# An Efficient Analysis Method for Nonlinear Networks*

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Abstract: By partitioning nonlinear elements an efficient method for the analysis of nonlinear networks is presented. The method is based upon techniques of Branin as well as resistive equivalents for dynamical elements. This involves the extraction of all nonlinear (one-port) elements as loads on a linear multiport, the loading occuring through inserted lossless transmission lines. The algorithm given solves the nonlinearities implicitly by iteration and combines this with an explicit solution by matrix inversion of the linear portion to obtain a complete solution.

## I. Introduction

In most general network-analysis programs the nonlinear elements are kept under many constraints and quite often limited to a certain class of elements. This is necessary in order to guarantee the convergence of an iterative method and hence the accuracy of the solution. In general, algorithms for analyzing networks fall into two categories well referenced in [1]: (a) those which first produce a matrix representation of the network by a simple transformation of the network into its Y or Z matrix (e.g. YV=I, with Y being formed [2] through general-admittance-matrix or cut-set methods) and solve the set of equations (YV=I say), (b) methods which represent the network in a form suitable to numerical analysis, e.g. $\frac{dv}{dt} = Av + Bi$ or state-space algorithms. Of course there are programs which employ a combination of both techniques, but they are usually task-oriented.

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Class (a) enjoys the capability of handling almost every type of circuit element for which in principle there are no more than two numerical methods to obtain the solution. These are [3] (i) matrix inversion or the whole class of matrix decompositions and (ii) iteration and all associated methods. As usual in other methods a combination of (i) and (ii) is used here and it is on this that we shall base the algorithm.

It should be mentioned that although class (b) has a much larger choice of numerical algorithms, it finds itself unable to present general methods for forming 'A' & 'B' matrices for non-linear elements; it is also paralyzed by long computation times associated with specific manipulations on A and B. Further, most nonlinear procedures require a matrix of derivatives, such as a Jacobian matrix, which in general is either found numerically or it is supplied by the user. We should note that numerical differentiation adds to the "noise level" of the data and therefore often makes these matrices useless as correctors; hence these methods are unattractive.

We shall use the method previously introduced in Ref. [2] which is based upon the nodal equations of the network to develop an algorithm for partitioning the admittance matrix for nonlinear elements. The nonlinear elements are assumed to have their graph in the first and third quadrants. It is assumed that the reader is familiar with these methods of generating the admittance matrix for passive elements with initial conditions [1] [2]. This paper concentrates on finding a solution by introducing transmission lines which isolate the nonlinear elements and rearranging the admittance matrix, after this matrix is found. Superscripts are used to indicate the iteration index, and subscripts indicate elements or the nature of the array or both.

## II. Analysis of the Network

The use of matrix inversion to solve $V=Y^{-1}I$, if values of $Y=\{y_{ij}\}$ depend on components of the voltage vector V, is not a fast method for finding the voltage, assumed to be the unknown, nor does it always guarantee a solution. In fact their convergence depends heavily on the presence of dominant leading diagonal elements which can only be guaranteed if there are no dependent sources, and the elements are all "physical". Under-relaxation and mean-value techniques are helpful in cases where there are large off-diagonal elements but the trade off is between accuracy and computation time, proportional to the value of these elements.

To quarantee a solution we shall separate the nonlinear elements from the rest of the circuit, by inserting lossless transmission lines between non-linear elements and the linear elements, so that we can invert the resultant admittance matrix representing the linear elements ( $f(v)=av$, a independent of v). This avoids the problem of iterative methods which have the disadvantage of not always converging[3].

In place of iteration on a set of simultaneous nonlinear equations we shall iterate on reflected waves which physically would be travelling on the inserted lines. Because of the physical nature of the lines a solution is intuitively feasible. At every state of the iteration, assuming $V^{(j)}$ is the solution, we can find the corresponding admittance matrix $Y^{(j)}$; by using this $Y^{(j)}$ we find the next values of voltage $V^{(j+1)}$ and so on. Note that if elementsare linear then $Y^{(j)}$ will be constant and will not vary with voltage.

(A) Analysis of a lossless transmission line:

At this point we review the results of Branin [4] placing them in matrix form for our purpose. Consider a single lossless transmission line with characteristic admittance $y_0$ and introduce the following notation:

$V_r = \{v_{rk}\}$ , k=1,2 reflected voltage at node k.

