For example, suppose $(x_1, y_1) \notin B_1$ and $(x_i, y_i) \in B_1$ for all $1 < i \leq n$. Then $\dot{V}_1 < 0$, but the other $\dot{V}_i$ may be sufficiently positive that $\sum_{i=1}^n \dot{V}_i > 0$.

Thus, although I and II are sufficient conditions for the boundedness of solutions of (3) in the uncoupled case, this result cannot be arrived at on the basis of the Lyapunov function given by (4), (8), and (9). It is by no means clear that in every case an α and β can be found such that $\dot{V}$ is negative outside of B (note that a change in α changes V). This situation would not occur if one were proving asymptotic stability. For if the Lyapunov functions $V_i(x_i, y_i)$ were such that $\dot{V}_i(x_i, y_i) < 0$ for $(x_i, y_i) \neq (0, 0)$, then $V(x, y) = \sum_{i=1}^n V_i(x_i, y_i)$ would have a negative definite derivative along solutions of (3). It is only when some of the V_i can have positive derivatives (as when proving boundedness) that weaker results are obtained. Obviously, the same difficulty illustrated above will occur when coupling exists, and therefore any results along these lines will be very conservative.

Example 2: Consider the case when $n = 2$ and

$$G(x_1, x_2) = \begin{bmatrix} \omega_1^2 + \xi & -\xi \\ -\xi & \omega_2^2 + \xi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{12}$$

Thus the coupling is linear and enters with coefficient ξ. Clearly, for $\dot{V}$ of (5) with $H(x)$ in the form of (8) and (9) to be negative for $\|x\|$ large, it is necessary that $G^t(x)F(x) \to \infty$ as $\|x\| \to \infty$. Suppose that the $F_i(x_i)$ are asymptotic to $\gamma_i x_i^p$ as $|x_i| \to \infty$ where p is odd. Then for large x_1, x_2, $G^t(x)F(x)$ is asymptotic to

$$Q(x_1, x_2, \xi) = [x_1 x_2] \begin{bmatrix} \omega_1^2 + \xi & -\xi \\ -\xi & \omega_2^2 + \xi \end{bmatrix} \begin{bmatrix} \gamma_1 x_1^p \\ \gamma_2 x_2^p \end{bmatrix}. \tag{13}$$

The partial derivative of Q with respect to ξ is then seen to be

$$\frac{\partial Q}{\partial \xi}(x_1, x_2, \xi) = \gamma_1 x_1^{p+1} - \gamma_1 x_2 x_1^p - \gamma_2 x_1 x_2^p + \gamma_2 x_2^{p+1}. \tag{14}$$

Consider the ray $x_2 = m x_1$. Along this ray

$$\frac{\partial Q}{\partial \xi}(x_1, m x_1, \xi) = \gamma_2 x_1^{p+1}(m - 1)\left(m^p - \frac{\gamma_1}{\gamma_2}\right). \tag{15}$$

Thus, unless $\gamma_1 = \gamma_2$, there will exist a slope m such that $\partial Q/\partial \xi(x_1, m x_1, \xi) < 0$. Since for a given x_1, x_2, $\partial Q/\partial \xi$ is constant, it is seen that for ξ sufficiently large there will exist a ray $(x_1, m x_1)$ such that $Q(x_1, m x_1, \xi) < 0$ for large x_1. Thus if $\gamma_1 \neq \gamma_2$, it follows that the condition $G^t(x)F(x) \to \infty$ as $\|x\| \to \infty$ will hold only for small coupling coefficient ξ.

Example 3: Consider again the case when $n = 2$ and G is given by (12). Further, assume that $F_1(x_1)$ is asymptotic to $\gamma_1 x_1^p$ and $F_2(x_2)$ is asymptotic to $\gamma_2 x_2^q$ as x_1 and $x_2 \to \infty$, p and q odd, and $p > q$. Along the ray $x_2 = m x_1$, $Q(x_1, x_2, \xi)$ given in (13) becomes

$$Q(x_1, m x_1, \xi) = x_1^{p+1}\gamma_1[(\omega_1^2 + \xi) - \xi m] + x_1^{q+1} m q[(\omega_2^2 + \xi)\gamma_2 m - \xi \gamma_2]. \tag{16}$$

Thus for $m > (\omega_1^2 + \xi)/\xi$, the coefficient of x_1^{p+1} is negative, and since $p > q$, it follows that $Q(x_1, m x_1, \xi) < 0$ for x_1 sufficiently large. Hence, in this case if $\xi \neq 0$, the condition $G^t(x)F(x) \to \infty$ as $\|x\| \to \infty$ cannot be met, and the Lyapunov function of (4) will not prove boundedness. Clearly, by exchanging the roles of x_1 and x_2, the same result can be shown to hold if $q > p$.

