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NONRECIPROCAL TRANSMISSION-LINE n-PORT SYNTHESIS

by

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ABSTRACT

Richards' theorem imparts its strength,
Using lines of integral length,
To realizations, type non-rational,
Of n-port circuits, non-reciprocal.
One-way transmission filter exampled
Details much of the theory included.

CONTENTS

ILLUSTRATIONS

I. INTRODUCTION

Until the advent of Richards' one-port transmission-line synthesis of 1948, [1], there was no rigorous means of designing distributed parameter networks. Following the appearance of this synthesis it was modified in various ways to cover specific structures, as well as transfer functions, [2], [3], [4], while other useful, but less exact methods also appeared, [5]. More recently, Saito, [6], gave a two-port transmission-line synthesis based upon Bayard's matrix extension of Richards' theorem, [7, p. 170]. However, none of these methods covers the synthesis of non-reciprocal, distributed parameter circuits, which are becoming of importance in many contexts. In particular, this is the case in parametric amplifier design.

Here we develop the theoretical aspects of non-reciprocal, resistor, lossless transmission-line n-ports, under the customary assumption of rationally related line lengths. Of course, we assume linearity, passivity and a finite number of elements. Transformers will generally be allowed but their number is to a large extent minimized.

In Section II we develop the properties of the types of networks under discussion. The n-port unit-element is defined and Kuroda's identity extended. In Section III the synthesis, which is based upon the recent generalization of Bayard's version of Richards' theorem, [8], is given.

Before proceeding, we introduce some necessary notation. 0_n and 1_n stand for the n x n zero and identity matrices, respectively; $\dotplus$ denotes the direct sum; PR stands for a rational positive-real matrix, [8, p. 11]; a superscript tilde, ~, denotes matrix transposition; a subscript asterisk, $_*$, denotes Hurwitz conjugation, that is, replacement of the complex frequency variable, p, by -p; finally, δ denotes McMillan's degree of a rational matrix, [9, p. 543].

II. PROPERTIES OF TRANSMISSION-LINE n-PORTS

Consider an n-port which is constructed from resistors, gyrators, transformers and lossless transmission-lines whose lengths are all rational multiples of each other. We can then choose the lengths such that they are all integral multiples of some base length, which is in turn one-half wavelength at some fixed frequency f_o. Any one of these lines of length ℓ, if we take ports at opposite ends of the line, can be considered as a two-port, as shown in Figure 1.

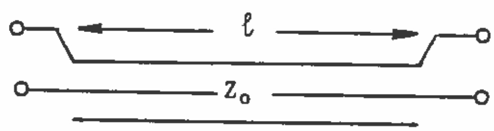

FIG. 1. LOSSLESS LINE

If the line inductance and capacitance are L henrys/meter and C farads/meter, the characteristic impedance is $Z_o = \sqrt{L/C}$. If the line is m half-wavelengths long (at f_o), then $\ell = m/(2f_o\sqrt{LC})$, and the line is described by the impedance matrix (Appendix A)

$$Z_o \begin{bmatrix} \text{ctnh}(ms/2f_o) & \text{csch}(ms/2f_o) \\ \text{csch}(ms/2f_o) & \text{ctnh}(ms/2f_o) \end{bmatrix} \tag{1}$$

where $s = \sigma + j\omega$, $\omega = 2\pi f$, is the actual frequency. When, as in the case under consideration, m is an integer, the impedance matrix of (1) is rational in $\exp[s/2f_o]$, and if we let

$$e^{s/2f_o} = [1+p]/[1-p] \tag{2a}$$

which is

$$p = \tanh(s/4f_o) \tag{2b}$$

(1) becomes rational in p. Consequently, under the transformation of (2), any of the standard linear descriptions of the n-port become rational in p (since only "rational" operations are involved with rational "loop" matrices). Further, these descriptions satisfy the normal realizability conditions, since p, by (2b), is a positive-real function of s; this would not be the case if the right of (2a) were replaced by p itself.

