the tth instant of time. Hence it can be shown that

$$\pi(t+1) = \pi(t)\,P(t) \qquad (2)$$

By iteration, eqn. 2 yields

$$\pi(t) = \pi(0) \prod_{j=0}^{t-1} P(j) \qquad (3)$$

where $\pi(0)$ is the automaton initial-state probability-distribution vector

Following the foregoing discussion, it can be shown that $P(t)$ can be written as

$$P(t) = P_s + \varepsilon f(t)\,P_T \quad \forall t \qquad (4)$$

where $\varepsilon f(t)$ replaces $\varepsilon_k f_k(t)$ of eqn. 1a and

$$P_s = P(t)|_{\varepsilon=0} \qquad (5)$$

and P_T can be found with the knowledge of the particular structure of the automaton.

Substituting eqn. 4 in eqn. 3 yields

$$\pi(t) = \pi(0) \prod_{j=0}^{t-1} \{P_s + \varepsilon_f(j) P_T\}$$

To facilitate the analysis, assume a small ε. The Maclaurin-series expansion of $\pi(t)$, taking only the 1st-order approximation, then yields

$$\pi(t) \simeq \pi(t)\Big|_{\varepsilon=0} + \varepsilon \frac{d}{d\varepsilon}\{\pi(t)\}\Big|_{\varepsilon=0} \qquad (6)$$

Now

$$\pi(t)\Big|_{\varepsilon=0} = \pi(0) \prod_{j=0}^{t-1} P_s = \pi(0)\,P_s^t \qquad (7)$$

where P_s^t is the matrix P_s raised to the tth power. Further, it can be shown that

$$\frac{d}{d\varepsilon}\{\pi(t)\}\Big|_{\varepsilon=0} = \pi(0) \sum_{j=0}^{t-1} P_s^j f(j)\,P_T\,P_s^{t-j-1} \qquad (8)$$

Hence, substituting eqns. 7 and 8 in eqn. 6 yields

$$\pi(t) = \pi(0)\left\{P_s^t + \varepsilon \sum_{j=0}^{t-1} P_s^j f(j)\,P_T\,P_s^{t-j-1}\right\} \qquad (9)$$

Assuming ergodicity, let π_s be the automaton steady-state probability-distribution vector. Also, without loss of generality, let $\pi(0) = \pi_s$. Then, $\forall k$, $\pi(0)\,P_s^k \rightarrow \pi_s$. Hence, eqn. 9 yields

$$\pi(t) = \pi_s + \varepsilon \pi_s \sum_{j=0}^{t-1} f(j)\,P_T\,P_s^{t-j-1} \qquad (10)$$

Let P_s^τ be the transpose of the matrix P_s, and assume that it is possible to find a nonsingular matrix B such that $P_s^\tau = BDB^{-1}$, where D is the Jordan canonical matrix. Further, assume that all the eigenvalues λ_i of P_s^τ are distinct, i.e. $D = \text{diag}\,[\lambda_i]$. Hence

$$P_s^\tau = B\,\text{diag}\,[\lambda_i]\,B^{-1} \qquad (11)$$

Taking the transpose of eqn. 10 yields

$$\{\pi(t)\}^\tau = \pi_s^\tau + \varepsilon \sum_{j=0}^{t-1} f(j)\,\{P_s^{t-j-1}\}^\tau\,P_T^\tau\,\pi_s^\tau$$

Finally, substituting eqn. 11 in this expression,

$$\{\pi(t)\}^\tau = \pi_s^\tau + \varepsilon B\,\text{diag}\left\{\sum_{j=0}^{t-1} f(j)\,\lambda_r^{t-j-1}\right\} B^{-1}\,P_T^\tau\,\pi_s^\tau \qquad (12)$$

As can be seen, eqn. 12 describes the automaton behaviour in a nonstationary random environment. To continue the analysis, it is necessary to know $f(t)$ and $P(t)$. However, although we leave the analysis at this point, the significance of the foregoing discussion should be noted in view of the introductory remarks to this letter.

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PASSIVE *N*-PATH-FILTER REALISATION

Indexing terms: Passive filters, Gyrators, State-space methods

Through the intermediary of state-variable equations, an equivalence is obtained which shows how passive *RC* circuits incorporating time-variable gyrators can be used to realise *N*-path filters. The effects of gyrator losses in the realisation of time-invariant transfer functions are briefly discussed to indicate the practical significance of the result.