From the above examples, one can conclude that the Lyapunov function of (4), (8), and (9) will 1) give very conservative results; 2) only prove boundedness for "small" coupling coefficients; and 3) will not give results in the case when the asymptotic behaviors of the $F_i(x_i)$ as the $x_i \to \infty$ are of different orders.

Numerical solutions of the equations

$$\begin{aligned} \dot{x}_1 &= y_1 - \mu_1(A_1 x_1^5/5 + A_2 x_1^3/3 + A_3 x_1) \\ \dot{y}_1 &= -x_1 - c(x_1 - x_2) \\ \dot{x}_2 &= y_2 - \mu_2(B_1 x_2^5/5 + B_2 x_2^3/3 + B_3 x_2) \\ \dot{y}_2 &= -x_2 + c(x_1 - x_2) \end{aligned}$$

for various values of parameters have been obtained by using a method due to Nordsieck [7]. It was observed that in each case the solution stays bounded [8]. From the conditions for the boundedness of a single oscillator, one may be led to the conjecture that the coupled oscillators are ultimately bounded if $G^t(x)F(x) \to \infty$ as $\|x\| \to \infty$. However, numerical results suggest that the solution for coupled oscillators are bounded under much weaker conditions.

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Multiple DC Solution One-Transistor Circuits

A. BARANYI AND ROBERT W. NEWCOMB

Abstract—**The necessary and sufficient conditions are given for the existence of multiple dc solutions of circuits consisting of one ideal Ebers–Moll transistor embedded in an admittance-described circuit consisting of dc sources, linear passive resistors, and ideal transformers. The results rest upon the previous theories of Sandberg and Willson.**

I. Introduction

Recently the nonlinear dc equations of networks containing transistors represented by the Ebers–Moll model have been studied by Sandberg and Willson [1]–[4]. Among the interesting results obtained is the fact that a circuit consisting solely of passive resistors, batteries, and one transistor has a unique dc solution [1, p. 29], [2, p. 1304], if a solution exists. However, if one adds transformers to the list of allowed components, then multiple dc solutions appear possible with only one transistor, as the blocking oscillator indicates [5, p. 251].

Here we study this situation. In particular, we use the formulation of Sandberg and Willson to obtain the conditions under which a circuit consisting of one transistor, passive resistors, transformers, and batteries has multiple solutions.

II. Review of the Sandberg–Willson Results

Consider a circuit containing one transistor connected to a network constructed of dc sources, ideal transformers, and (linear, passive) resistors, as shown in Fig. 1. Then if the Ebers–Moll model [5, p. 100] is used to describe the transistor, the resulting equations most often can be written as [1, p. 22]

$$F(x) + Ax = B \tag{1a}$$

$$A = T^{-1}G, \quad T = \begin{bmatrix} 1 & -\alpha_f \\ -\alpha_r & 1 \end{bmatrix}, \quad \begin{matrix} 0 < \alpha_f < 1 \\ 0 < \alpha_r < 1. \end{matrix} \tag{1b}$$

Here G is the admittance matrix of the connecting two-port of Fig. 1, x is the two-vector of base-to-emitter and collector voltages, B is a two-vector linearly proportional to source parameters, and $F(\cdot)$ is a strictly monotone increasing two-vector (Ebers–Moll equation exponentials) mapping.

Of key importance to the uniqueness of solutions of (1) is the set P_0 of matrices, all principal minors of which are nonnegative [1, p. 5]. In particular, if $A \in P_0$, then any solution of (1a) is unique [1, p. 10], while if $A \notin P_0$, then there exists a B for which there are at least two solutions [3, p. 105].

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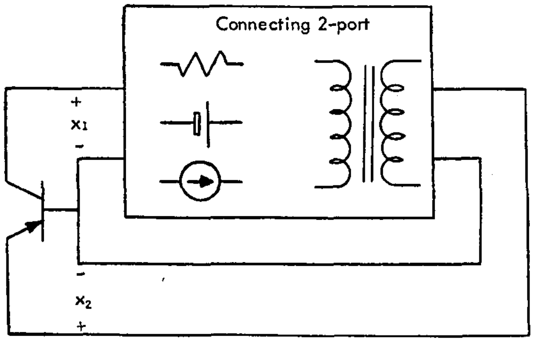


Fig. 1. Transistor connection.