Of course (1) is valid for non-integer m and the quarter-wavelength line, for which m = 1/2, has considerable interest. For it (1), under (2), becomes

$$Z_{UE}(p) = \begin{bmatrix} Z_o/p & \sqrt{1-p^2}\,Z_o/p \\ \sqrt{1-p^2}\,Z_o/p & Z_o/p \end{bmatrix} \qquad (3)$$

Even though this isn't rational, it is of considerable importance, since it forms the basis for the n-port unit element. To see this, consider the 2n-port of Figure 2 where all lines are a quarter-wavelength long and initially assumed uncoupled.

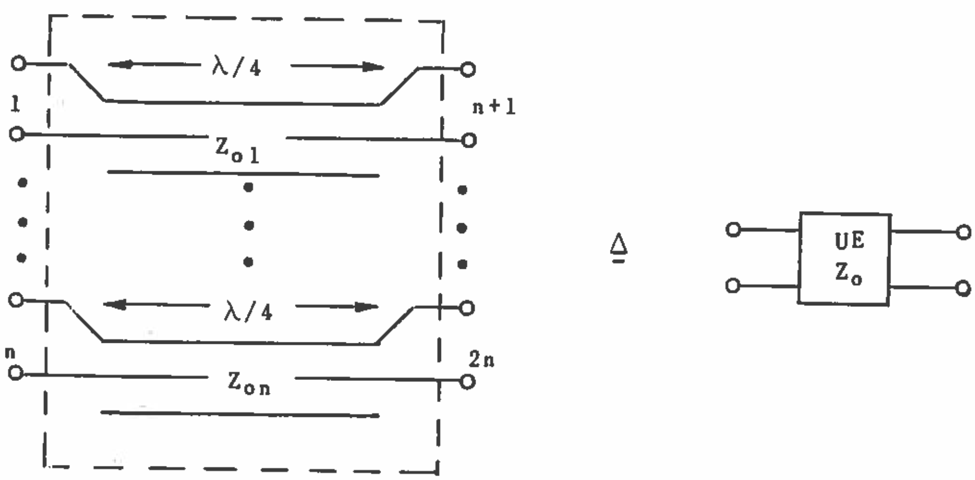

FIG. 2. n-PORT UNIT ELEMENT

Defining the diagonal characteristic impedance matrix by

$$Z_o = Z_{o1} \dotplus \cdots \dotplus Z_{on} \qquad (4)$$

(3) shows that Figure 2 is also described by (3) with Z_{UE} now a 2nx2n matrix written in partitioned form. If the lines in Figure 2 are mutually coupled, then (3) can still be used to describe the 2n-port, when Z_o is taken as a (symmetric) positive semi-definite matrix. In either case, coupled or not, the 2n-port of Figure 2, when considered in the p plane, will be called a unit-element and denoted as shown on the right of the figure. It is of interest to recognize that the admittance matrix of the unit element has the same form as (3) except that Z_o is replaced by $Y_o = Z_o^{-1}$ and the (1,2) and (2,1) terms are multiplied by -1.

If now we connect an n-port on a unit-element, as shown in Figure 3 where the impedance matrices Z_i and Z_ℓ are defined, we find (Appendix B)

$$Z_i(p) = Z_o[pZ_\ell(p)+Z_o]^{-1}[Z_\ell(p)+pZ_o] \tag{5a}$$

which is

$$Z_\ell(p) = [pZ_o-Z_i(p)][pZ_i(p)-Z_o]^{-1}Z_o \tag{5b}$$

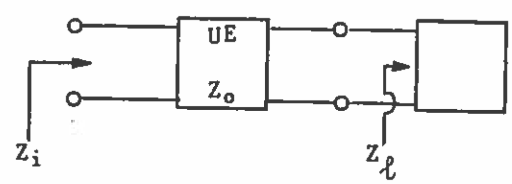

FIG. 3 LOADED UNIT-ELEMENT

From (5a) it is observed that Z_i is rational in p, when Z_ℓ is, even though there are some quarter-wavelength lines present. Some simple equivalences result from the connection of Figure 3. For instance, if the load consists of shorts, then (5a) with $Z_\ell = 0_n$ gives $Z_i = pZ_o$, and the configuration acts as an inductor n-port, in the p-plane. If the load consists of opens, a dual treatment shows that the input behaves as a capacitor n-port, $Y_i = pY_o$. These equivalences are illustrated in Figure 4.

