*On the Problem of Degree Reduction of a Scattering Matrix by Factorization**

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ABSTRACT: *Belevitch* (**1**) *has shown that, starting from a given passive, rational,* $n \times n$ *scattering matrix* $S(p)$ *of degree* δ, *one can proceed to a realization by factoring it in the form*

$$S(p) = S_1(p)\,S_2(p),$$

where $S_2(p)$ *is an* $n \times n$, *lossless scattering matrix of degree one, while the degree of* $S_1(p)$ *is reduced to* $\delta - 1$. *Some sufficient conditions allowing the stated factorization were developed by Youla* (**2**) *and Belevitch* (**3**) *but complete necessary and sufficient conditions were not obtained. Complete conditions are derived here by two different and complementary methods, one based on Hankel matrices, the other on the Smith–MacMillan form. Moreover, several errors of the above-mentioned papers are corrected. The resulting conditions are quite simple and only involve the structure of the resistivity matrix of the given network in the neighborhood of a singularity. Finally, the conditions clarify certain aspects of the cascade synthesis of passive n-ports and increase the similarity of this process with the Darlington one-port synthesis.*

I. Introduction

The scattering matrix $S(p)$ of a linear passive time-invariant lumped n-port is bounded and rational. The matrix is, in addition, para-unitary if the n-port is lossless. One method of network synthesis, initiated by Belevitch (**1**), operates by factoring the prescribed $n \times n$ matrix S, of degree δ, into the form

$$S = S_1 S_2, \tag{1}$$

where S_2 is a bounded para-unitary matrix of degree 1 and where the degree of S_1 is reduced to $\delta - 1$. Some sufficient conditions allowing such a factorization are described by Youla (**2**) and Belevitch (**3**), but the conditions stated

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as necessary and sufficient in the above references are incomplete and partially incorrect. The main purpose of this paper is to correct and complete the latest and most detailed publication (**3**) with respect to the factorization conditions and their consequences for synthesis. Frequent reference will thus be made to (**3**) (under the abbreviation CNT).

We will use the upper star (*) to indicate a complex conjugate, the upper tilde (~) a Hermitian conjugate and the upper circumflex (^) a para Hermitian conjugate, that is

$$\hat{A}(p) = \tilde{A}(-p^*).$$

The $\dotplus$ will indicate a direct sum of matrices.

As mentioned in CNT, p. 330, there is no restriction in assuming for S_2 the form

$$S_2 = 1_n - \frac{2\alpha_0 u\tilde{u}}{p+p_0^*}, \tag{2}$$

where u is an n-vector normalized to

$$\tilde{u}u = 1 \tag{3}$$

and where $\alpha_0 = \operatorname{Re} p_0 > 0$. It is also known that S_1 is bounded with S if and only if the vector u satisfies

$$S(p_0)\,u = 0. \tag{4}$$

The following discussion is therefore limited to the determination of the necessary and sufficient condition for degree reduction.

II. The Degree of a Rational Matrix

The degree of a rational matrix has been defined by McMillan (**4**) and further discussed by many authors (**5–8**). A different approach, originating from dynamical system theory and using the Hankel matrix, is due to Youla (**9**), Ho and Kalman (**10**) and Ho (**11**). A detailed account of both methods can be found in CNT, Ch. 8. Since a bounded matrix is regular at infinity, we only quote here below the essential results holding in that case.

(a) Let p_i denote the distinct poles of a rational matrix A and let A_i be the corresponding principal parts (unipolar components) of A. The degree of A_i (called the degree of A at p_i) is noted as δ_i. One has $\deg A = \sum \delta_i$.

(b) Let $A = PMQ$ where M is the Smith–McMillan form of A and where P and Q are unimodular (see CNT, p. 406). If A is regular at infinity, $\deg A = \deg M$. This equality also holds for the local degree at each p_i.

(c) The degree of A at p_i is the highest possible power of $p-p_i$ occurring in the denominator of any minor of A.