III. Main Results

In view of the last mentioned results, we wish to find the conditions on the 2×2 matrices G, T, A such that $A \notin P_0$; G is PR and symmetric and $0<\alpha_f, \alpha_r<1$. We first note that

$$G = TA = \begin{bmatrix} a_{11} - \alpha_f a_{21} & a_{12} - \alpha_f a_{22} \\ a_{21} - \alpha_r a_{11} & a_{22} - \alpha_r a_{12} \end{bmatrix} \tag{2a}$$

$$A = T^{-1}G = \frac{1}{1-\alpha_f\alpha_r}\begin{bmatrix} g_{11} + \alpha_f g_{21} & g_{12} + \alpha_f g_{22} \\ g_{21} + \alpha_r g_{11} & g_{22} + \alpha_r g_{12} \end{bmatrix} \tag{2b}$$

$$\det G = (1 - \alpha_f\alpha_r) \det A \tag{2c}$$

where det denotes the determinant. From this last part we note that $\det A \geq 0$ since G, being PR and symmetric, requires $\det G \geq 0$ [6, pp. 117, 357]. Consequently, we concentrate on forcing $a_{11}<0$ or $a_{22}<0$ and, by an interchange of ports it is immaterial which, to obtain $A \notin P_0$.

Theorem 1

Given a PR G, α_f and α_r $(0<\alpha_f, \alpha_r<1)$ exist such that $a_{11}<0$ $(A\notin P_0)$ if and only if

$$0 \leq g_{11} < -g_{21}. \tag{3}$$

(The proof is immediate from the (1, 1) entry of (2b) and by then choosing any α_r, $0<\alpha_r<1$ with $1>\alpha_f>g_{11}/-g_{21}$.) Physically, Theorem 1 says that given a passive resistive network (even allowing gyrators), an Ebers–Moll transistor exists that, when connected as in Fig. 1 with an appropriate set of dc sources, yields at least two dc solutions. Fig. 2, for example, shows two two-ports satisfying (3), but we rule out the nonreciprocal gyrator since its physical construction would most likely use transistors [7, p. 39], violating the one-transistor assumption.

Corollary

Given an Ebers–Moll transistor of parameters α_f and α_r $(0<\alpha_f, \alpha_r<1)$, there exists a PR symmetric G such that $A\notin P_0$.

We can simply take

$$G = g\begin{bmatrix} 1 & -n \\ -n & n^2 \end{bmatrix}, \quad \begin{matrix} n > 1/\alpha_f \\ g > 0 \end{matrix} \tag{4}$$

in which case $a_{11}<0$ results from the single resistor transformer circuit of Fig. 2(a).

Theorem 2

Given $A\notin P_0$, an ideal Ebers–Moll transistor $(0<\alpha_f, \alpha_r<1)$ exists, resulting in a symmetric PR G if and only if (possibly after an interchange of port numbers)

$$a_{11} < 0 \tag{5a}$$

$$a_{22} > 0 \tag{5b}$$

$$0 < \frac{a_{11}}{a_{21}} < 1 \tag{5c}$$

$$\det A \geq 0 \tag{5d}$$

$$a_{12} - a_{21} < a_{22} - a_{11}. \tag{5e}$$

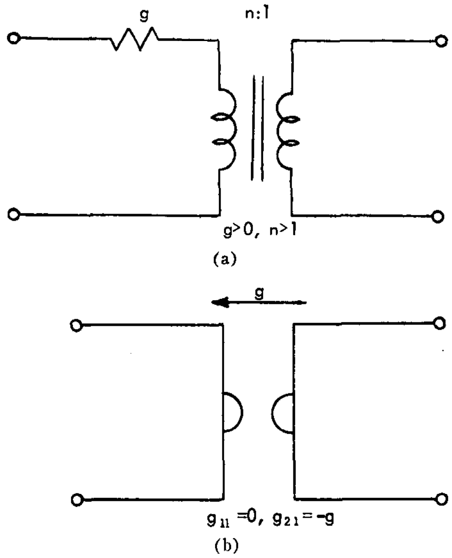


Fig. 2. Possible passive portions of connecting 2-ports for multiple dc solutions. (a) Reciprocal. (b) Nonreciprocal.