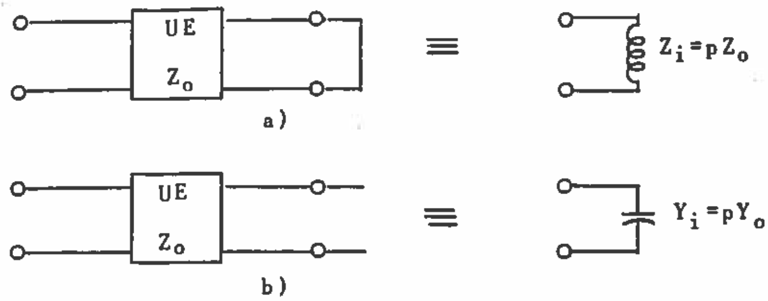

FIG. 4. p-PLANE EQUIVALENCES

Also of some interest are the n-port Kuroda identities illustrated in Figure 5. The first of these is proven in Appendix C, while the second follows in several ways from the first, for instance by duality or exchange of input and output.

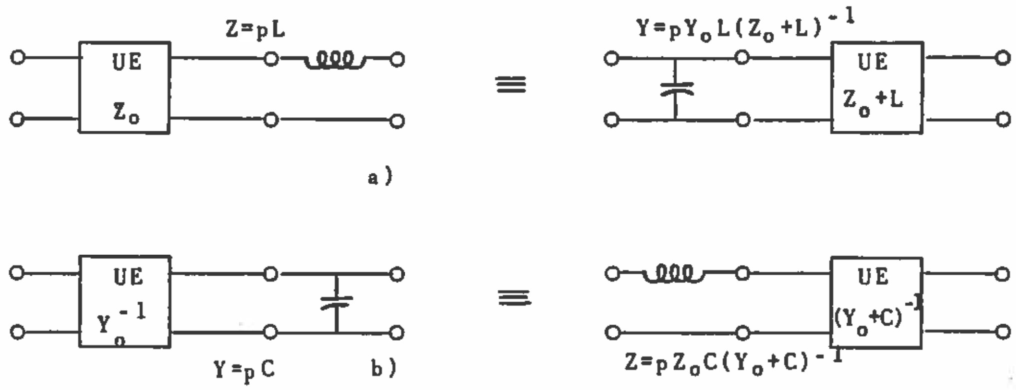

FIG. 5. n-PORT KURODA IDENTITIES

III. SYNTHESIS

The types of networks under consideration can be specified in several ways, the most general of which is through the scattering matrix.

We will therefore assume that an $m \times m$ scattering matrix, $S(p)$, is given which has been made rational by the transformation of (2b). However, the synthesis will be based upon the impedance matrix, $Z(p)$. Consequently, we find a T, as can always be done, such that $TS\tilde{T} = S_o \dotplus 1_{m-n}$, for which the nxn scattering matrix S_o has $1_n - S_o$ non-singular, [10, p. 152]. The impedance matrix is then found from

$$Z(p) = 2[1_n - S_o(p)]^{-1} - 1_n \tag{6}$$

The network for S then results from that for S_o by adjoining open-circuits through a transformer realization of T, [10, p. 153]; of course for the important case of m=n no transformers are present.

Given such a $Z(p)$, which is PR, we then apply the ideas of a previous report, [11], and derive from Z the $(n+r)x(n+r)$ impedance matrix $X(p)$ of a lossless network, that is, $X = -\tilde{X}_*$ and X is PR; r is the rank of $Z + \tilde{Z}_*$. A realization for Z then results from one for X by terminating the last r ports in resistors.