(d) If

$$A_i = \sum_{s=1}^{m} \frac{K_{m-s}}{(p-p_i)^s}, \tag{5}$$

δ_i is the rank of the Hankel matrix

$$T = \begin{bmatrix} K_0 & 0 & \dots & 0 \\ K_1 & K_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ K_{m-1} & K_{m-2} & \dots & K_0 \end{bmatrix} \qquad (6)$$

of dimension mn. This matrix will be called the *Hankel* matrix at the pole p_i.

The following additional properties correct certain statements of (**2**, **3**):

(e) Express M as a direct sum $M_a \dotplus M_b$ where all entries of M_a are infinite, whereas M_b is finite, at p_i. The dimension t of M_a is the span of A at p_i. In relation with (c), poles of the minors of A, when these are computed by the Binet–Cauchy theorem, only reach the maximum order δ_i when they use *at least* the first t rows and columns of M, so that only minors of order greater than t of A (the inequality cannot be replaced by an equality) can have a pole of order δ_i.

(f) Although the principal value of M at p_i is $M_a \dotplus 0_{n-t}$, this matrix is generally not the Smith–McMillan form of A_i. In other words, if P_i and Q_i are the values of P and Q at p_i, respectively, one does not have

$$A_i = P_i(M_a \dotplus 0_{n-t})Q_i$$

because some entries of P or Q may vanish at p_i and thus destroy some principal values in entries of A_i. This occurs for instance at $p = 0$ in the example

$$\begin{bmatrix} 1/p^2 & 1/p^2+1/p \\ 1/p & 1/p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \begin{bmatrix} 1/p^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1+p \\ 0 & -1 \end{bmatrix}.$$

III. First Criterion for Degree Reduction

Since (1) is equivalent to $\hat{S} = \hat{S}_2\hat{S}_1$ and since $\hat{S}_2 = 1_n + 2\alpha_0 u\tilde{u}/(p-p_0)$ is finite for $p \neq p_0$ and nonsingular for $p \neq -p_0^*$, all poles distinct from p_0 in $\hat{S}$ and $\hat{S}_1$ are identical with identical degrees. Using the fact that $\hat{S}_2 = S_2^{-1}$ an increase of degree from $\hat{S}_1$ to $\hat{S}$ (hence a decrease from $\hat{S}$ to $\hat{S}_1$) can thus only occur at p_0, and is at most of one unit, by the Binet–Cauchy theorem. It is therefore sufficient to investigate the conditions of degree reduction in the form $\deg S_2\hat{S} < \deg \hat{S}$, locally at p_0, on the matrix $\hat{S}$ to be noted A.

The condition is given by the following theorem: The inequality

$$\deg\left(1_n - \frac{2\alpha_0 u\tilde{u}}{p+p_0^*}\right)A < \deg A \qquad (7)$$

at $p_0 \neq \infty$ is equivalent to the existence of a vector $v(p)$, analytic near p_0, such that

$$\lim Av = u \neq 0; \quad \lim v = 0, \qquad (8)$$

all limits being taken for p tending to p_0.

Before proving the theorem we note that (8) implies $A(p_0) = \infty$, a known necessary condition, for otherwise one would have $\lim Av = A(p_0) \lim v = 0$.

Let $A = PMQ$ with $M = M_a + M_b$ as in Section II. By the transformation $P^{-1} \ldots Q^{-1}$, (7) is equivalent to $\deg BM < \deg M$ with

$$B = 1_n - 2\alpha_0 P^{-1} u\tilde{u} P/(p + p_0^*).$$

This inequality is satisfied if and only if all minors of BM have degrees smaller than the one of M at p_0. Since the only minors of BM which could reach the maximum degree originate from minors of B using at least the first t columns, the degree inequality holds if and only if the first t columns of B have rank $< t$ at p_0. This is equivalent to the existence of a vector x partitioned into

$$x = \begin{bmatrix} x_a \\ x_b \end{bmatrix}, \quad x_a \neq 0, \quad x_b = 0, \tag{9}$$

such that $Bx = 0$ at p_0. This condition is $(1_n - P_0^{-1} u\tilde{u} P_0) x = 0$ where $P_0 = P(p_0)$ is finite nonsingular and is thus equivalent to $(1_n - u\tilde{u}) P_0 x = 0$. Since $1_n - u\tilde{u}$ has rank $n-1$ and since (3) gives $(1_n - u\tilde{u}) u = 0$, the only solution of the last equation is

$$P_0 x = u. \tag{10}$$

Moreover, the first condition (9) is redundant, since $x = 0$ would give $u = 0$ in contradiction with (8).

