To illustrate the procedure, we realize the impedance

$$Z = \frac{(s^2+1)(s^2+3)(s^2+8)}{s(s^2+2)(s^2+6)} = \frac{s^6+12s^4+35s^2+24}{s^5+8s^3+12s} \quad (10)$$

by the cycle of Fig. 2.

The appropriate Cauer development of Z is shown in Fig. 12. The admittance of the boxed two-port M' is found to be

$$y_{22} = \frac{1}{2\rho^2}\frac{s^4+\frac{57}{7}s^2+48}{s(s^2+2)} = \frac{1}{2\rho^2}\frac{\left(s^2+\frac{26}{7}+\frac{2}{7}\sqrt{85}\right)\left(s^2+\frac{26}{7}-\frac{2}{7}\sqrt{85}\right)}{s(s^2+2)}. \quad (11)$$

The values of L, C, ω_1^2 and ω_2^2 are

$$L = 1 \qquad \omega_1^2 = \frac{26}{7}+\frac{2}{7}\sqrt{85}$$

$$C = \tfrac{1}{2} \qquad \omega_2^2 = \frac{26}{7}-\frac{2}{7}\sqrt{85}.$$

The proper value for the turns ratio is

$$\rho = \pm\frac{1}{2}\left(\frac{4}{7}\sqrt{85}\right) = \pm\frac{2}{7}\sqrt{85}. \quad (12)$$

The choice of turns ratio (12) produces the immittances

$$y_{22} = \frac{49}{680}\frac{s^4+\frac{57}{7}s^2+48}{s(s^2+2)}, \quad \text{and} \quad Z_r = \frac{272}{21}\left(s+\frac{7}{25}\right).$$

The elements of the two-port N', which is equivalent to M', now are found by realizing y_{22} on N' by Foster's procedure. Z_r can be realized by inspection. The final realization of Z is shown in Fig. 13.

As is the case for the cycle of Fig. 1, the most involved mathematical operation which must be performed to execute the cycle at Fig. 2 is the factorization of a second degree polynomial.

Concluding Remarks

The new canonic cycles serve to enlarge the family of known canonic networks. By mixing together these cycles, and those due to Foster and Cauer, one can construct a considerably greater variety of canonic networks than is obtainable through use of the Foster and Cauer cycles alone.

The new cycles are easily extended to the RC and RL cases. This is done most simply by means of the familiar Cauer transformations.

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A Bayard-Type Nonreciprocal n-Port Synthesis*

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Summary—**The impedance synthesis of Bayard is extended to the case of nonsymmetric impedance matrices. Two such extensions are given with an example of each illustrated. Computationally the method seems to be the simplest available for n-port synthesis. A method of obtaining equivalent networks is mentioned.**

I. Introduction

At PRESENT several synthesis methods exist for the realization of finite, passive, nonreciprocal n-ports. However, these all suffer from calculation difficulties which usually makes their use cumbersome. Thus, the original nonreciprocal synthesis of Oono and Yasuura[1] requires converting to a scattering matrix, if the original specification is in immittance terms. Likewise, the Brune synthesis of Belevitch[2] requires minimization of Hermitian matrices as well as a chain matrix calculation with its accompanying synthesis. When considering reciprocal networks, these inconveniences can be bypassed by using the method of Bayard.[3,4]

Here we extend the impedance synthesis ideas of Bayard to nonreciprocal networks. That is, given a rational, but not necessarily symmetric, positive-real impedance matrix, we obtain an auxiliary lossless impedance matrix by "para-Hermitian" part factorization.

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[1] Y. Oono and K. Yasuura, "Synthesis of finite passive $2n$-terminal networks with prescribed scattering matrices," *Memoirs of the Faculty of Engineering, Kyushu University*, vol. 14, pp. 125–177, p. 153; May, 1954.

[2] V. Belevitch, "On the brune process for n-ports," IRE Trans. on Circuit Theory, vol. CT-7, pp. 280–296; September, 1960.

[3] M. Bayard, "Résolution du problème de la synthèse des réseaux de Kirchoff par la détermination de réseaux purement réactifs," *Câbles et Transmission*, no. 4, pp. 281–296; October, 1950.

[4] M. Bayard, "Synthesis of N-terminal pair networks," Proc. Brooklyn Polytechnic Symposium on Modern Network Synthesis, Brooklyn, N. Y., pp 66–83; 1952. This is a summary of ref. 3 in somewhat more readable form. However, details, precautions and limitations are generally omitted.