Several methods are available for synthesizing X. The first, and somewhat trivial way, is to synthesize X in the standard manner using L's, C's, gyrators and transformers (in the p plane), [12, p. 155]. The inductors and capacitors are then converted into transmission lines through the equivalences of Figure 4. This method generally uses an excessive number of transformers and the transmission lines don't occur in as useful combinations as in the following cascade type synthesis.

Observing (5b) we see that the loaded unit-element gives Z_i in terms of Z_ℓ through the use of the Richards' theorem for matrices, [8, Eq. (4f)], if $k = 1$ is chosen. Noting that Richards' theorem requires the matrix to be symmetric at $p = k$, we first extract series gyrators by writing

$$\overline{X}(p) = X(p) - X_{ss}(1) \tag{7}$$

where $2X_{ss} = X - \tilde{X}$ is the skew-symmetric part of X. A unit-element is then extracted by forming

$$X_\ell(p) = [p\overline{X}(1)-\overline{X}(p)][p\overline{X}(p)-\overline{X}(1)]^{-1}\overline{X}(1) \tag{8}$$

whenever the required inverse exists. X_ℓ is PR, by the cited Richards' theorem, and Figure 3 shows that forming X_ℓ does correspond to the extraction of a unit-element. Further, $\delta(X_\ell) = \delta(Z)-n$, [8, p. 7], and, as a consequence, the repetition of this procedure leads to a matrix of degree zero, which can only be 0_n, since it must be constant and $X = -\tilde{X}_*$. This has assumed that the inverse present in (8) exists, which may not always be the case. However, singularity of $[p\overline{X}(p)-\overline{X}(1)]$ can be taken care of as follows. We can extract any pole at zero of $\overline{X}(p)$ by writing, using a partial fraction expansion, say,

$$\overline{X}(p) = \overline{X}_o(p) + A/p \tag{9a}$$

where $\overline{X}_o$ is analytic at $p = 0$. Then

$$p\overline{X}(p)-\overline{X}(1) = p\overline{X}_o(p)-\overline{X}_o(1) \tag{9b}$$

which has the rank of $\overline{X}_o(p)$, as is seen by considering a Taylor series expansion near $p = 1$ and noting that this then has the rank of $\partial\overline{X}_o(1)/\partial p$ which is the same rank as that of $\overline{X}_o(p)$ by rationality and analyticity. Then, if $\overline{X}_o$ has rank $k < (n+r)$, the A/p term can be separately realized by Figure 4b) and connected in series with a realization for $\overline{X}_o$. $\overline{X}_o$ can be realized by writing, [10, pp. 66, 84],

$$\tilde{T}_o\overline{X}_o(p)T_o = \overline{X}'_o \dotplus 0_{n+r-k} \tag{10}$$

The Richards' transformation can be applied to the PR matrix $\overline{X}'_o$ and the above procedure continued with T_o being realized by transformers. Kuroda's identities then sometimes become useful for converting the series combinations into cascade connections. The form of the realization for X, when the Richards' transformation can be performed at every step, is sketched in Figure 6.

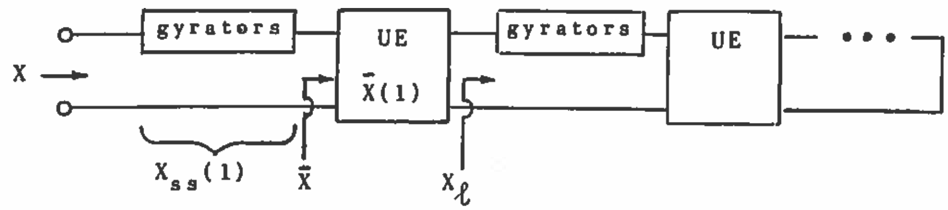

FIG. 6. CANONICAL REALIZATION OF X

In the matrix case, sometimes the Richards' transformation yields a reduction in degree when applied directly to Z, [8, p. 7]. In such cases the extra steps involved in finding X can be avoided. However, as yet, the conditions on Z for this degree reduction aren't known.