The existence of a vector x with $x_b = 0$ is equivalent to the existence of a vector $y(p)$, analytic near p_0, such that

$$\lim My = x \neq \infty; \quad \lim y = 0. \tag{11}$$

From (10), one deduces $x_b = \lim M_b y_b = M_b \lim y_b = 0$ since all factors are finite. Conversely, to deduce (11) from the existence of a finite x with $x_b = 0$, define $y(p)$ by $y_a = M_a^{-1} x_a$, $y_b = 0$. Since all entries of M^{-1} vanish at p_0 whereas x_a is finite, this gives $\lim y = 0$. On the other hand the first equation above, equivalent to $x_a = M_a y_a$ for $p \neq p_0$, has the analytic continuation $x_a = \lim M_a y_a$ at p_i. Since $x_b = y_b = 0$ and since M_b is finite, the last condition can be extended into $x = \lim My$.

By combining (10) and (11) one obtains $P_0(\lim My) = u$. Since both factors are finite, (11) is equivalent to

$$\lim PMy = u; \quad \lim y = 0. \tag{12}$$

By the substitution

$$y = Qv \tag{13}$$

the conditions (8) and (12) are equivalent, since Q is finite non-singular.

IV. Second Criterion for Degree Reduction

A different, but equivalent form of the condition for degree reduction can be derived from the ranks of the Hankel matrices. The condition is: inequality (7) at $p_0 \neq \infty$ is equivalent to the existence of a non-zero vector x with nm components such that

$$Tx = z, \tag{14}$$

where T is the Hankel matrix of A at p_0 defined as in Eqs. (5)–(6) and where

$$z = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix}. \tag{15}$$

The unipolar component of $S_2 A = C = [1_n - 2\alpha_0 u\tilde{u}/(p+p_0^*)]A$ at p_0 is deduced from the unipolar component of $\hat{S} = A$, of the form (5) with $p_i = p_0$, by replacing $(p+p_0^*)^{-1}$ in the expression of C by its Taylor expansion in powers of $p-p_0$ and by regrouping the terms of identical powers. The Hankel matrix T_1 of C is then deduced from the one (6) of A by

$$T_1 = UT \tag{16}$$

with

$$U = \begin{bmatrix} 1_n - u\tilde{u} & 0 & 0 & 0 & \cdots \\ u\tilde{u}/2\alpha_0 & 1_n - u\tilde{u} & 0 & 0 & \cdots \\ -u\tilde{u}/(2\alpha_0)^2 & u\tilde{u}/2\alpha_0 & 1_n - u\tilde{u} & 0 & \cdots \\ u\tilde{u}/(2\alpha_0)^3 & -u\tilde{u}/(2\alpha_0)^2 & u\tilde{u}/2\alpha_0 & 1_n - u\tilde{u} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^m u\tilde{u}/(2\alpha_0)^{m-1} & \cdots & \cdots & \cdots & 1_n - u\tilde{u} \end{bmatrix}. \tag{17}$$

The condition for degree reduction is rank $T_1 <$ rank T, hence the existence of non-zero vectors x and z such that (14) holds with

$$T_1 x = 0. \tag{18}$$

By Eqs. (16) and (14), Eq. (18) is

$$Uz = 0. \tag{19}$$

If the mn-vector z is partitioned into a number m of n-vectors $z_0, z_1, \ldots, z_{m-1}$, Eq. (19) becomes, owing to (17),

$$\left.\begin{aligned} (1_n - u\tilde{u})\, z_0 &= 0, \\ u\tilde{u} z_0/2\alpha_0 + (1_n - u\tilde{u})\, z_1 &= 0, \\ \ldots . \end{aligned}\right\} \tag{20}$$

The general solution of the first equation (20) is $z_0 = \lambda u$, where λ is an arbitrary scalar and where u is normalized by Eq. (3). Substituting this value of z_0 in the second equation (20) premultiplied by $\tilde{u}$, and using Eq. (3), one obtains $\lambda = 0$, hence $z_0 = 0$, so that Eq. (20) is reduced to a similar system with one equation less. By recurrence one proves that the solution of Eq. (20) is Eq. (15).

