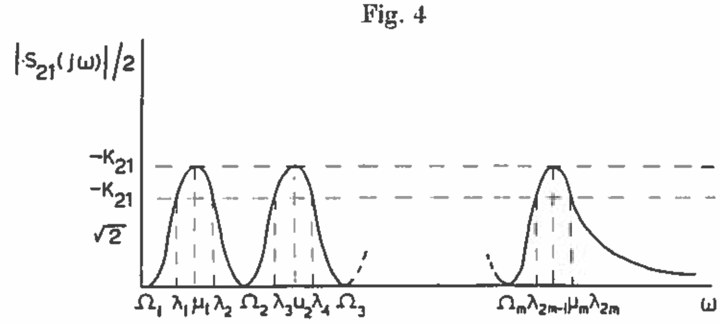
$$\exp[j\theta_i(\omega)] = \cos\theta_i + j\sin\theta_i = \left[\frac{p^2-\alpha_i p+\gamma_i}{p^2+\alpha_i p+\gamma_i}\right]_{p=j\omega} \tag{18 a}$$

The magnitude of S_{21} is therefore given by

$$|S_{21}(j\omega)| = \sqrt{2}|K_{21}| \sqrt{\left[1-\cos\left(\sum_{i=1}^{m}\theta_i(\omega)\right)\right]} \tag{18 b}$$

From this equation it is seen that as $\sum_{i=1}^{m}\theta_i(\omega)$ varies monotonically between 0 and $2m\pi$, the continuous function $|S_{21}|$ becomes equal to zero $m+1$ times, passing m times through its maximum value of $-2K_{21}$. The zeros and maxima of $|S_{21}|$ alternate along the $j\omega$ axis, being also interleaved with those frequencies for which $|S_{21}| = \sqrt{2}|K_{21}|$, typically as shown in fig. 4. This exhibits the comb filter characteristic of the circuit. Although the specifications of such comb filter characteristics can be made in several ways, we shall concentrate on one of most interest to the practical point of view, giving comments on another.

Fig. 4



Typical lossless cavity filter response.

Practically a specification of the zeros and maxima of $|S_{21}|$ is of interest and, as we know from fig. 4, these must alternate. Thus, assume that the peaks and valleys of $|S_{21}|$ are given by $\pm\mu_1, \pm\mu_2, \ldots, \pm\mu_m$ and $\Omega_1 = 0, \pm\Omega_2, \pm\Omega_3, \ldots, \pm\Omega_m, \infty$, respectively, with

$$\Omega_1 = 0 < \mu_1 < \Omega_2 < \mu_2 < \Omega_3 < \ldots < \Omega_m < \mu_m < \infty \tag{19}$$

From eqn. (18 b) it is clear that these frequencies are solutions of

$$\cos\left(\sum_{i=1}^{m}\theta_i(\omega)\right)=\begin{cases}-1 \text{ for } \omega=\mu_i \text{ (peak)}\\ +1 \text{ for } \omega=\Omega_i \text{ (valley)}\end{cases} \tag{20 a}$$

with, from eqn. (18 a),

$$\tan\frac{\theta_i(\omega)}{2}=\frac{\alpha_i\omega}{\Omega^2-\gamma_i},\quad i=1,\ldots,m \tag{20 b}$$

These last two equations are equivalent to

$$2\sum_{i=1}^{m}\tan^{-1}\frac{\alpha_i\omega}{\omega^2-\gamma_i}=\begin{Bmatrix}(2k+1)\pi & \text{for } \omega=\mu_i\\ 2k\pi & \text{for } \omega=\Omega_i\end{Bmatrix},\quad k=0,\ \pm 1,\ \pm 2,\ldots \tag{21}$$