This lossless matrix is synthesized by standard methods and yields the desired network when terminated in resistors. As with the symmetric synthesis the calculations are simple and straightforward.

Before proceeding we introduce some necessary notation. A superscript asterisk,*, will denote complex conjugation; a subscript asterisk, $_*$, denotes Hurwitz conjugation (replacement of $p = \sigma + j\omega$ by $-p$); a superscript tilde, $\tilde{}$, denotes matrix transposition; a subscript H denotes the Hermitian part of a matrix (*i.e.*, $2A_H = A + \tilde{A}^*$); 1_n and 0_n denote, respectively, the identity and zero matrices of order n; and $\dotplus$ denotes the direct sum of two matrices.[5] A polynomial will be called real if all coefficients are real and a network realization will be called positive if it uses a finite number of passive elements.

We also recall that an $n \times n$ matrix is positive-real if

1) $A(p)$ is analytic in $Re\ p > 0$,
2) $A^*(p) = A(p^*)$ in $Re\ p > 0$,
3) $A_H(p)$ is positive semi-definite in $Re\ p > 0$.

If $A(p)$ is positive-real and rational, it will simply be called *PR*.

II. Decomposition and Properties of a Given Network

Before beginning a synthesis, we investigate some related analysis ideas, which give some insight into the method of synthesis. Consider a positive realization, N, of a given *PR* impedance matrix Z. Assuming m resistors in N we can isolate these in a subnetwork N_r and look upon N as being realized by a lossless subnetwork N_l terminated in N_r. By, perhaps, incorporating suitable transformers in N_l we can require the resistors to be isolated and of value $+1$. This decomposition is shown in Fig. 1(a).

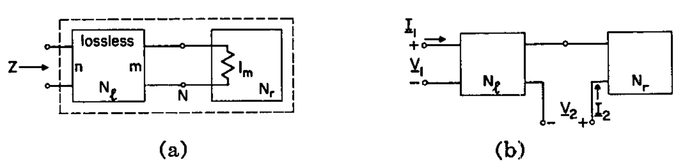


Fig. 1—Decomposition of N.

Assuming that N_l has an impedance matrix X, we can write

$$X = -\tilde{X}_* \tag{1a}$$

$$X = \begin{bmatrix} U & V \\ -\tilde{V}_* & W \end{bmatrix}; \quad \begin{matrix} U \text{ is } n \times n \\ V \text{ is } n \times m \\ W \text{ is } m \times m. \end{matrix} \tag{1b}$$

Here (1a) follows from the lossless property,[6] X is $(n + m) \times (n + m)$ and U pertains to the original n-ports. Note that the existence of X is an assumption, which, however will always be valid for the following synthesis method. We can create a new $(n + m)$-port as shown in Fig. 1(b) by breaking a lead to each resistor. This $(n + m)$-port is described by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} U & V \\ -\tilde{V}_* & W + 1_m \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{2}$$

By placing shorts on the last m ports of Fig. 1(b), N results, and, hence solving (2) when $V_2 = 0$, yields

$$Z = U + V(W + 1_m)^{-1}\tilde{V}_*. \tag{3}$$

In this $W + 1_m$ is nonsingular, as shown in Appendix I, since W is *PR*.

Now form the para-Hermitian part of Z, Z'', defined by

$$2Z'' = Z + \tilde{Z}_*. \tag{4a}$$

From (1) and (3) we get

$$Z'' = V(W + 1_m)^{-1}(\tilde{W}_* + 1_m)^{-1}\tilde{V}_*. \tag{4b}$$

$Z''(p)$ is analytic on the $j\omega$ axis, since all $j\omega$ axis poles cancel when (4a) is formed. Since $Z''(j\omega) = Z_H(j\omega)$ whenever this Hermitian part is defined, we see that $Z''(j\omega)$ is positive semi-definite, because Z is *PR*.[7] Further, (4b) shows that, at least when X exists, the rank, r, of Z'' satisfies

$$[\text{rank } Z''] = r \leq m = [\text{number of } R\text{'s in } N] \tag{5}$$

since the rank of a product is no greater than that of any term in the product. By using the scattering matrix in a manner similar to that used for reciprocal networks, it can be seen that the rank of Z'' is at most equal to the number of resistors in N even when X doesn't exist.[8] Physically, (5) corresponds to the fact that in the sinusoidal steady state, energy can be channeled independently to only r of the m resistors. Thus we would suspect that only r resistors are needed for synthesis. Using the analysis results obtained in this section, we will obtain such a minimum resistor realization.