Introduction: One of the basic reasons proposed for the introduction of *N*-path filters is low sensitivity to parameter variations,[1] this sensitivity being compared with that available in active *RC* circuits. In achieving this result, however, other, and possibly more significant, advantages became apparent, such as the possibility of obtaining periodic filtering characteristics and the possibility of having low-frequency filters free of magnetic elements.[1] Since it is also intuitively true that passive circuits have inherently low sensitivities when compared with active ones, it would seem most profitable to obtain passive realisations of *N*-path filters. Here we show that this is possible through an application of state-variable synthesis methods.[2,3]

Our procedure is to obtain results for a specific but practically important case, that of a degree-2 filter. Then, by physically reasoning on this, we see that any *N*-path filter can be realised by a similar configuration. The configuration obtained is actually ideal for integrated-circuit construction, since it allows all components to have a common earth, and the components (*R*, *C* and time-variable gyrators) are passive and conveniently realised by integrated circuits.[4]

Passive realisations: The *N*-path-filter configuration is as shown in the block-diagram form of Fig. 1 (Reference 1, p. 1322) where the $p_i(t)$ and $q_i(t)$ are appropriate modulating functions; in practice, the filters inserted between modulators are lowpass *RC* filters.

As an example to be considered in the development here, a practical choice for the parameters yields the 2-path filter of Fig. 2a. A straightforward analysis of this latter configuration yields the state-variable description

$$\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\alpha & 0\\ 0 & -\alpha\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\sin\gamma t\\ \cos\gamma t\end{bmatrix} v_1 = Ax + Bv_1 \qquad (1a)$$

$$v_2 = [\sin\gamma t, \cos\gamma t]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = Cx \qquad (1b)$$

Consideration of the weighting pattern $C(t)\exp\{A(t-\tau)\}\,B(\tau)$ shows that the system is time-invariant, and one finds the transfer function to be

$$\frac{V_2}{V_1}(p) = H(p) = \frac{p+\alpha}{(p+\alpha)^2+\gamma^2} \qquad (2)$$

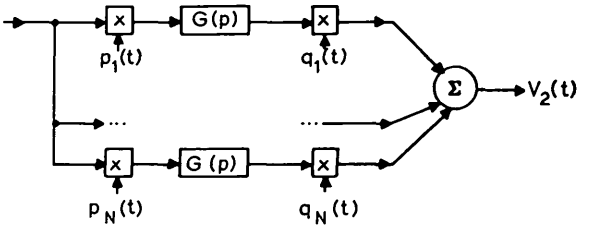

Fig. 1 *General N-path filter*

Given the state-variable equations (eqns. 1) we can synthetise them, and thus the transfer function of eqn. 2 by a standard state-variable synthesis method (Reference 2, p. 127). This yields the passive circuit of Fig. 2*b* which is obtained by loading in unit capacitors a synthesis of the admittance matrix ($\sim$ = transpose)

$$Y_c = \begin{bmatrix} 0 & 0 & -\tilde{B} \\ 0 & 0 & -C \\ B & \tilde{C} & -A \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin\gamma t & -\cos\gamma t \\ 0 & 0 & -\sin\gamma t & -\cos\gamma t \\ \sin\gamma t & \sin\gamma t & \alpha & 0 \\ \cos\gamma t & \cos\gamma t & 0 & \alpha \end{bmatrix} \quad \ldots\ldots (3)$$

On obtaining the circuit of Fig. 2*b*, we can notice the correspondence between elements of the circuit and elements of the blocks in Fig. 2*a*. This correspondence is as labelled underneath the circuit and is discussed as follows.