The approximation problem is to determine the unknown α_i and γ_i for this last equation given μ_i and Ω_i satisfying eqn. (19). Towards this, let us define the polynomial $F(p)$ by

$$F(p)=\prod_{i=1}^{m}(p^2+\alpha_i p+\gamma_i)=Q(p)+K_0P(p) \tag{22 a}$$

where $Q(p)$ and $P(p)$ are the even and odd parts of F respectively with $K_0>0$ a constant to make P monic. We have by eqn. (18 a)

$$\tan\left[\sum_{i=1}^{m}\tan^{-1}\frac{\alpha_i\omega}{\omega^2-\gamma_i}\right]=jK_0\frac{P(j\omega)}{Q(j\omega)} \tag{22 b}$$

and, therefore, comparing eqns. (21) and (22 b), it is evident that the zeros of $P(p)$ occur at $\pm j\Omega_i$ and those of $Q(p)$ occur at $\pm j\mu_i$. Thus

$$Q(p)=\prod_{i=1}^{m}(p^2+\mu_i^2),\quad P(p)=p\prod_{i=1}^{m}(p^2+\Omega_i^2) \tag{22 c}$$

and K_0 is a constant free to be chosen. Since the poles and zeros of $X(p)=K_0P(p)/Q(p)$ alternate and are simple on the $j\omega$ axis with $X(1)\geqslant 0$, $X(p)$ is a reactance function. Thus $F(p)$ is a Hurwitz polynomial (Guillemin 1949, p. 398), in which case α_i and γ_i are positive. Under our lossless approximation assumption of $\beta_i=-\alpha_i$ we can see that the $S_{21}(p)$ realizability conditions of eqn. (11) are satisfied by choosing any real K_{21} with $0\leqslant -K_{21}\leqslant\frac{1}{2}$. Any positive K_0 can be chosen at eqn. (22 b), but since, as experience shows, the α_i decrease with K_0, a suitably small K_0 can conveniently be used to obtain suitable large Q_{ex} (eqn. (14)).

In conclusion, any comb filter curve of the type shown in fig. 4 can be realized by specifying its alternating zeros and (equal) peaks the latter of magnitude not greater than $1=2|K_{21}|_{max}$, by eqn. (18 b).

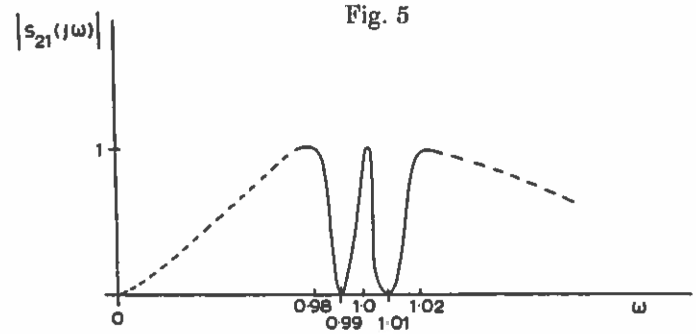
Further, if other than the zeros and peaks are specified, as for example the 3 dB frequencies, then the interpolation of Youla and Saito (1967, p. 107) may be applicable to finding the reactance function P/Q. However, this latter method does not guarantee that P is odd and of degree less than Q, nor does it apply to the case treated above since the A matrix of Youla and Saito (1967) has infinite entries. If the interpolation of Youla and Saito (1967) were ever applicable it would generally use close to twice as many cavities as seem necessary.

7. Example

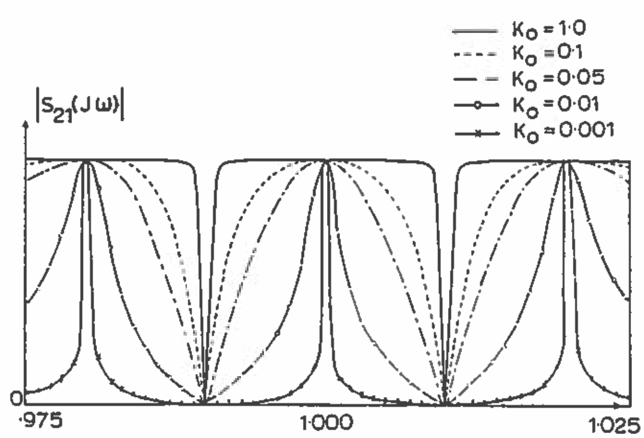
This example demonstrates the design of a narrow X-band waveguide comb filter. The normalized angular frequency peaks and valleys of the scattering matrix coefficient S_{21} are specified as 0·98, 1·0, 1·02 and 0·99, 1·01 respectively as shown in fig. 5 (*a*). The peaks and valleys alternate as required by eqn. (19). K_{21} in eqn. (9) is taken to be $-\frac{1}{2}$, which gives $t_1 = t_2$ and hence equal coupling holes in each port of the cavity. From these zeros and peaks the polynomial $F(p)$ is formed as in eqn. (22 *a*):