III. Synthesis Method

Now consider a given impedance matrix $Z(p)$ which is assumed to be *PR* and for which a positive realization, N, is desired. For such a synthesis we first form Z'' as defined in (4a); this has rank r as defined by (5). As discussed in the Section II, Z'' has no poles on the $j\omega$ axis and thus the least common denominator of the elements of Z'' have no zeros on the $j\omega$ axis. Further, $Z''(j\omega)$ is positive semidefinite and consequently the least common denominator can be written as dd_* with d a monic positive Hurwitz polynomial. The matrix $A(p) = dd_*Z'' = \tilde{A}_*$

[5] C. C. MacDuffee, "The Theory of Matrices," Chelsea Publ. Co., New York, N. Y., p. 81; 1956. The direct sum of two matrices, A and B, is defined as $A \dotplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$.

[6] Y. Oono and K. Yasuura, *op. cit.*, pp. 155–158.

[7] R. W. Newcomb, "On network realizability conditions," Proc. IRE, vol. 50, p. 1995; September, 1962.

[8] V. Belevitch, "Synthèse des réseaux électriques passifs a n paires de bornes de matrice de répartition prédéterminée," *Annales des Télécommunications*, vol. 6, pp. 302–312, esp. p. 305; November, 1951.

TABLE 1
SUMMARY OF SYNTHESIS

Step		Operation		Eq. No. Comments
1)	Factor	$Z'' = \frac{1}{2}[Z + \tilde{Z}_*] = N(dL)^{-1}(dL)_*^{-1}\tilde{N}_*$		(4a), (6), Appendix 2, $dd_* = L.\ C.\ D.\ Z''$; $N = N_*$ if $Z = \tilde{Z}$
		Case 1 $W = \text{Ev}(dL)$ $Y = \text{Od}(dL)$	Case 2 $W = \text{Od}(dL)$ $Y = \text{Ev}(dL)$	(10) To avoid Y singular in Case, 1, can augment L and N.
2)	Form	$V = N[\text{Od}(dL)]^{-1}$	$V = N[\text{Ev}(dL)]^{-1}$	(7a), (9a), (10b), (10d)
3)	Form	$W = [\text{Ev}(dL)][\text{Od}(dL)]^{-1}$	$W = [\text{Od}(dL)][\text{Ev}(dL)]^{-1}$	(7b), (10)
4)	Form	$U = Z - V(W + 1_r)^{-1}\tilde{V}_*$		(11)
5)	Form	$X = \begin{bmatrix} U & V \\ -\tilde{V}_* & W \end{bmatrix}$		(1b) $X = -\tilde{X}_*$, X is PR; if $Z = \tilde{Z}$, case 1 gives $X = \tilde{X}$
6)		Synthesize X as a lossless N_l and terminate last r ports by unit resistors.		Fig. 1(a) Case 2 always requires gyrators.

is (real) polynomial, has rank r and is positive semidefinite on the $j\omega$ axis. Consequently, it can be factored by the Gauss process, see Appendix II, to yield

$$Z'' = N(dL)^{-1}(dL)_*^{-1}\tilde{N}_* \tag{6}$$

where the $n \times r$ matrix N is (real) polynomial and the $r \times r$ matrix L is diagonal with entries being real Hurwitz polynomials. If Z is symmetric then N is even, that is $\tilde{N}_* = \tilde{N}$.

We then wish to identify (4b) and (6). For this we identify r and m and introduce polynomial matrices $\underline{V}$, $\underline{W}$ and $\underline{Y}$, $\underline{Y}$ nonsingular, such that

$$V = \underline{V}\underline{Y}^{-1} \tag{7a}$$

$$W = \underline{W}\underline{Y}^{-1}. \tag{7b}$$

In terms of these (4b) takes the form

$$Z'' = \underline{V}(\underline{W} + \underline{Y})^{-1}(\tilde{\underline{W}}_* + \tilde{\underline{Y}}_*)^{-1}\tilde{\underline{V}}_*. \tag{8}$$

Now comparing (8) with (6) we make the identifications

$$\underline{V} = N \tag{9a}$$

$$\underline{W} + \underline{Y} = dL. \tag{9b}$$

In order to completely specify $\underline{W}$ and $\underline{Y}$, several choices are open. We investigate two cases, from which others can be derived.