$V_i = \{v_{ik}\}$ , k=1,2 incident voltage at node k.

& $V = \{v_k\}$ , node voltage with respect to ground

We have

$$V_i + V_r = V \tag{1a}$$

It is known [4] that for the purpose of transient analysis we can replace the transmission line by a current source of strength $2V_i y_0$, and an internal admittance of $y_0$, and further

$$v_{i1}^{(n)d} = v_{r2}^{(n-1)d} \tag{1b}$$

where d is the delay of the line.

We shall assume d=1, because this delay will be seen to be independent of the time response of the network. So the circuit of Fig. 2a) becomes that of Fig. 2b) at the $n^{th}$ iterative step (i.e. t=nd).

Now let $I=\{i_k\}$, k=1,2, be the total current entering node k, then the equations for current for Fig. 2b) will be

$$I^{(n)} = I^{(0)} + 2y_0 CV_r^{(n-1)} \tag{2}$$

where, for this single line,

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad I^{(0)} = \begin{bmatrix} I_g \\ 0 \end{bmatrix}$$

From Eq.'s (1)

$$V_r^{(n)} = V^{(n)} - CV_r^{(n-1)} \tag{3}$$

and for the over-all 2 node network

$$Y^{(n)}V^{(n)} = I^{(n)} \tag{4}$$

where, from Fig. 2b)

$$Y^{(j)} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} , \qquad y_{12} = y_{21} = 0$$

$$y_{11} = y_g + y_0 \qquad y_{22} = y_L + y_0$$

Note that C is the connection matrix and Eq. (1b) has the form

$$V_i^{(n)} = CV_r^{(n-1)} \tag{1c}$$

(B) <u>Application of transmission line and convergence criteria</u>:

We shall first show the application of lossless transmission lines to isolating a single one port element in two steps;

case a for a linear one-port and case b for a nonlinear one-port load. Then we will generalize to handle all nonlinear elements.

We begin by taking the network of Fig. 3a and isolating the load through an added transmission line, Fig. 3b). Let $V_r$ be the true solution and $e^{(n)}$ be the error after n iterations then we will show that

$$e^{(n)} = V_r - V_r^{(n)} \tag{5}$$

tends to zero as $n \to \infty$, and also that the end voltages satisfy

$$v_1 = v_2 \quad \text{when } e^{(n)} \to 0 \tag{6}$$

This will serve as a proof in matrix form of Branin's result [4]. In Eq. (6) the two ends of the line are at the same voltage, which indicates that the line is indeed short circuited. Equation (5) indicates a stable state for the transmission line (as $e^{(n)} \to 0$) and hence the end of the iteration.

Proof that $e^{(n)} \to 0$:

From Eq.'s (2) & (4) we have

$$Y^{(n)}V^{(n)} = I^{(n)} = I^{(0)} + 2y_0 C V_r^{(n-1)} \tag{7}$$

Substituting Eq. (7) in Eq. (3) we have

$$Y^{(n)}V_r^{(n)} = I^{(0)} + 2y_0 C V_r^{(n-1)} - Y^{(n)} C\, V_r^{(n-1)} \tag{8}$$

The true solution, $V_r$, must also satisfy Eq. (8), so

$$Y^{(n)}V_r = I^{(0)} + [2y_0 I - Y^{(n)}] C V_r \tag{9}$$

Subtracting Eq. (8) from Eq. (9) we get

$$Y^{(n)}(V_r - V_r^{(n)}) = [2y_0 I - Y^{(n)}] C (V_r - V_r^{(n-1)}) \tag{10}$$

and by definition of Eq. (5)

$$Y^{(n)}e^{(n)} = [2y_0 I - Y^{(n)}] C e^{(n-1)} \tag{11}$$

$$\text{or } e^{(n)} = [Y^{(n)}]^{-1} [2y_0 I - Y^{(n)}] C e^{(n-1)}$$

( $[Y^{(n)}]^{-1}$ always exists for "physical" elements. )

$$e^{(n)} = [2y_0 (Y^{(n)})^{-1} - I] C e^{(n-1)}$$

$$\text{and } e^{(n)} = [2y_0 (Y^{(n)})^{-1} - I] C [2y_0 (Y^{(n-1)})^{-1} - I] C e^{(n-2)} \tag{12a}$$

$$e^{(n)} = \left( \prod_{i=1}^{n} \left| [2y_0 (Y^{(i)})^{-1} - I] \right| \right) \left| e^{(0)} \right| \qquad (12b)$$

'| |' is the norm of the vector or matrix; we shall consider Eq.'s (12a) and (12b) in the following two cases.