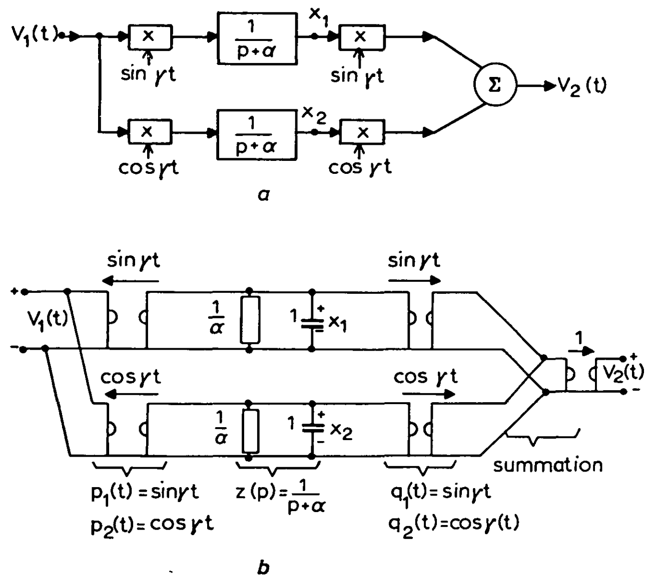

Fig. 2 *Degree-2 time-invariant 2-path filter*
a Block schematic
b Passive realisation

The input voltage $v_1(t)$ appears directly across each of the left-hand gyrator inputs. By virtue of the gyrator actions, the currents entering the *RC* circuits from the left are $(-\sin\gamma t)\,v_1(t)$ and $(-\cos\gamma t)\,v_1(t)$, respectively, for the top and bottom circuits. The resultant voltages developed, actually the state variables $x_1(t)$ and $x_2(t)$, are then changed into output gyrator currents $(+\sin\gamma t)\,x_1(t)$ and $(+\cos\gamma t)\,x_2(t)$ flowing into the input (summation) node of the final gyrator on the right. This node sums the two currents, by Kirchhoff's current law, and the final gyrator turns these summed currents into a voltage. The points to notice are then as follows:

(*a*) The time-varying gyrators truly act as multipliers in this configuration.

(*b*) The summation node truly realises the block diagram summer.

(*c*) The *RC* circuits yield their $z(p)$ as the transfer function $z(p) = G(p)$ for the corresponding blocks in the block diagram (since, by virtue of zero output current reflected through the three right-hand gyrators, there is no current flowing to the right out of the *RC* circuits).

Having made the observations of the last paragraph, we see that we can immediately realise the general *N*-path filter of Fig. 1 by the same process. Each multiplier, of multiplication factor $p_i(t)$ or $q_i(t)$, is obtained as a time-varying gyrator of gyration conductance $p_i(t)$ or $q_i(t)$. The $q_i(t)$ gyrators have their final (output) ports all connected in parallel to obtain the summation of currents, and this is followed by the cascade of a single unit-gyration conductance gyrator to obtain conversion to voltage. Each $G(p)$ is realised by a driving-point impedance $z(p) = G(p)$ placed in parallel with the common ports of the multipliers for a given path; if $z(p)$ is *RC*-realisable by a passive circuit, a completely passive *N*-path filter results.

Effect of losses: Although very-high-quality gyrators can be fabricated,[5] there will always be some losses present. For practical use of the passive *N*-path structure given, the effect of these losses must be evaluated. Since, for low-frequency performance, gyrator losses can be taken into account by shunt resistors across the gyrator ports (Reference 2, p. 154), even for time-varying gyrators, the effects of these losses are not too difficult to consider.

As an example, let us consider that all the gyrators in Fig. 2*b* have equal and fixed (time-invariant) input and output shunt-loss conductance g_L. The effects are then as follows. The two left-hand input conductances g_L play no role, since they are shunted by the source. The four g_L on either side of the *RC* circuits combine with the conductances α to yield resistances $(\alpha+2g_L)^{-1}$, with a corresponding replacement for α in eqn. 2. The three conductances g_L shunting the summation node are reflected through the unit output gyrator as a series resistor $r = 3g_L$ which, with the output loss conductance g_L, acts as a voltage divider of gain

$$(1/g_L)/\{3g_L+(1/g_L)\} = 1/(1+3g_L^2)$$

However, besides acting as a voltage divider at the output, these conductances also load the *RC* circuits by allowing currents to flow into the final multiplier gyrators. By writing either state-variable equations or time-domain superposition integral equations, one can determine that the overall voltage transfer characteristic is time-invariant and obtained from a voltage source