Case 1

This is essentially the case covered by Bayard;[9] here we choose $\underline{W}$ even and $\underline{Y}$ odd, that is

$$\underline{W} = \underline{W}_* = \text{Ev}\,(dL) \tag{10a}$$

$$\underline{Y} = -\underline{Y}_* = \text{Od}\,(dL). \tag{10b}$$

Here Ev and Od denote the even and odd parts, respectively; the right sides of (10) are obtained directly from (9b). This choice has the disadvantage that $\underline{Y}$ may be singular, since $\text{Od}(1) = 0$. This singularity can, however, be bypassed by inserting surplus factors in L and N.

Case 2

Here we choose $\underline{W}$ odd and $\underline{Y}$ even, that is

$$\underline{W} = -\underline{W}_* = \text{Od}\,(dL) \tag{10c}$$

$$\underline{Y} = \underline{Y}_* = \text{Ev}\,(dL). \tag{10d}$$

In this case $V = N[Ev\,(dL)]^{-1}$, and, as a consequence, when Z is symmetric, this has the disadvantage of requiring a nonreciprocal network, since $V \neq -V_*$.

In either case, (7) and (9a) determine V and W while (3) is then used to determine U as

$$U = Z - V(W + 1_r)^{-1}\tilde{V}_*. \tag{11}$$

Eq. (1b) is used to form X, with $m = r$, which is PR and satisfies $X = -\tilde{X}_*$, as shown in Section IV. X is then synthesized by known techniques[6] and the resulting N_l is terminated in r unit resistors to yield N. The process is summarized in Table 1 and illustrated by the following simple example.

Example: Consider the PR matrix

$$Z(p) = \begin{bmatrix} 4 & 8\dfrac{p-1}{p+1} \\ 0 & 4 \end{bmatrix} \tag{12}$$

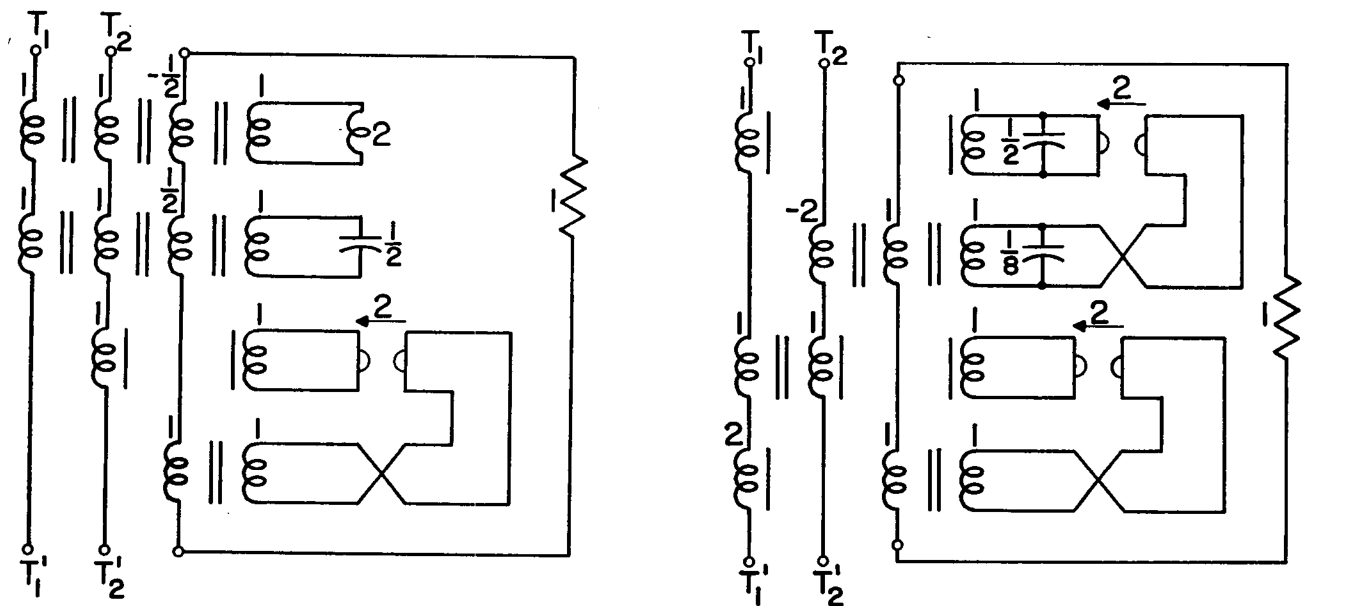
then $d = p + 1$ and we wish to apply the Gauss factorization to