(a) In the <u>linear</u> case where $Y^{(j)}=Y$ is independent of V for all j:

$$Y^{-1} = \begin{bmatrix} y_0+y_g & 0 \\ 0 & y_0+y_L \end{bmatrix}^{-1} = \begin{bmatrix} 1/(y_0+y_g) & 0 \\ 0 & 1/(y_0+y_L) \end{bmatrix} \qquad (13)$$

Let $\rho=[2y_0Y^{-1}-I]$ be a reflection coefficient matrix associated with Fig. 2b) or

$$\rho = \begin{bmatrix} \rho_{11} & 0 \\ 0 & \rho_{22} \end{bmatrix} \qquad (14a)$$

where $\rho_{11}=(y_0-y_g)/(y_0+y_g)$, has $|\rho_{11}| \leq 1$

and $\rho_{22}=(y_0-y_L)/(y_0+y_L)$, has $|\rho_{22}| \leq 1$ by passivity.

But $|\rho_{ii}|=1$ practically never occurs within this theory[2, pp. 26-27]; so we will assume $|\rho_{ii}| \neq 1$.

Then $|e^{(n)}| = |\rho|^n |e^{(0)}|$ where the term $|\rho|^n$ denotes the $n^{th}$ power of $|\rho|$.

But $$|\rho|^n = \begin{vmatrix} |\rho_{11}|^n & 0 \\ 0 & |\rho_{22}|^n \end{vmatrix} \to 0 \text{ as } |\rho_{ii}| \neq 1 \quad (n \to \infty) \qquad (15a)$$

and from Eq. (5) $V_r - V_r^{(n)} = e^{(n)}$, $V_r^{(n)} \to V_r$ as $n \to \infty$

Substituting $V_r$ in Eq. (3) gives $V_r = V^{(n)} - CV_r$ or solving for $V^{(n)}$

$$V^{(n)} = (I+C)V_r \quad , \quad n \to \infty \qquad (16)$$

Therefore

$$V^{(n)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{r1} \\ v_{r2} \end{bmatrix} = \begin{bmatrix} v_{r1}+v_{r2} \\ v_{r1}+v_{r2} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

hence $v_1=v_2$, and Eq. (6) is demonstrated.

(b) In the <u>nonlinear</u> case $Y^{(j)}=Y(V^{(j)})$

We will assume for the nonlinear devices that they have an admittance $y=f(v)/v$ which is positive and finite for finite nonzero v, $f(0)=0$. As in Eq. (13) we still can invert each $Y^{(j)}$ and Eq. (12b) becomes

$$\left|e^{(n)}\right| = \left(\prod_{i=1}^{n} \left|\rho^{(j)}\right|\right)\left|e^{(0)}\right| \qquad (12c)$$

and $$\rho^{(j)} = [2y_0(Y^{(j)})^{-1}-I]$$

Now $y_g=f_g(v_g)/v_g$ and $y_L=f_L(v_L)/v_L$ in the non-linear case, so let

$$y_g^{(j)}=f_g(v_g^{(j)})/v_g^{(j)} \quad \text{and} \quad y_L^{(j)}=f_L(v_L^{(j)})/v_L^{(j)}$$

As before $$\rho^{(j)} = \begin{bmatrix} \rho_{11}^{(j)} & 0 \\ 0 & \rho_{11}^{(j)} \end{bmatrix} \qquad (14b)$$

where $\rho_{11}^{(j)}=(y_0-y_g^{(j)})/(y_g^{(j)}+y_0)$ and $\rho_{22}^{(j)}=(y_0-y_L^{(j)})/(y_L^{(j)}+y_0)$

and Eq. (15a) becomes

$$\prod_{j=1}^{n}\left|\rho^{(j)}\right| = \begin{vmatrix} \prod_{j=1}^{n}\left|\rho_{11}^{(j)}\right| & 0 \\ 0 & \prod_{j=1}^{n}\left|\rho_{22}^{(j)}\right| \end{vmatrix} \to 0 \quad (n\to\infty) \qquad (15b)$$

and $\left|\rho_{kk}^{(j)}\right|<1$ for all j's, k=1,2 so $\prod_{j=1}^{n}\left|\rho_{kk}^{(j)}\right| \to 0$ , $n\to\infty$