An alternative proof of the second criterion is now given by establishing its equivalence with the first criterion. Write $p - p_0 = \varepsilon$ and consider the unipolar component

$$K_0/\varepsilon^m + K_1/\varepsilon^{m-1} + \dots K_{m-1}/\varepsilon$$

of $\hat{S}$ at p_0. Owing to the second relation (8), the Taylor expansion of v is of the form $\varepsilon x_0 + \varepsilon^2 x_1 + \dots$. The first relation (8) is then verified if and only if all terms with negative powers of ε cancel in the product Av and if the constant term is u. This proves that Eq. (8) is equivalent to the relations

$$\left.\begin{aligned} K_0 x_0 &= 0, \\ K_1 x_0 + K_0 x_1 &= 0, \\ &\dots \\ K_{m-1} x_0 + \dots + K_0 x_{m-1} &= u, \end{aligned}\right\} \tag{21}$$

which are simply the expanded form of (14–15). Consequently, by solving Eq. (21) we obtain v, and have hence interrelated the two methods. Note that Eq. (21) shows that the vector z, which gives u, is a linear combination of the columns of T; this allows for convenient computation of u.

V. Degree Reduction for Impedance Matrices

Instead of applying (8) to the matrix $\hat{S}$ considered in Section III one may add an arbitrary constant finite matrix. In order to obtain relations in terms of the impedance matrix, we set $A = \hat{S} - 1_n$ and $w = Av$ so that the first relation (8) becomes $\lim w = u$. If $(\hat{S} - 1_n)^{-1}$ exists (if the n-port has an impedance matrix), v is $(\hat{S} - 1_n)^{-1} w$, and the second relation (8) becomes $\lim (\hat{S} - 1_n)^{-1} w = v$. Finally, (8) is equivalent to

$$\lim (1_n - \hat{S})^{-1} w = 0; \quad \lim w = u \neq 0. \tag{22}$$

In the cascade synthesis [CNT, sec. 11.24] of a passive n-port of prescribed (positive) impedance matrix $Z(p)$, S is normalized to $\tilde{Z}_0$, where $Z_0 = Z(p_0)$ with $\operatorname{Re} p_0 > 0$, by

$$S = R_0^{-\frac{1}{2}}(Z - Z_0)(Z + \tilde{Z}_0)^{-1} R_0^{\frac{1}{2}} \tag{23}$$

where $R_0 = (Z_0 + \tilde{Z}_0)/2$ and where $R_0^{\frac{1}{2}}$ denotes the hermitian square root matrix. Since (23) produces $S(p_0) = 0$, (4) is satisfied with any u. By the criterion of Section IV, the only condition for degree reduction is then the existence of a pole of $\hat{S}$ at p_0: one then has $K_0 \neq 0$ in (21) and a solution $u \neq 0$ is produced, for instance, by $x_0 = x_1 = \dots = x_{m-2} = 0$ and $x_{m-1} \neq 0$ an arbitrary vector which does not anihilate K_0. Equivalently $1_n - \hat{S}$ must have a pole at p_0. Since Eq. (23) gives

$$(1_n - \hat{S})^{-1} = R_0^{-\frac{1}{2}}(\hat{Z} + Z_0) R_0^{-\frac{1}{2}}/2 \tag{24}$$

the only condition for degree reduction is that $(\hat{Z}+Z_0)^{-1}$ must have a pole at p_0, and it only remains to determine the vector u by Eq. (22) which certainly has a solution. With the notations

$$h = \alpha_0^{\frac{1}{2}} R_0^{-\frac{1}{2}} u;\ g = \alpha_0^{\frac{1}{2}} R_0^{-\frac{1}{2}} w \tag{25}$$

and remarking that from Eq. (4) we had to choose the normalization such that

$$Z_0 h = \hat{Z}(p_0)\, h \tag{26}$$

the conditions (22) are transformed by (24) into

$$\lim\,(Z+\hat{Z})\, g = 0;\ \lim g = h \neq 0 \tag{27}$$

where Z_0 has been replaced by $\lim Z$, for Z is analytic in $\mathrm{Re}\, p > 0$. We now see that an all-pass section of degree one can be extracted from Z at any pole p_0 of $(Z+\hat{Z})^{-1}$ in $\mathrm{Re}\, p > 0$, and its parameters are determined by (27), or an equivalent condition in terms of Hankel matrices.