$$dd_*Z'' = 4\begin{bmatrix} dd_* & -d_*^2 \\ -d^2 & dd_* \end{bmatrix}.$$

A possible factorization yields

$$N = \begin{bmatrix} dd_* \\ -d^2 \end{bmatrix}; \qquad dL = [d^2/2].$$

[9] M. Bayard, *op. cit.*, p. 73.



(a) (b)

Fig. 2—Case 1, (a), and Case 2, (b), realizations of (12).

For Case 1 we get $\underline{Y} = [p]$, $\underline{W} = [(p^2 + 1)/2]$ and hence

$$X_1 = p\begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & \frac{1}{2} \end{bmatrix} + \frac{1}{p}\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}. \tag{13a}$$

For case 2 we get $\underline{Y} = [(p^2 + 1)/2]$, $\underline{W} = [p]$ and hence

$$X_2 = \frac{1}{p^2 + 1}\begin{bmatrix} 8p & -8 & 4 \\ 8 & 8p & -4p \\ -4 & -4p & 2p \end{bmatrix} + \begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}. \tag{13b}$$

The final realizations are shown in Fig. 2 where it is seen that Case 2 has used one more gyrator than Case 1. Most of the transformer windings in the figure can be omitted, but are kept for simplicity of drawing.

IV. Validity of the Method

It remains to be shown that the procedure yields a positive realization. Such will be the case if X is lossless (that is X is PR with $X = -\tilde{X}_*$), since then (and only then) lossless synthesis procedures apply.[6]

To see that $X = -\tilde{X}_*$ note that this is true of W, since it is a diagonal matrix of reactance functions. $-\tilde{V}_*$ is inserted as the (2, 1) term in X to guarantee the desired property and $U = -\tilde{U}_*$ since, as shown in Appendix III,

$$U = \tfrac{1}{2}[Z - \tilde{Z}_*] + N(dL)^{-1}W(dL)_*^{-1}\tilde{N}_*. \tag{14}$$

To see that X is PR we apply a known n-port PR test[7] which shows, since $X = -\tilde{X}_*$, that X will be PR if (and only if) it has simple poles lying only on the $j\omega$ axis with positive semidefinite residue matrices. Now W itself has these properties, its entries being reactance functions. Further, the finite poles of V are simple and on the $j\omega$ axis, being those of W, since N is polynomial. Now Z'' can have no pole at infinity and hence neither can $N(dL)^{-1}$. Consequently, by observing the form for V it can have at most a simple pole at infinity, since Ev (dL) or Od (dL) is at most one "degree" smaller than dL (the latter consisting of Hurwitz polynomials). Considering U, we see by the comments just made on V that U has no poles in $Re\ p > 0$, and hence none in $Re\ p < 0$ by $U = -\tilde{U}_*$, since $(W + 1_r)^{-1}$ is clearly analytic in $Re\ p > 0$ (Appendix I). By (14) the poles of U are therefore those of W or of Z and consequently are simple.

To investigate the residue matrices, we form

$$\phi(p) = \bar{x}^*X\bar{x} - \bar{x}_1^*Zx_1 \tag{15}$$

with $\bar{x} = [\bar{x}_1, \bar{x}_2]$ an arbitrary complex $(n + r)$-vector, x_1 an n-vector. Then define

$$y_1(p) = \bar{x}_1^*N \tag{16a}$$

$$y_2(p) = \tilde{N}_*x_1 \tag{16b}$$

which gives

$$y_2(j\omega) = \tilde{y}_1^*(j\omega). \tag{16c}$$

If the results of Table 1 are substituted into (15), using (16) one gets by direct calculation in terms of the separate components (after multiplying numerators and denominators by $(dL)_i$)