Thus Eq. (16) is still valid, so as in the linear case

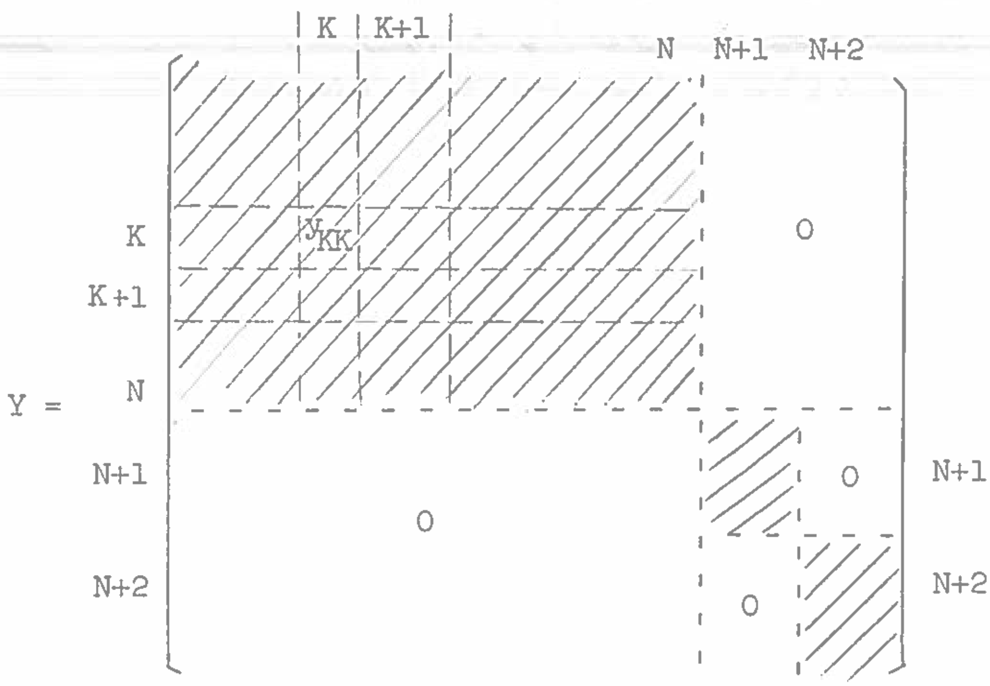
$$V^{(n)} = (I+C)V_r \quad \text{or} \quad v_1=v_2$$ Q.E.D.

Note that in this last case even if $f(v)=0$ for finite nonzero v, $v_1=v_2$ still results but physically this is not an interesting situation.

So far we have shown the existance of the convergence of the method, and that the input and output voltages on the transmission line are identical. Now we shall use the complete network by first considering one non-linear element and extending it to more elements.

(C) Network partition:

Let the network have N+1 nodes with only one nonlinear element, $y_L = f(v)/v$, connected between nodes K and K+1 as shown in Fig. 4); also take one node to be the ground node which is number zero. We place two transmission lines of delay d=1 between the nonlinear element and its terminal nodes K & K+1 and number the new nodes, between the device and the lines, N+1 & N+2. Figure 2b) is then applied to extract the nonlinearity, and the linear part of the circuit is analyzed [2, p. 26] to set up its nodal admittance matrix. Using the notation in (B) the admittance matrix $Y(\dot{V})$ for $I=Y(V)V$, will become (after the partitioning by the transmission lines)




where $y_{KK} = y_0 + \hat{y}_{KK}$ , $y_{(K+1)(K+1)} = y_0 + \hat{y}_{(K+1)(K+1)}$

$$y_{(N+1)(N+1)} = y_0 + \frac{f(v_{N+1}-v_{N+2})}{v_{N+1}} , \quad y_{(N+2)(N+2)} = y_0 - \frac{f(v_{N+1}-v_{N+2})}{v_{N+2}}$$