$$V_s(p) = [(p+\alpha+2g_L)/\{(p+\alpha+2g_L)^2+\gamma^2\}]\,V_1(p)$$

of internal (Thévénin's-equivalent) impedance

$$z(p) = (p+\alpha+2g_L)/\{(p+\alpha+2g_L)^2+\gamma^2\}$$

in series with the voltage divider mentioned above (series $r = 3g_L$ with $r = 1/g_L$ as output). Analysis of this circuit yields exactly

$$\left.\frac{V_2}{V_1}(p)\right|_{\substack{\text{with equal}\\ \text{gyrator losses}}} = H_L(p) = \frac{1}{1+3g_L^2} \times \frac{p+\alpha+2g_L}{\{(p+\alpha+2g_L)^2+\gamma^2\}+\dfrac{g_L}{1+3g_L^2}(p+\alpha+g_L)} \quad (4a)$$

For reasonably small losses, this becomes

$$H_L(p) \simeq \frac{p+\alpha+2g_L}{(p+\alpha+2g_L)^2+\gamma^2} \quad \ldots\ldots (4b)$$

from which we note, on comparison with $H(p)$ of eqn. 2, that the primary effect of gyrator loss is to shift the poles and zeros uniformly to the left. Depending on the original α satisfying $\alpha \geqslant 2g_L$, in the small-loss case this effect of gyrator loss can be completely compensated for by predistorting, or, equivalently, choosing the *RC* circuit conductances originally to be $g = \alpha-2g_L$, rather than $g = \alpha$. Also, on noting the method of calculation, the same type of result holds for unequal losses as long as the total conductances in the middle of the two paths are identical.

It is interesting to note that the state-variable equations for the equal-loss case of Fig. 2*b*, as treated above, are

$$\dot{x}_L = \begin{bmatrix} -\alpha-g_L\left(2+\dfrac{\sin^2\gamma t}{1+3g_L^2}\right) & -g_L\dfrac{\sin\gamma t\cos\gamma t}{1+3g_L^2} \\ -g_L\dfrac{\sin\gamma t\cos\gamma t}{1+3g_L^2} & -\alpha-g_L\left(2+\dfrac{\cos^2\gamma t}{1+3g_L^2}\right) \end{bmatrix} x_L + \begin{bmatrix} \sin\gamma t \\ \cos\gamma t \end{bmatrix} v_1 \quad (5a)$$

$$v_2 = \frac{1}{1+3g_L^2}\,[\sin\gamma t, \cos\gamma t]\,x_L \quad \ldots\ldots (5b)$$

For comparison, a direct state-variable realisation of $H_L(p)$ of eqn. 4*a* yields (Reference 2, p. 58)

$$\dot{z}_L = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} Z_L + \begin{bmatrix} 1 \\ (\alpha+2g_L)-a_1 \end{bmatrix} v_1 \qquad (6a)$$

$$v_2 = \frac{1}{1+3g_L{}^2}[1, 0] z_L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6b)$$

with

$$\left.\begin{aligned} a_1 &= 2(\alpha+2g_L) + \frac{g_L}{1+3g_L{}^2} \\ a_2 &= \{(\alpha+2g_L)^2+\gamma^2\} + \frac{g_L}{1+3g_L{}^2}(\alpha+2g_L) \end{aligned}\right\} \qquad (6c)$$

It is then relatively easy to check that these two sets of state-variable equations describing the same lossy system transfer function are related by the state transformation $x_L = T(t) z_L$, where

$$T(t) = \begin{bmatrix} \sin\gamma t + \dfrac{K}{\gamma}\cos\gamma t & \dfrac{1}{\gamma}\cos\gamma t \\ \cos\gamma t - \dfrac{K}{\gamma}\sin\gamma t & -\dfrac{1}{\gamma}\sin\gamma t \end{bmatrix} \qquad (6d)$$

and

$$K = \alpha + 2g_L + \frac{g_L}{1+3g_L{}^2}$$

For reference purposes, we comment that det. $T = -1/\gamma$.

As a consequence, we have determined that the lossy system is indeed described by the time-invariant transfer function $H_L(p)$ of eqn. 4*a*.