Case 1

$$\phi(p) = \sum_{i=1}^{r} \frac{(dL)_i}{[\text{Od}\,(dL)_i]}\left\{\frac{y_{1i}y_{2i}}{(dL)_i^2} + \frac{y_{1i}x_{2i} + x_{2i}^*y_{2i}}{(dL)_i} + x_{2i}^*x_{2i}\,\frac{[\text{Ev}\,(dL)_i]}{(dL)_i}\right\}. \tag{17a}$$

Case 2

$$\phi(p) = \sum_{i=1}^{r} \frac{(dL)_i}{[\mathrm{Ev}\,(dL)_i]} \left\{ -\frac{y_{1i}y_{2i}}{(dL)_i^2} + \frac{y_{1i}x_{2i} - x_{2i}^* y_{2i}}{(dL)_i} + x_{2i}^* x_{2i} \frac{[\mathrm{Od}\,(dL)_i]}{(dL)_i} \right\}. \tag{17b}$$

A pole at $p_0 = j\omega_0$ is a pole of the terms outside the brackets (including ∞), and at such a pole this term has its residue, k_i, that of a reactance function (and hence positive). At such a pole the odd part (for Case 1) or the even part (for Case 2) of $(dL)_i$ vanishes. For Case 1 the residue of $\phi(p)$ at such a pole is then, by the use of (16c),

Case 1

$$k_0 = \sum_i k_i \left[\frac{y_{1i}^*}{[\mathrm{Ev}\,(dL)_i]} + x_{2i}^* \right] \cdot \left[\frac{y_{1i}}{[\mathrm{Ev}\,(dL)_i]_*} + x_{2i} \right] \Bigg|_{p=j\omega_0} \tag{18}$$

where the sum is taken over those i pertinent to the pole; for case 2, Ev $(dL)_i$ is replaced by Od $(dL)_i$ in (18). Clearly for both cases $k_0 \geq 0$ which proves the semi-definiteness of the residue matrices for X. Consequently X is a lossless impedance matrix.

V. Comments and Conclusions

In this paper we have presented an extension of Bayard's method to cover the synthesis of nonreciprocal networks. The method is based upon a Gauss factorization of Z'' and is summarized in Table 1. Several possible methods of proceeding are open, two of which were covered in detail. The first Case corresponds to that used by Bayard and reduces to his synthesis when Z is symmetric. We do wish to point out that this method may require augmentation (of L) before it can be carried out. However, the situations requiring augmentation are somewhat trivial, since for them $d = 1$. The Case 2 synthesis can always be directly carried out, but an excess number of gyrators may be required, since gyrators are needed when Z is symmetric. In either case the minimum number r = rank Z'' of resistors is used. In contrast to this, more than the minimum number of L's and C's may be required. This can be seen from the example treated, where two reactive elements are required. Since Z for the example has "degree" one, the results of Oono and Yasuura show that it can be realized by one reactive element.[10]

One can make some comments on equivalent circuits. Many equivalent circuits can be obtained by using different types of factorizations of Z'', the one given by Youla can be used for example.[11] Or one can factor by the Gauss process and insert para-unitary matrices (ones satisfying $\theta\tilde{\theta}_* = 1_r$, θ being $r \times r$). That is, another factorization is

$$Z'' = [N(dL)^{-1}\theta][\tilde{\theta}_*(dL)_*^{-1}\tilde{N}_*]. \tag{19}$$

In fact many such θ can be written in the form

$$\theta = (1_r - \phi)^{-1}(1_r + \phi) \tag{20a}$$

$$\phi = -\tilde{\phi}_* \tag{20b}$$

with ϕ otherwise arbitrary (however $\theta = -1_r$ isn't of this form); here if θ is given, $\phi = (\theta - 1_r)(\theta + 1_r)^{-1}$. Note that there are many equivalent circuits which can't be obtained this way, for instance those with no X matrix or more than r resistors. To obtain a desired realization the most reasonable procedure would be to use the synthesis method described here. This yields a synthesis method, requiring relatively simple calculations, to which the results of Oono and Yasuura can be applied to obtain (almost) every equivalent circuit and in particular the one with desired properties.[1]

Also we point out that the method yields a synthesis of realizable scattering matrices, S. For this, one forms $Z = 2(1_n - S)^{-1} - 1_n$ if $1_n - S$ is nonsingular. If $1_n - S$ is singular, one forms $T_0 S \tilde{T}_0 = S_0 \dotplus 1_{n-\rho}$ with T_0 a constant orthogonal matrix (representing transformers) and $1_\rho - S_0$ is nonsingular. This is followed by a realization of $Z_0 = 2(1_\rho - S_0)^{-1} - 1_\rho$.[12]

It should also be observed that the method requires transformers. However, if a transformerless network exists it will be among those found by using Oono's technique. Further if $Z'' = 0_n$, Z is already lossless and $X = Z$.