The following example illustrates almost all of the interesting points of the synthesis.

EXAMPLE:

It is desired to design a low-pass, one-way, matched, 3rd order Butterworth, transmission-line filter; perhaps for an antenna feed. Such is described by

$$S(p) = \begin{bmatrix} 0 & 0 \\ \frac{1}{p^3+2p^2+2p+1} & 0 \end{bmatrix}$$

which has, by (6),

$$Z(p) = \begin{bmatrix} 1 & 0 \\ \frac{2}{p^3+2p^2+2p+1} & 1 \end{bmatrix}$$

then, using the Case 1 method of [11],

$$X(p) = \frac{1}{p(p^2+2)} \left[\begin{array}{cc|cc} 2p^2+1 & 1 & p^2+2p^2+2p+1 & 0 \\ 1 & 2p^2+1 & 1 & p^3 \\ \hline -p^3+2p^2-2p+1 & 1 & 2p^2+1 & 0 \\ 0 & -p^3 & 0 & 2p^2+1 \end{array} \right]$$

This has

$$X_{ss}(1) = \frac{1}{3} \left[\begin{array}{cc|cc} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right]$$

and

$$\overline{X}(p) = \frac{1}{p(p^2+2)} \left[\begin{array}{cc|cc} 2p^2+1 & 1 & 2p^2+1 & 0 \\ 1 & 2p^2+1 & 1 & \frac{2}{3}\, p(p^2-1) \\ \hline 2p^2+1 & 1 & 2p^2+1 & 0 \\ 0 & -\frac{2}{3}\, p(p^2-1) & 0 & 2p^2+1 \end{array} \right]$$

This is singular with the third row and column equal to the first. After deleting this row and column, (8) gives

$$X_\ell = p \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 1/3 & 1/9 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] + \frac{1}{p} \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array} \right] + \left[\begin{array}{cc|c} 0 & 0 & -1 \\ 0 & 0 & -5/3 \\ \hline 1 & 5/3 & 0 \end{array} \right]$$

The final realization is shown in Figure 7, where one of the gyrators and one of the shorted lines, both in the X_ℓ portion, could be saved by the use of appropriate transformers. Because of the singularity of the inductance and capacitance matrices, the series combination used to realize X_ℓ can't be reduced by the Kuroda identities to cascade form.