Outline of Proof: Equation (5a) is a simple consequence of $A\notin P_0$. Equations (5a) and (5c) are necessary and sufficient to take, for α_f,

$$0 < \frac{a_{11}}{a_{21}} < \alpha_f < 1 \tag{6}$$

resulting in a positive (1, 1) entry of G. The necessity of (5d) is clear from (2c).

We prove the necessity of (5b) by contradiction. Suppose that $a_{22}<0$; then for a positive (2, 2) entry in G, we need, similarly to (6),

$$0 < \frac{a_{22}}{a_{12}} < \alpha_r < 1. \tag{7}$$

Equations (6) and (7) result in negative det A; hence we see the necessity of (5b).

The necessity of (5e) is a consequence of the reciprocity of G, together with (5a):

$$a_{12} - a_{21} = \alpha_f a_{22} - \alpha_r a_{11} < a_{22} - a_{11}.$$

To prove the sufficiency of (5b), (5d), and (5e), we calculate α_r from the reciprocity relation

$$\alpha_r = \frac{a_{21} - a_{12} + \alpha_f a_{22}}{a_{11}}. \tag{8}$$

Applying (6), (5d), and (5e), we have the following inequalities:

$$0 < \frac{a_{21}^2 - a_{12}a_{21} + a_{22}a_{11}}{a_{11}a_{22}} < \alpha_r < \frac{a_{21} - a_{12} + a_{22}}{a_{11}} < 1.$$

Hence (6) and (8) result in an Ebers–Moll transistor with $0<\alpha_r<1$. To complete the proof, we show that the (2, 2) entry is positive. From (5e) and (5c)

$$a_{22} - a_{21} > a_{11} - a_{21} > 0.$$

As a consequence of (5a), (5b), (5c), and (5d), a_{12} is positive; thus

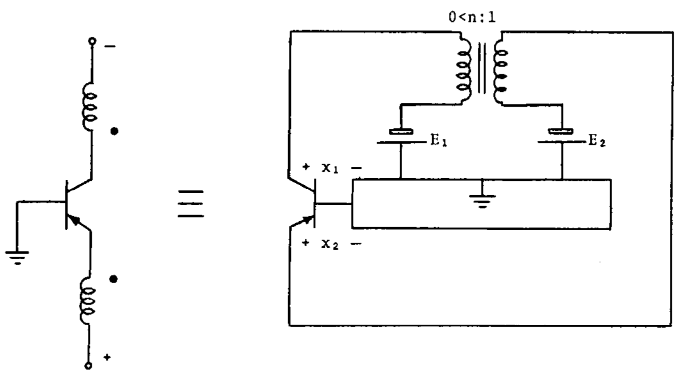
$$a_{22}/a_{12} > 1 > \alpha_r.$$

IV. Examples

Example 1

$$A = \begin{bmatrix} -1 & 4 \\ -4 & 12 \end{bmatrix}.$$

This satisfies the conditions of Theorem 2 for which $1>\alpha_f>a_{11}/a_{21} = 1/4$ while $a_{22}/a_{12}=3>\alpha_r>(a_{12}-a_{21}-a_{22})/-a_{11}=-4$. The reciprocity constraint (8) requires $\alpha_f=(8/12)-(1/12)\alpha_r$, which shows the linear relationship between α_r and α_f. Choosing $\alpha_r=1/2$ gives



(a) (b)

Fig. 3. Example circuit with no admittance description.

$\alpha_f = 0.625$. Hence (2a) gives

$$G = \begin{bmatrix} 1.5 & -3.5 \\ -3.5 & 10 \end{bmatrix}$$

which is seen to be PR. Note that its realization requires another resistor in parallel with the right-port of Fig. 2(a).

Example 2

$$A = \begin{bmatrix} -1 & x \\ -2 & 2 \end{bmatrix}.$$

For large $x > 0$ this satisfies (5) except for the reciprocity constraint of (5e). Since the determinant of the symmetric part varies as $-x^2/4$ for large x, the resultant G cannot be PR. Hence even if a nonreciprocal connecting network is allowed, there is no transistor to give multiple solutions through a passive connecting two-port for this A for large x.