Appendix

I. Nonsingularity of $W + 1_m$, W PR

Consider any PR $m \times m$ matrix W; $W + 1_m$ can be seen to be nonsingular as follows.

Consider any fixed p_0 in $Re\ p > 0$ and assume that $W + 1_m$ is singular at p_0. Then there exists a nonzero, complex constant m-vector x_0 such that $(W + 1_m)x_0 = 0$. Consequently $\tilde{x}_0^*(W + 1_m)x_0 = \tilde{x}_0^* W x_0 + \tilde{x}_0^* x_0 = 0$ which is a contradiction, since W being PR requires $\tilde{x}_0^* W x_0 \geq 0$ while clearly $\tilde{x}_0^* x_0 > 0$. Since this holds for any p in $Re\ p > 0$, the desired result follows. Note that $W + 1_m$ is PR being the sum of two PR matrices and consequently $(W + 1_m)^{-1}$ is PR and thus analytic in $Re\ p > 0$. This result is of course familiar to those accustomed to the scattering matrix.[13]

II. Gauss Factorization

Here we extend the Gauss factorization, given by Bayard[4] for symmetric matrices, to nonsymmetric matrices. Consider an $n \times n$ matrix $A(p)$ of real polynomials, such that $A = \tilde{A}_*$, A has rank r, and $A(j\omega)$ is positive semi-definite. Then, as we now show, we can write

$$A = NL^{-1}L_*^{-1}\tilde{N}_* \tag{21}$$

[10] Y. Oono and K. Yasuura, *op. cit.*, p. 168.

[11] D. C. Youla, "On the factorization of rational matrices," IRE Trans. on Information Theory, vol. IT-7, pp. 172–189; July, 1961.

[12] Y. Oono and K. Yasuura, *op. cit.*, p. 155.

[13] D. C. Youla, L. J. Castriota and H. J. Carlin, "Bounded real scattering matrices and the foundations of linear passive network theory," IRE Trans. on Circuit Theory, vol. CT-6, pp. 102–124, esp. p. 123; March, 1959.

where L is a $r \times r$ diagonal matrix of real Hurwitz polynomials and N is a $n \times r$ matrix of real polynomials.

To see that this is possible we proceed in the following recursive manner where a superscript i will refer to the ith step. Since A has rank r, there is a $r \times r$ nonsingular submatrix which we permute by the permutation matrix P to the upper left $r \times r$ corner; $A = PA^1\tilde{P}$. By the semidefiniteness of $A^1 = [a^1_{ij}]$ any $j\omega$ axis zeros of a^1_{11} are even (those at zero are of order $4k$, k = integer), and thus by the realness such factors can be combined as $\phi_1\phi_{1*}$ where $\phi_1 = \phi_{1*}$ are real polynomials. By the semidefiniteness of A^1, ϕ_1 factors its first row and column (consider 2×2 submatrices which must have nonnegative determinants). Consequently we can write $A^1 = (\phi_1 \dotplus 1_{n-1})Q^1(\phi_{1*} \dotplus 1_{n-1})$ with $Q^1 = [q^1_{ij}]$ having all the properties of A^1, and hence of A, except q^1_{11} has no $j\omega$ axis zeros. Q^1 can be factored, as is easily checked by direct multiplication, into

$$Q^1 = (1/q^1_{11}) \cdot \left[\begin{array}{c|c} q^1_{11} & 0 \cdots 0 \\ \hline q^1_{21} & \\ \vdots & 1_{n-1} \\ q^1_{n1} & \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & [a^2_{ij}] \\ 0 & \end{array}\right] \left[\begin{array}{c|c} q^1_{11*} & q^1_{21*} \cdots q^1_{n1*} \\ \hline 0 & \\ \vdots & 1_{n-1} \\ 0 & \end{array}\right] \tag{22a}$$

$$a^2_{ij} = q^1_{11}q^1_{ij} - q^1_{i1}q^1_{1j}; \qquad i, j, = 2, \cdots, n. \tag{22b}$$