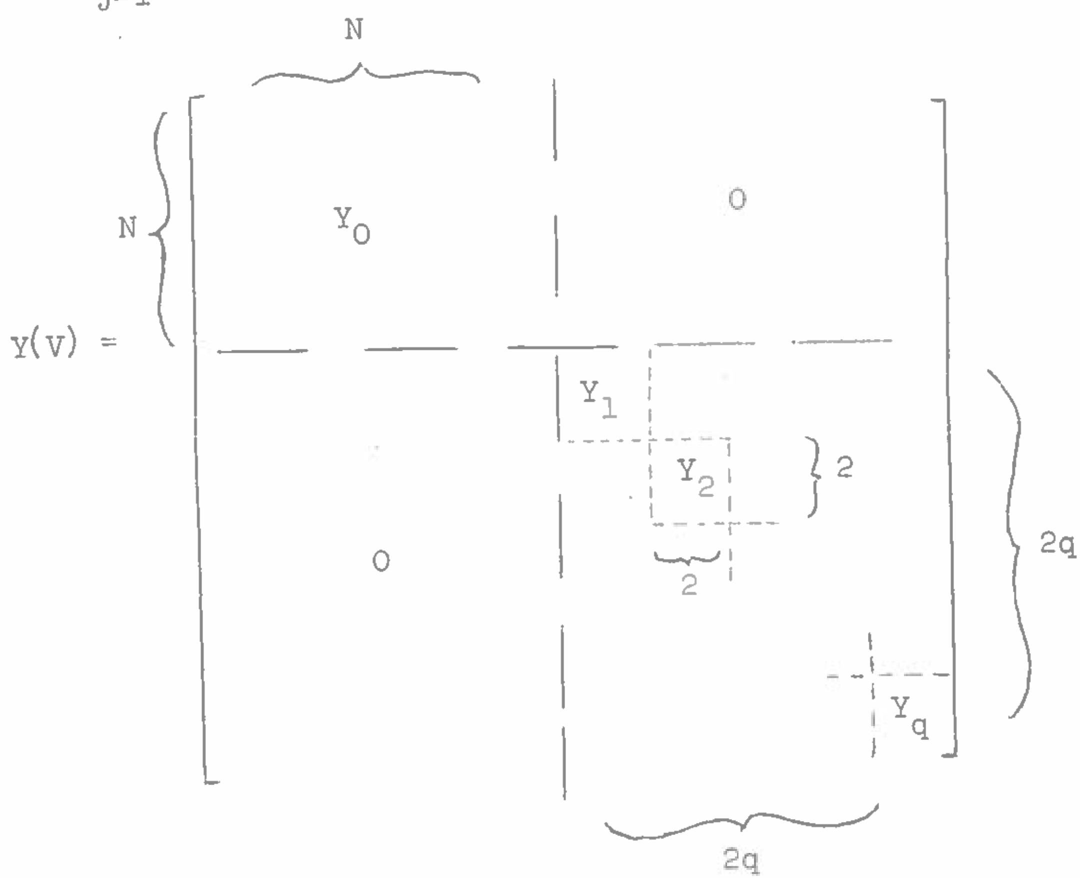
and the $\hat{y}$ represent admittances without the nonlinear element. For this nonlinear element the C matrix is a (N+2) by (N+2) matrix with all elements zero except for the entries describing the transmission lines' positions

$$c_{K,N+1} = c_{N+1,K} = c_{K+1,N+2} = c_{N+2,K+1} = 1 \tag{17a}$$

The upper-left N by N submatrix of the Y matrix will look as if the nonlinear element is removed and two resistors of value $y_0$ are connected between the K & K+1 nodes and ground. If there is more than one nonlinear element, then in general we change an N by N matrix to a (N+2q) by (N+2q) matrix as shown below, where q is the number of the nonlinear elements (q=1 for the above case); the first N by N elements of the new Y matrix, $Y_0$, will be linear (that is, independent of V). Note that if one of the terminals of the device is connected to the ground node then we do not need a transmission line between ground and that terminal of the device.

Now the general C matrix is the sum of all the C's called here $C_j$, of which there are q, one for each transmission lines (connected between the two nodes $K_j$ and $K_j+1$, where the device was located in the N-terminal network, and the nodes N+2j-1 and N+2j) and with non-zero entries as indicated by Eq. (17a),

$$C = \sum_{j=1}^{q} C_j \quad , \tag{17b}$$




(17c)

For this the notation is as follows:

$Y_0$ = upper left N by N elements of Y, all constant.