Example 3

A reviewer has suggested the circuit of Fig. 3(a), which when idealized to the dc situation, becomes that of Fig. 3(b). In this idealized situation we obtain

$$\begin{bmatrix} 0 & 0 \\ n - \alpha_r & 1 - \alpha_f n \end{bmatrix} F(x) + \begin{bmatrix} -1 & n \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} E_1 - nE_2 \\ 0 \end{bmatrix}$$

which is outside the scope of (1) but is covered in [4]. Writing this as $AF(x) + Bx = C$, the condition for multiple solutions is that det $(AD + B) = 0$ for some diagonal $D > 0$ [4, p. 1727]. Expanding this relation, we find that for some $d_1, d_2 > 0$ we require

$$(1 - \alpha_f n)d_2 + (n - \alpha_r)nd_1 = 0$$

from which follow the conditions for multiple dc solutions: $n < \alpha_r$ or $n > 1/a_f$.

V. Discussion

We have shown the necessary and sufficient conditions for obtaining multiple dc solutions with one ideal Ebers–Moll transistor embedded in a resistive network consisting of batteries, ideal transformers, and passive resistors, for which the latter two form a two-port described by an admittance matrix. From previous results it is known that at least two solutions exist for appropriate choices of batteries [3, p. 105], but from the examples we have tried, it appears that most often three solutions result.

Previously it had been proven that if G is dominant, then at most one solution exists [1, p. 29], and this is reflected in Theorem 1, where (3) shows that G must be nondominant. But this is the only constraint on G over and above its assumed PR nature. But since nonsymmetric PR G matrices require gyrators in a synthesis using only passive components, we have sought symmetric G at (4) and (5e) to rule out gyrators. As a consequence, the resulting connecting network requires a transformer, as shown by Fig. 1(a), to obtain the nondominant G.

Concerning the use of nonsymmetric G, more general A could be realized by their use since (5e) is relaxed [but only it among (5)], but as shown by Example 2, not all A satisfying (5a)–(5d) can be realized by a PR nonsymmetric G. However, as mentioned earlier, the nonsymmetric G case would undoubtedly use further transistors in constructing the gyrators (or their equivalents needed for nonreciprocity), invalidating the one-transistor assumption.

It should be noted that Fig. 1 will allow n-p-n transistors, as well as p-n-p, by a reversal of x polarities, in which case the theory is unchanged. Further, if $\alpha_f > \alpha_r$, as desired for physical transistors, does not result from the theory, this can be obtained by a reversal of emitter collector terminals.

The treatment rests upon ideal Ebers–Moll transistors. However, transistor series lead resistance can be accounted for by letting G' replace G where [2, p. 1300] (1_2 is the 2×2 identity)

$$G' = [1_2 + GR]^{-1}G \tag{9a}$$

$$R = \begin{bmatrix} r_c + r_b & r_b \\ r_b & r_e + r_b \end{bmatrix} \text{(lead resistance).} \tag{9b}$$

As a consequence, the present methods hold, except that rather than requiring G' to be simply PR, we require $R^{-1} - G'$ to be PR (since $G = G'(1_2 - RG')^{-1}$ and G' describes the connecting network with R embedded in it and G the connecting network with R removed).

The work of man doth sparkle and resound
Enduring but an hour, and then is gone.
Madách [8, p. 18]

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An Exponential Voltage-Controlled Oscillator Improves the Performance of a Phase-Locked Loop

JAMES C. ROCK

***Abstract*—A design example is worked out to demonstrate how to stabilize a phase-locked loop containing an exponential voltage-controlled oscillator. It is shown that this kind of loop can be designed with an arbitrarily large capture range, a wide lock-in range, and a short transient receovery-time constant.**

Introduction

The phase-locked loop reported here represents a portion of the "Recursive Plane Plotter" (RPP) designed and constructed by the author to aid his study of phase locking in nonperiodic signal-dependent samplers [5]. One of the design goals met in the RPP was the construction of a sawtooth generator to serve as the phase reference voltage for any periodic input signal with a frequency between 15 Hz and 15 kHz. A new digital phase-frequency detector (Motorola MC 4044) was chosen for this application because its unique static transfer characteristic prevents push-away and provides a capture range limited only by the voltage-controlled oscillator (VCO) dynamic range [2]–[4]. Because the dynamic range of the output of the active filter contained on the MC 4044 is limited to 10 dB, logarithmic compression in the form of the exponential VCO was chosen to achieve the desired 60-dB output frequency range.

The Analytic Model

A small-signal linearized model for the phase-locked ramp generator shown in Fig. 1(a) is derived in this section.

The phase detector has a three-state output. When the negative transitions of the signals applied to its variable input (VI) and ref-

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