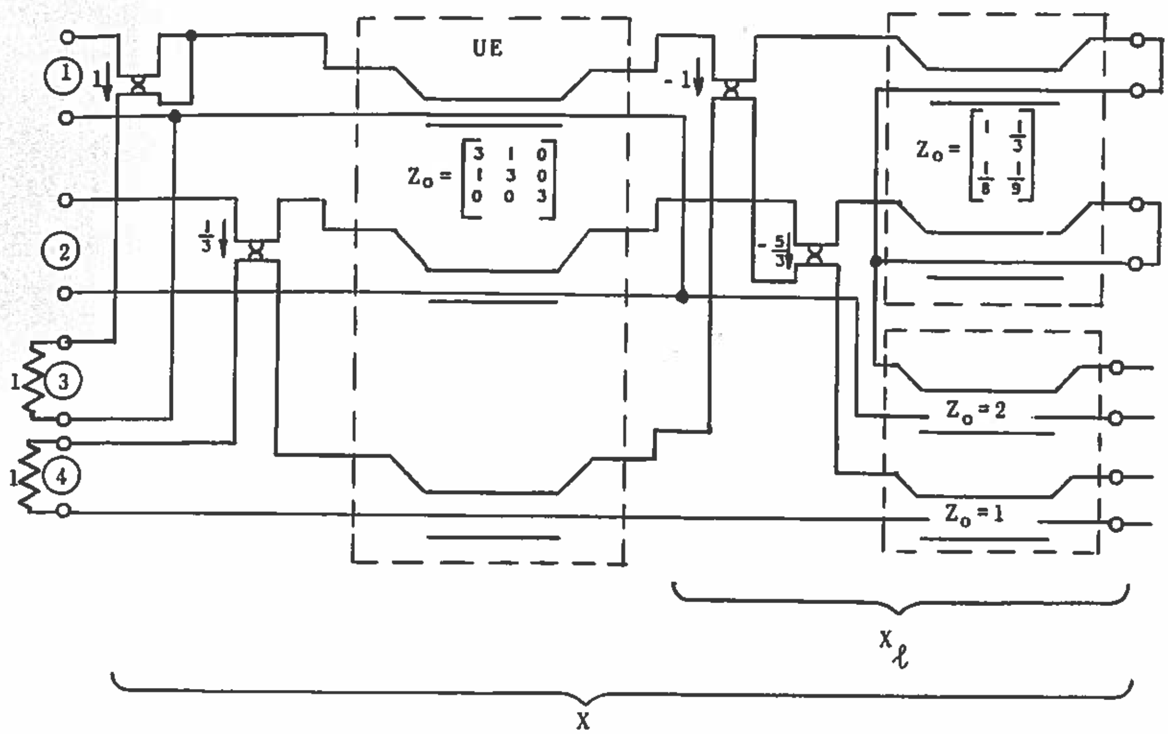

FIG. 7. 3RD ORDER BUTTERWORTH ONE-WAY FILTER

IV. CONCLUSIONS

In this report a synthesis of non-reciprocal resistor transmission-line synthesis has been given. Although several alternatives exist, the method using the Richards' transformation seems the most useful, since a cascade type realization is obtained. Although, at higher frequencies transformers aren't such a nuisance as at lower frequencies, they may be required at several places. Transformers may be required when converting S to Z or, at a more bothersome point, in realizing some Z_o's for unit-elements. As yet we are unaware of any study on which Z_o's can be physically realized without transformers, but a congruency transformation allows any symmetric, positive semi-definite Z_o to be realized from a diagonal one by the use of transformers (Appendix A shows how L and C are affected by such a transformation in the positive definite case).

The use of Kuroda's identity as mentioned in Section III and illustrated in Figure 8, will sometimes lead to a more useful structure. However, as was seen in the last cycle of the example, singular matrices may hamper its application. But one does feel that the trouble caused by such singularities could somehow be avoided. It would also be useful if all the gyrators could be removed at an initial or final step, since then the unit-elements shown in Figure 6 could be more conveniently connected together physically. However, series and shunt connections can be physically realized by the use of Grayzel's connection, [3, p. 173], generalized to n-ports, where possible.