$V_r$ = reflected voltage, N+2q elements but only 4q are nonzero.

$Y_1$ to $Y_q$ = 2 by 2 matrices for nonlinear elements.

$V_0$ and $I_0$ = first N component of V and I.

$V_1$ to $V_q$ = q last 2-tuple components of V.

$I_1$ to $I_q$ = q last 2-tuple components of I.

The equations are now written for the new network described by Y and current sources that would be inserted as in Fig. 2b).

The general formula, YV = I, becomes:

$$\begin{cases} Y_0^{(n)} V_0^{(n)} = I_0^{(n)} \quad , \quad Y_0^{(n)} \equiv Y_0 \\ Y_1^{(n)} V_1^{(n)} = I_1^{(n)} \\ Y_q^{(n)} V_q^{(n)} = I_q^{(n)} \end{cases} \tag{17d}$$

Equation (2) generalizes to the following with C now (N+2q)x(N+2q)

$$I^{(n)} = I^{(0)} + 2y_0 C V_r^{(n-1)} \tag{18}$$

From Eq. (17d)

$$V_0^{(n)} = (Y_0)^{-1} I_0^{(n)} \tag{19}$$

$$Y_k^{(n)} V_k^{(n)} = I_k^{(n)} \quad , \quad k=1,2,\ldots,q \tag{20}$$

From Eq. (3)

$$V_r^{(n)} = V^{(n)} - C V_r^{(n-1)} \tag{21}$$

where C is as defined in Eqs. (17a,b).

Equations (18) to (21) are a set of difference equations to be solved which show the order of execution of the algorithm. As at Eq. (3) $V_r^{(o)}=0$. After Eq. (21) a test of the per unit value of the norm of $V_r^{(n)}$ should be made and depending on the size of the result, either the iteration terminates or it continues by going back to Eq. (18) with the index of iteration increased by one.

The final solution is the first N elements of V; Eq. (12c) shows that the potential difference across theline is within a given bound which can be determined from the iteration index and the maximum reflection coefficients $s_{ii}$ which the nonlinear elements can possess.

(D) Solution of the nonlinear 2 by 2 matrices

The linear portion of the network has been solved by Eq. (19) by any standard linear systems technique. Thus it remains to solve the q nonlinear 2 x 2 matrix equations (20), shown in part (C) and derived from Fig. 2b).

$$\begin{bmatrix} \frac{f(v_k - v_{k+1})}{v_k} + y_0 & 0 \\ 0 & y_0 - \frac{f(v_k - v_{k+1})}{v_{k+1}} \end{bmatrix} \begin{bmatrix} v_k \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} i_k \\ i_{k+1} \end{bmatrix}$$

Here the index of iteration is ignored for clarity. Multiplying out and letting $v = v_k - v_{k+1}$

$$f(v) + y_0 v_k = i_k$$

$$-f(v) + y_0 v_{k+1} = i_{k+1}$$

$$\text{or } 2f(v) + y_0(v_k - v_{k+1}) = i_k - i_{k+1} \text{ so}$$

$$2f(v) + y_0 v = (i_k - i_{k+1}) \qquad (22)$$

Equation (22) describes the circuit shown in Fig.5 which can be solved in many ways. A Newton-Raphson method can be applied to Eq. (22), instead of solving the above 2 x 2 nonlinear matrix.

One of the advantages of partitioning is that the user can supply a subroutine to find $v_k$ and $v_{k+1}$ when $i_k$ and $i_{k+1}$ is given for Fig. 5 and in doing so he can choose the most suitable method for the device and even specify '$y_o$' such that Eq. (12c) gives the fastest estimated convergence. Note that multiple valued functions f(.) can be handled, including hysteresis, etc., if the user has available a subroutine for such as just mentioned.

## III. Conclusions

An iterative algorithm has been given for analyzing nonlinear networks. In the situations covered by this paper the nonlinear elements are two-terminal elements described by $i=f(v)$ with $f(v)/v$ finite and nonnegative with $f(0)=0$.