VI. Transmission Zeros

For any two complex frequencies p_α and p_β of a lossless n-port such that

$$p_\beta = -p_\alpha^* \tag{28}$$

corresponding voltages and currents satisfy:

$$\tilde{i}_\beta v_\alpha + \tilde{v}_\beta i_\alpha = 0. \tag{29}$$

The proof consists in verifying (29) for all separate lossless components (L, C, imaginary resistances, ideal transformers and gyrators) and noting that interconnections merely cancel pairs of terms in the sum of the separate equations. Relation (29) is obviously equivalent to $2R(p) = Z+\hat{Z} = 0$ if Z exists.

For a dissipative n-port, the above theorem is not true, but may be valid locally for some p_α (then also for $-p_\alpha^*$). The n-port is then *locally lossless* if i and v are both not 0. Let

$$Av = Bi \tag{30}$$

be the equations defining the n-port, where A and B are square polynomial matrices of order n, left coprime if the n-port is completely controllable. Relations (30) for frequency p_α and (29), form a homogeneous system of $n+1$ equations in the $2n$-vector formed by v_α and i_α. The matrix of the system of equations must then have rank n at most, and a non-zero row-vector $[\xi^*, \tilde{x}]$ exists, where ξ is a scalar and x an n-vector, such that

$$[\xi^*\, \tilde{x}] \begin{bmatrix} \tilde{i}_\beta & \tilde{v}_\beta \\ A_\alpha & -B_\alpha \end{bmatrix} = 0$$

or

$$\xi^* \tilde{i}_\beta = -\tilde{x}A_\alpha; \quad \xi^* v_\beta = \tilde{x}B_\alpha. \tag{31}$$

Moreover, one has $\xi \neq 0$ in (31), otherwise A_α and B_α would not be left coprime, so that one can set $\xi = 1$. On the other hand, the conjugate transpose of Eq. (30) for frequency p_β is

$$\tilde{i}_\beta \tilde{B}_\beta - \tilde{v}_\beta \tilde{A}_\beta = 0. \tag{32}$$

Eliminating $\tilde{i}_\beta$ and $\tilde{v}_\beta$ between Eqs. (32) and (31) with $\xi = 1$, one obtains a homogeneous system in the row-vector x, and this yields the condition

$$\det(A_\alpha \tilde{B}_\beta + B_\alpha \tilde{A}_\beta) = 0. \tag{33}$$

Since A_α is $A(p_\alpha)$ to be simply noted A, A_β is the conjugate transpose of $A(-p_\alpha^*)$ which is $\hat{A}$. Consequently the points of local losslessness are the roots of

$$\det(A\hat{B} + B\hat{A}) = 0. \tag{34}$$

We now prove that an n-port having an impedance matrix Z, such that Z and $Z + \hat{Z}$ have normal rank n, is locally lossless at the poles of $(Z+\hat{Z})^{-1}$, and at the poles of Z on the imaginary axis, and nowhere else. Since $Z = A^{-1}B$ and

$$(Z + \hat{Z})^{-1} = \hat{A}C^{-1}A \tag{35}$$

with $C = A\hat{B} + B\hat{A}$, every finite pole of $(Z+\hat{Z})^{-1}$ is a zero of $\det C$, because A and C are polynomial matrices. Conversely, a zero of $\det C$ certainly produces a pole of $(Z+\hat{Z})^{-1}$ when A and $\hat{A}$ are both non-singular. We prove, *ab absurdo*, that this is still true if only one of the matrices A or $\hat{A}$ is singular, for instance the second one (the proof is similar in the opposite case). If $G = (Z+\hat{Z})^{-1}$ were then finite, so would be $F = \hat{A}^{-1}G = C^{-1}A$. From the resulting equation $A = CF$ and $\det C = 0$, one concludes that a row-vector $\tilde{x} \neq 0$ exists such that $\tilde{x}C = 0, \tilde{x}A = 0$, hence also $\tilde{x}B = 0$ by the definition of C and the non-singularity of $\hat{A}$. The last two relations are, however, impossible since A and B are left coprime.