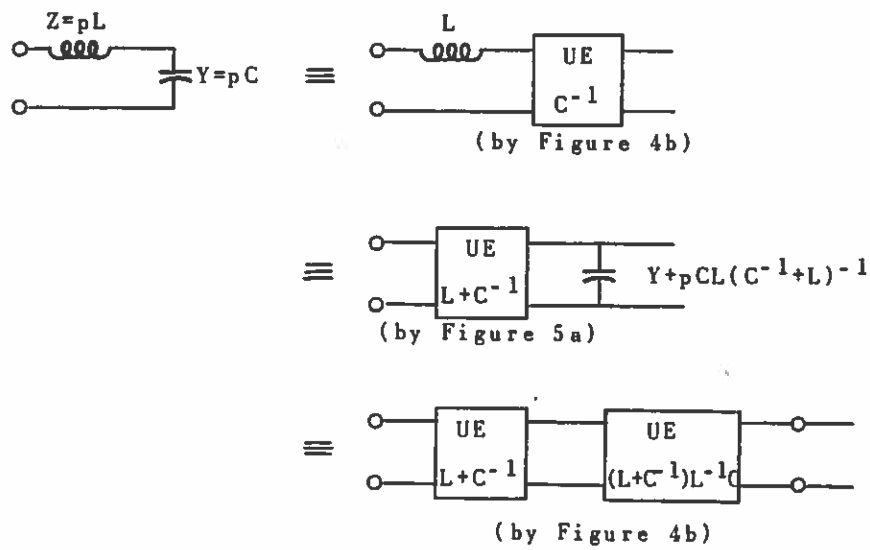

FIG. 8. APPLICATION OF KURODA'S IDENTITY

At (2b) we used the transformation $p = \tanh(s/4f_o)$. This is appropriate to low- or high-pass designs. However, the transformation $p = \text{ctnh}(s/4f_o)$ also can be used and it seems to be the most appropriate for band-pass or band-stop designs. The use of this latter transformation replaces p by $1/p$ and thus gives dual results in the p-plane.

ACKNOWLEDGEMENT

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APPENDICES

APPENDIX A: Lossless Transmission Lines

The lossless transmission line of Figure 1 is described by the two (dual) equations

$$\frac{\partial v(x,t)}{\partial x} = -L\,\frac{\partial i(x,t)}{\partial t} \qquad \text{(A-1a)}$$

$$\frac{\partial i(x,t)}{\partial x} = -C\,\frac{\partial v(x,t)}{\partial t} \qquad \text{(A-1b)}$$

where v and i are the current and voltage on the line, t is time, and x is distance down the line measured with respect to port one.

Taking the bilateral Laplace transform with respect to time yields

$$\frac{dV(x,s)}{dx} = -sLI(x,s) \qquad \text{(A-2)}$$

and its dual. Here we have replaced the partial derivatives by total derivatives, which allows the time differentiation to go into multiplication by s, and is justified, since x and t are independent. Differentiating (A-2) with respect to x and using the dual equation yields .

$$\frac{d^2V(x,s)}{dx} = s^2LCV(x,s) \qquad \text{(A-3)}$$

This can be solved using the unilateral Laplace transform in conjunction with (A-2) evaluated at x = 0 to give

$$V(x,s) = \cosh(s\sqrt{LC}x)V(0,s) - \sqrt{LC^{-1}}\,\sinh(s\sqrt{LC}x)I(0,s) \qquad \text{(A-4a)}$$

$$I(x,s) = \cosh(s\sqrt{CL}x)I(0,s) - \sqrt{CL^{-1}}\,\sinh(s\sqrt{CL}x)V(0,s) \qquad \text{(A-4b)}$$

where the second follows by duality from the first. Defining

$$Z_o = \sqrt{LC^{-1}} \qquad \text{(A-5)}$$

letting $V_2(s) = V(\ell,s)$, $V_1(s) = V(0,s)$, $I_2(s) = -I(\ell,s)$, $I_1(s) = I(0,s)$ and solving (A-4) for the voltages in terms of the currents yields

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Z_o \begin{bmatrix} \operatorname{ctnh}(s\sqrt{LC}\ell) & \operatorname{csch}(s\sqrt{LC}\ell) \\ \operatorname{csch}(s\sqrt{LC}\ell) & \operatorname{ctnh}(s\sqrt{LC}\ell) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \text{(A-6)}$$

which gives the impedance matrix for the line. A wavelength is defined at a given frequency f by $\lambda = 1/(f\sqrt{LC})$, and thus, if the line is m half-wavelengths long at the frequency f_o, then $\ell = m/(2f_o\sqrt{LC})$.

If we have a 2n-port of coupled lines, then (A-1) remain valid with v and i n-vectors and L and C coefficient matrices. Assuming these latter positive definite L and C^{-1} can be simultaneously diagonalized by properly choosing T in the following equations, [13].