The admittance matrix is partitioned by employing lossless transmission lines to isolate the nonlinearities. This in turn has avoided the generation of ill-conditioned nonlinear matrices which is often a source of trouble in the solution of simultaneous equations [3] because the iteration results do not converge. By inverting the linear portion $Y_0$ of the admittance matrix we can speed up the computation and limit the iteration to the nonlinear elements, as in Part II (D). In the case of an ill-conditioned matrix $Y_0$ for the linear portion Gaussian nodal elimination has this advantage over iterative methods in that it rearranges the equations as it proceeds. Because of its capability in choosing the largest element of the array as its pivot, it is one of the best known algorithms for matrix inversion in terms of generality and accuracy. So, by partitioning, we have been able to produce a submatrix of linear elements which we can invert once by Gaussian elimination and use in all of the iterations following. Note, however, Gaussian elimination is not directly applicable to nonlinear systems. As a consequence, it was shown that by partitioning nonlinear elements in blocks of 2 by 2 matrices the nonlinear portion can be reduced to a single nonlinear equation for each block; this guarantees a result at each step of iteration. Finally the criteria for convergence can be obtained from Eq. (12c).

In this work all characteristic admittances were chosen equal, but this is not necessary. Likewise, the delay of each line can be different, but this difference will cause a fair amount of bookkeeping.

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Figure Titles

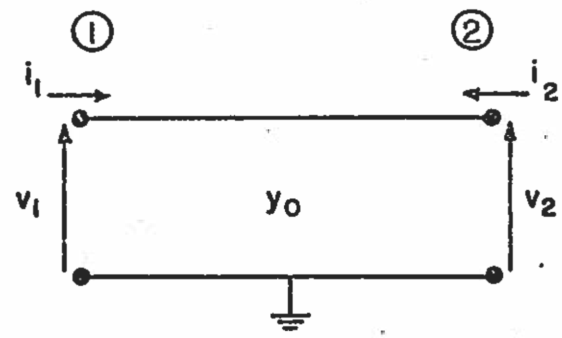



Fig 1

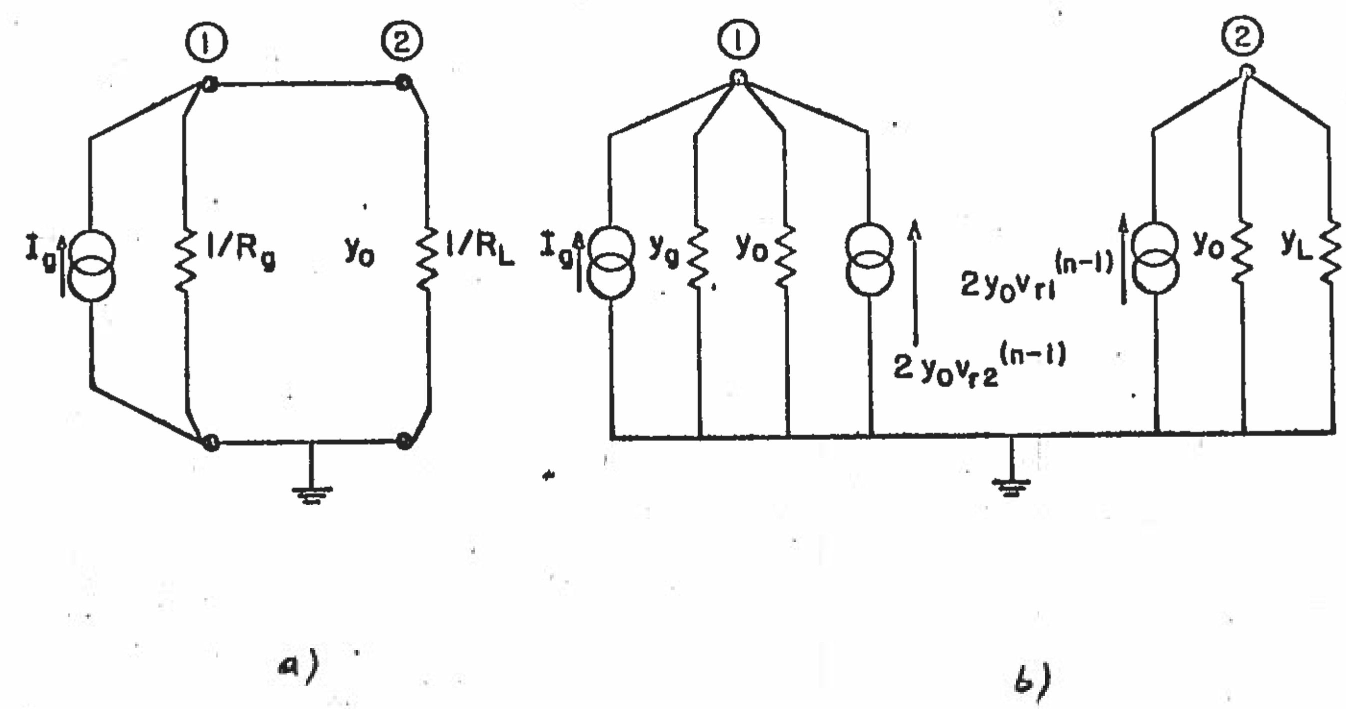
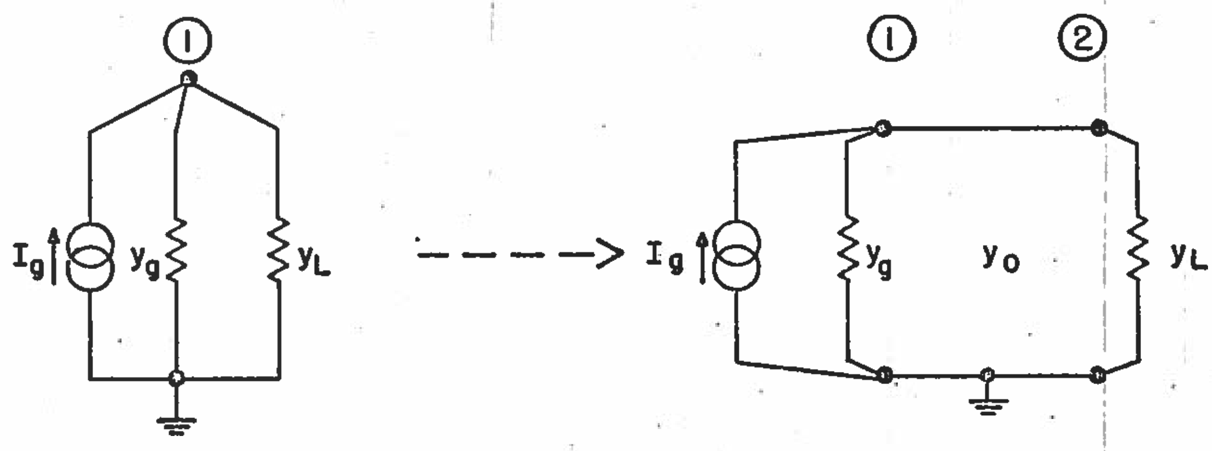