$$F(p) = Q(p) + K_0 P(p) = \prod_{i=1}^{m} (p^2 + \alpha_i p + \gamma_i)$$

Fig. 5



(*a*)



(*b*)

(*a*) Example specification. (*b*) Details of response for different K_0.

clearly we require m to be equal to 3. The constant multiplier K_0 of the odd part of $F(p)$ can be arbitrarily chosen. Although the peaks and valleys of the S_{21} response do not change with K_0, these responses are of different shapes. The larger K_0, the broader the maxima and the narrower the minima, and vice versa, as shown in fig. 5 (*b*) which shows the effect of variation of K_0 on the shape of the S_{21} response. It is true that different values of K_0 result in different external quality factors Q_{ex} for each cavity as shown in table 2 (as calculated through table 1).

Table 2

K_0	External quality factor		
	Cavity No. 1	Cavity No. 2	Cavity No. 3
1	2·0018	3282·68	3381·39
0·1	23·03	301·41	309·64
0·05	100·46	130·95	135·035
0·01	528·89	541·77	790·98
0·001	5269·32	5400·89	8002·40

In order to have aperture dimensions that are not large in comparison with the waveguide dimensions, the ranges of Q_{ex} from 50 to 500 are practical. Thus we choose $K_0 = 0{\cdot}05$ which gives an adequate range of external quality factors. By performing a computer-aided factorization on $F(p)$ into quadratic factors, we obtain $\alpha_1 = 0{\cdot}01506$, $\alpha_2 = 0{\cdot}019918$, $\alpha_3 = 0{\cdot}015019$, $\gamma_1 = 0{\cdot}97268$, $\gamma_2 = 0{\cdot}99896$, $\gamma_3 = 1{\cdot}10283$. Assume that the comb filter is to operate at a centre frequency of 10 GC in WR–90 waveguide of rectangular cross section ($a = 0{\cdot}9$ in., $b = 0{\cdot}4$ in.). At this frequency $\lambda_0 = 1{\cdot}18$ in., and the lengths, $2l_i = \lambda_{g_i}$, of the cavities are, in inches, $l_1 = 0{\cdot}817$, $l_2 = 0{\cdot}782$, $l_3 = 0{\cdot}757$. From table 1, the corresponding magnetic polarizabilities are found to be, in cubic inches, $M_1 = 6{\cdot}93 \times 10^{-3}$, $M_2 = 7{\cdot}42 \times 10^{-3}$, $M_3 = 6{\cdot}12 \times 10^{-3}$. By using the curve in Matthaei *et al.* (1964, p. 234) an elliptic aperture with $d_2/d_1 = 0{\cdot}25$ is chosen which gives $M/d_1^3 = 7{\cdot}15 \times 10^{-3}$. Knowing M, we then solve for d_1 and hence d_2 to give table 3 which shows the aperture dimensions for the three cavities.