The above procedure, with the permutation omitted, can now be repeated on $[a^2_{ij}]$. Since $\tilde{Q}^i_* = Q^i$, q^i_{ii} is an even polynomial and can be factored into $q^i_{ii} = \theta_i\theta_{i*}$ where θ_1 is a real Hurwitz polynomial. We can then finally obtain

$$A = N\{(1/q^1_{11}) \dotplus (1/q^1_{11}q^2_{22}) \dotplus \cdots \dotplus (1/q^1_{11}q^2_{22} \cdots q^r_{rr})\}\tilde{N}_* \tag{23a}$$

where

$$N = P\begin{bmatrix} q^1_{11}\phi_1 & 0 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ q^1_{21} & q^2_{22}\phi_2 & 0 & & \vdots \\ q^1_{31} & q^2_{32} & q^3_{33}\phi_3 & & 0 \\ \vdots & \vdots & \vdots & \ddots & q^r_{rr}\phi_r \\ \vdots & \vdots & \vdots & & \vdots \\ q^1_{n1} & q^2_{n2} & q^3_{n3} & \cdots & q^r_{nr} \end{bmatrix} \tag{23b}$$

and writing diag for diagonal,

$$L = \text{diag.}\,[\theta_1, \theta_1\theta_2, \cdots, \theta_1\theta_2 \cdots \theta_r]. \tag{24}$$

We note that N is not unique, since 1) many permutations P can be used, 2) the multipliers to choose with the ϕ_i and θ_i are optional. A somewhat different factorization has been given by Youla.[11] However Youla's differs in several aspects, the most important being in that ours has the right inverse, N^{-1}, analytic in $Re\ p < 0$. Note that if $Z = \tilde{Z}$ then (23b) has $N = N_*$, that is the entries of N are even polynomials when Z is symmetric.

III. Evaluation of U

Here we show that in both the Case 1 and Case 2 synthesis (14) is valid. We have by (11), $U = Z - V(W + 1_r)^{-1}\tilde{V}_*$ which can be rewritten as

$$\begin{aligned} V &= \tfrac{1}{2}[Z - \tilde{Z}_*] + Z'' - V(W + 1_r)^{-1}\tilde{V}_* \\ &= \tfrac{1}{2}[Z - \tilde{Z}_*] + N(dL)^{-1}(dL)^{-1}_*\tilde{N}_* \\ &\qquad - V(W + 1_r)^{-1}\tilde{V}_*. \end{aligned} \tag{25}$$

For Case 1, inserting V and W as given in Table 1,

$$\begin{aligned} U &= \tfrac{1}{2}[Z - \tilde{Z}_*] + N(dL)^{-1}(dL)^{-1}_*\tilde{N}_* \\ &\qquad - N(dL)^{-1}[\text{Od}\ (dL)]^{-1}_*\tilde{N}_* \\ &= \tfrac{1}{2}[Z - \tilde{Z}_*] + N(dL)^{-1} \\ &\qquad \cdot\{(dL)^{-1}_* - [\text{Od}\ (dL)]^{-1}_*\}\tilde{N}_* \\ &= \tfrac{1}{2}[Z - \tilde{Z}_*] + N(dL)^{-1} \\ &\qquad \cdot\{1_r - [\text{Od}\ (dL)]^{-1}_*(dL)_*\}(dL)^{-1}_*\tilde{N}_*. \end{aligned} \tag{26}$$

But

$$\begin{aligned} 1_r - [\text{Od}\ (dL)]^{-1}_*(dL)_* &= [\text{Od}\ (dL)_*]^{-1}[\text{Od}\ (dL)_* - (dL)_*] \\ &= [\text{Od}\ (dL)_*]^{-1}[\tfrac{1}{2}(dL)_* - \tfrac{1}{2}(dL) - (dL)_*] \\ &= [\text{Od}\ (dL)]^{-1}[\text{Ev}\ (dL)]. \end{aligned}$$

This last is equal to W since the matrices commute, being diagonal. Similarly, if we consider Case 2, the Od in (26) is replaced by Ev and we then show by identical calculations that $W = 1_r - [Ev\ (dL)]^{-1}_*\ (dL)_*$. Consequently, in both cases we obtain

$$U = \tfrac{1}{2}[Z - \tilde{Z}_*] + N(dL)^{-1}W(dL)^{-1}_*\tilde{N}_*. \tag{27}$$

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