Fig 2




a) b)

Fig 3

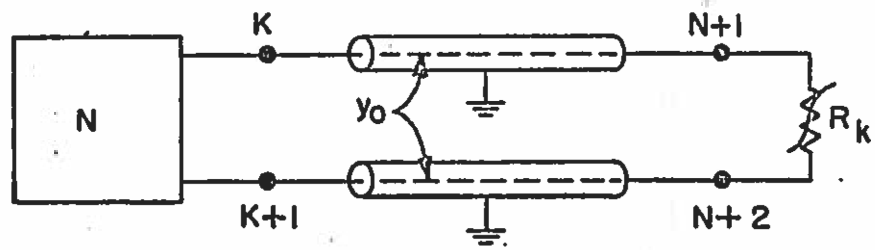



Fig 4

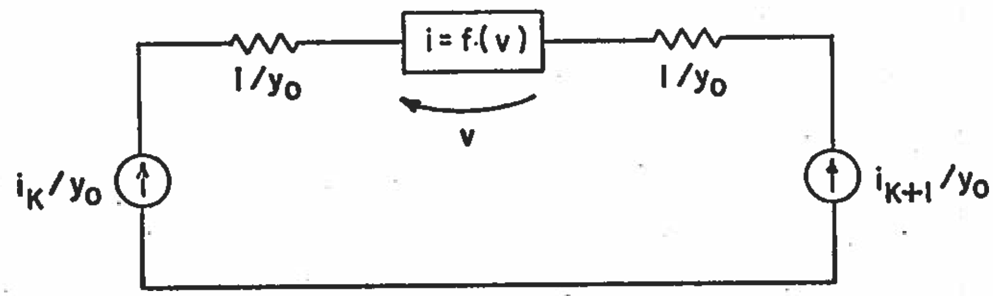



Fig 5