From the right side of eqn. (15 *c*), knowing d_1, λ_c is found and eqn. (15 *b*) used with the M just found to obtain the resonance correction ; the new

Table 3. Initial aperture dimensions

Cavity No.	d_1 (in.)	d_2 (in.)
1	0·459	0·11475
2	0·47	0·1175
3	0·44	0·11

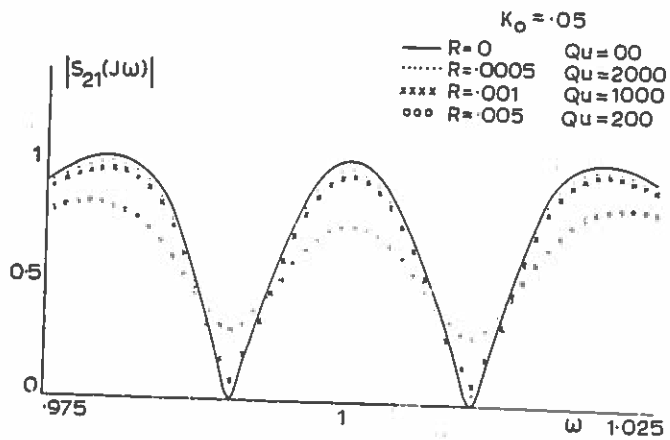
elliptic aperture dimensions are obtained after three iterations using this same process with the new d_1 giving table 4.

Table 4. Corrected aperture dimensions

Cavity No.	d_1 (in.)	d_2 (in.)
1	0·403	0·10075
2	0·407	0·10175
3	0·386	0·0965

The final cavities can be constructed with WR–90 waveguide of lengths given above and identical apertures at each end of dimensions shown in table 4. The three cavities are then connected to three matched 6-port circulators (which can be made of twelve 3-port circulators) according to fig. 2. Then S_{21}, for fig. 5, results by loading the final port in a matched load and feeding at port one by a matched source.

Fig. 6



Effects of cavity losses.

Finally the effects of cavity losses on the S_{21} response are shown in fig. 6 for the example at hand. As expected, the maxima decrease and the minima increase as the losses increase. This is illustrated by the curves of fig. 6 which show the response for unloaded Q's of ∞, 2000, 1000 and 200.

8. Discussion

This paper presents results directly applicable to the design of microwave filters using resonant cavities as the dynamical elements. Although the microwave cavity gives the practical motivation for the study, it should be clear that the results hold for any multiport having an equivalent circuit in the form of fig. 1 (*b*).

Indeed the main theoretical results hold for large R in the equivalent circuit of fig. 1 (*b*). However, for practical microwave structures only low-loss materials are ever considered, so that our synthesis results are of most use when lossless or nearly lossless matrices are on hand, this being the significant reason why only lossless approximation is seriously considered.

The design of the coupling holes into the cavity is rather fascinating. If small enough circular holes do not result from the design formula, eqn. (15), then elliptic holes will fit for which the theory is outlined in Kretzschmar (1970). However, practical designs are more readily constructed for rectangular-like, cigar-shaped, holes. These latter approximate elliptical ones with large eccentricity and considerable empirical design data exist for cigar-shaped holes yielding magnetic polarizability in the range $0 < M < \infty$ (Matthaei *et al.* 1964, p. 234). Likewise physical structures have non-zero guide wall thickness whereas the results presented here assume zero thickness. Consequently, thickness corrections (Matthaei *et al.* 1964, p. 243) can be simply applied to M, say at eqn. (15), to get more accurate designs.

For comb filters with normal insertion loss requirements, the results given here can yield practical designs. However, if high insertion loss is required, the limiting factor undoubtedly becomes the isolation of the circulators used in fig. 2 to realize the product of scattering matrix factors. Indeed, other methods of realizing such products exist (Belevitch 1968, p. 329); however, these cannot be applied here because of the necessary form of S required at eqn. (7 *a*).

Since for a cavity with more than two ports only the number of turns ratios increase, in fig. 1 (*b*), over the 2-port case and since these turns ratios are designed through appropriate coupling apertures, the design calculations of table 1 are seen to hold for the n-port case; the only caution being on the placement of the aperture holes which should not be too close together such that the assumptions under which the magnetic polarizability of eqn. (15) is derived are not upset.

Acknowledgments

The authors very gratefully acknowledge the comments and assistance of Dr. A. Atia concerning the work, especially with regard to aperture design.

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