Conclusions: On applying a state-variable synthesis method, we have shown that any *N*-path filter, of the form of Fig. 1, has a passive-circuit realisation where the modulating signals are inserted through a time-varying gyrator gyration-conductance variation. This result was obtained by simple generalisation from the practical 2-path filter realised in Fig. 2. Further observation of this Figure shows that all components can be obtained earthed, and that they are also realisable by integrated-circuit technology. Indeed, the sinusoidal modulating signals can be obtained from capacitor–gyrator resonant circuits used to control resistors realising the time-varying gyration conductances of the gyrators shown in Fig. 2.

When, as in Fig. 2, appropriate sinusoidal signals are present, gyrator (and capacitor) losses can be taken into account by the methods of the last Section. There, it was shown that these gyrator losses do not drastically affect the performance of the filter, since they essentially only uniformly shift the poles and zeros of the intended transfer function, at least when the losses are reasonably small. Consequently, *N*-path filters designed, as in Fig. 2*b*, using passive components should prove of practical significance.

Various extensions can, of course, be made, one of which is to replace the middle $G(p)$ section 2-ports by 2*N*-ports. Doing this, one can obtain transfer functions which are insensitive to the modulating-signal frequency, in contrast to the situation discussed here [note that γ appears in $H(p)$ of eqn. 2]. This and other results will be reported later.

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SCATTERING-MATRIX MEASUREMENTS ON 3-PORT ELETRONIC CIRCULATORS

Indexing term: Circulators

The evaluation of the 3-port circulator using differential-input operational amplifiers has been performed by scattering-matrix analysis. The *s* parameters are easily determined over the frequency range tested (1–300 kHz) and accurately depict the transmission and reflection properties of the circulator.

The design concept of a 3-port circulator using differential-input operational amplifiers has been previously described by Keen, Glover and Harris,[1] and by Rollett and Greenaway.[2]

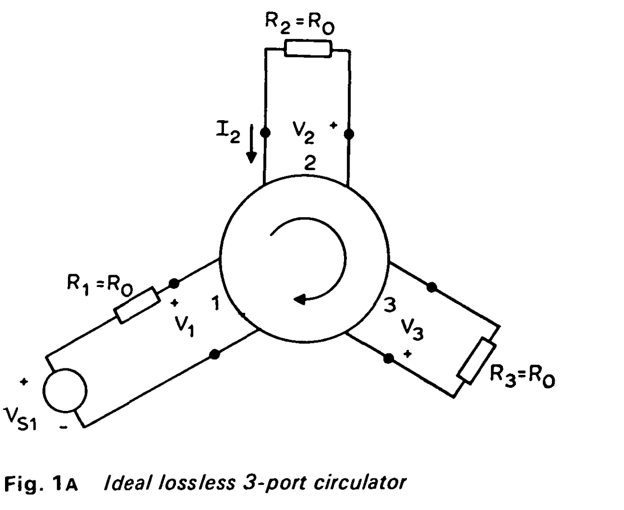

Fig. 1A *Ideal lossless 3-port circulator*

The potential uses of these devices lie in the fields of telephone systems and low-frequency radio transceivers.

Circulators have been widely used in microwave systems, and the performance of the microwave circulator is usually

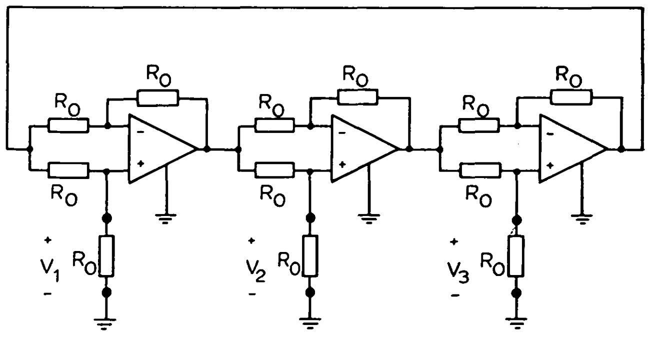

Fig. 1B *3-port circulator realisation*

determined by the evaluation of its scattering matrix. It should then be possible to determine the scattering matrix of the electronic circulator and thus investigate its transmission and reflection properties.

For an ideal lossless circulator, the scattering matrix is

$$[S] = \begin{bmatrix} 0 & 0 & S_{13} \\ S_{21} & 0 & 0 \\ 0 & S_{32} & 0 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The electronic circulator will be terminated with purely resistive loads at its ports, and it is assumed that reactance