$$v_d = \tilde{T}v \qquad \text{(A-7a)}$$

$$i = Ti_d \qquad \text{(A-7b)}$$

$$L_d = \tilde{T}LT \qquad \text{(A-7c)}$$

$$C_d = T^{-1}C\tilde{T}^{-1} \qquad \text{(A-7d)}$$

We also define the diagonal characteristic impedance matrix

$$Z_{o_d} = [L_d C_d^{-1}]^{1/2} \qquad \text{(A-8)}$$

If all lines are a quarter wavelength long at some frequency f_o, we then obtain, in partitioned form,

$$\begin{bmatrix} V_{1d} \\ V_{2d} \end{bmatrix} = \begin{bmatrix} \operatorname{ctnh}(s/4f_o)Z_{o_d} & \operatorname{csch}(s/4f_o)Z_{o_d} \\ \operatorname{csch}(s/4f_o)Z_{o_d} & \operatorname{ctnh}(s/4f_o)Z_{o_d} \end{bmatrix} \begin{bmatrix} I_{1d} \\ I_{2d} \end{bmatrix} \qquad \text{(A-9)}$$

where the functional multipliers of Z_{o_d} can be taken as scalars. Defining

$$Z_o = \tilde{T}^{-1} Z_{o_d} T^{-1} \quad \text{(A-10)}$$

shows that, in this case, the coupled lines can be described by the impedance matrix in (A-9) when the subscript d's are omitted. Note that here $Z_o = \tilde{T}^{-1}[\tilde{T}LT\tilde{T}C^{-1}T]^{1/2}T^{-1}$ which is not (A-5) with scalars replaced by matrices; this latter replacement causes trouble in previous works, [14], [15].

APPENDIX B: Loaded Unit-Element

In this appendix we derive equations (5). For this define voltage and current variables for the network of Figure 3, such that

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_o/p & \sqrt{1-p^2}Z_o/p \\ \sqrt{1-p^2}Z_o/p & Z_o/p \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \text{(B-1)}$$

$$V_2 = -Z_\ell(p)I_2 \quad \text{(B-2)}$$

Eliminating V_2 from these two gives

$$[-Z_\ell(p)-Z_o/p]I_2 = [\sqrt{1-p^2}Z_o/p]I_1 \quad \text{(B-3a)}$$

Substituting this in the first of (B-1) gives

$$V_1 = [(Z_o/p) + (\sqrt{1-p^2}Z_o/p)(-Z_\ell(p)-Z_o/p)^{-1}(\sqrt{1-p^2}Z_o/p)]I_1 \quad \text{(B-3b)}$$

$$= Z_o[Z_\ell(p)+Z_o/p]^{-1}[Z_\ell(p)+(Z_o/p)-(1-p^2)Z_o/p]I_1 \quad \text{(B-3c)}$$

$$= Z_o[pZ_\ell(p)+Z_o]^{-1}[Z_\ell(p)+pZ_o]I_1 \quad \text{(B-3d)}$$

which is (5a). Setting the coefficient matrix of I_1 equal to Z_i gives

$$[pZ_\ell+Z_o]Y_oZ_i = Z_\ell+pZ_o \quad \text{(B-4a)}$$

or

$$Z_\ell[pY_oZ_i-1_n] = pZ_o-Z_i \tag{B-4b}$$

which is (5b) after multiplying by the appropriate inverse.

APPENDIX C: n-Port Kuroda Identity

Consider the left-hand network of Figure 5a). This is described by the impedance matrix

$$Z(p) = \begin{bmatrix} Z_o/p & \sqrt{1-p^2}Z_o/p \\ \sqrt{1-p^2}Z_o/p & pL+Z_o/p \end{bmatrix} \tag{C-1}$$

Inverting this gives

$$Y(p) = \begin{bmatrix} (Z_o+L)^{-1}(LY_op+1_n/p) & -\sqrt{1-p^2}(Z_o+L)^{-1}/p \\ -\sqrt{1-p^2}(Z_o+L)^{-1}/p & (Z_o+L)^{-1}/p \end{bmatrix} \tag{C-2}$$

Since Z is symmetric, we see that the (1,1) term of this is also equal to $Y_oL(Z_o+L)^{-1}$ since Z_o and L are symmetric, by the PR assumption. This matrix is easily seen to describe the right-hand network of Figure 5a).

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