If A and $\hat{A}$ are both singular, and this can only occur on the imaginary axis (including infinity) so that upper circumflexes can be replaced by upper tildes, Z, and hence $\hat{Z}$, is necessarily infinite: for a row-vector $\tilde{x}$ such that $\tilde{x}A = 0$, one would deduce $\tilde{x}B = 0$ from $B = AZ$ with Z finite, which is again impossible for left coprime matrices. Conversely, at a finite pole of Z on the imaginary axis, A is singular and one has $\tilde{x}A = 0$ for some $x \neq 0$, hence $\tilde{x}A\tilde{B} = 0$, hence $\tilde{x}Cx = 0$, so that $\det C = 0$ because C is hermitian non-negative definite. The same result holds for a pole at infinity by the transformation of p into $1/p$.

If Z is minimum reactance of degree m, the only points of local losslessness are the $2m$ poles of $(Z+\hat{Z})^{-1}$. Since they are associated in pairs $(p_i, -p_i^*)$ and are of even multiplicity on the imaginary axis, it is convenient to halve the set and call *transmission zeros*, in accordance with Youla (**2**), the following:

(a) the poles of $(Z+\hat{Z})^{-1}$ in $\mathrm{Re}\, p > 0$, with multiplicities equal to their degrees

(b) the poles of $(Z+\hat{Z})^{-1}$ in $\mathrm{Re}\, p = 0$ (including infinity), with multiplicities equal to half their degrees.

If Z is not minimum reactance, we define as additional transmission zeros

(c) the poles of Z on $\mathrm{Re}\, p = 0$, with multiplicities equal to their full degrees. With the above convention, the total number of transmission zeros is m in all cases. The above results are a natural extension of the one-port treatment [CNT, sec. 9.36]. The transmission zeros of type (b) and (c) can both be found as poles of $(1_n - \hat{S}S)^{-1}$.

VII. Cascade Synthesis

We consider the cascade synthesis [CNT, chapters 10 and 11] of a passive n-port for which Z and $Z+\hat{Z}$ have normal rank n. Three kinds of lossless sections can be extracted, corresponding to the three types of transmission zeros discussed in Section VI: type (a) corresponds to the Darlington process of section V, type (b) corresponds to the Brune process [CNT, Chap. 10] whereas type (c) is a Foster extraction. We will prove that, after any extraction of a section of degree one, the set of transmission zeros in $\mathrm{Re}\, p > 0$ [type (a)] on the one hand, and the set of transmission zeros on $\mathrm{Re}\, p = 0$ [type (b) and (c)] on the other, remain separately invariant, except for the suppression (or multiplicity reduction by one unit) of the zero used in the extraction process. There is, however, no separate invariance for the transmission zeros of type (b) and type (c), as already known in the one-port case [CNT, sec. 9.8].

The invariance of the transmission zeros is a consequence of the following theorem: if N is an n-port formed by a lossless $2n$-port N_2 terminated on a (dissipative) n-port N_1, every transmission zero of N_1 is a transmission zero of N. This results immediately from the fact that (29) is satisfied identically for N_2. In the case of cascade synthesis with degree reduction by one unit, where $\deg N = m$, $\deg N_2 = 1$, $\deg N_1 = m-1$, the sets of transmission zeros of N and N_1 can only differ by one member, and this proves the invariance of $m-1$ zeros. As for the multiplicity of the different transmission zeros, it will appear in the sequel [see after (47)] that no increase in degree can occur at any transmission zero, while the only transmission zero where a degree reduction actually occurs is fixed by the section extracted.

By contrast, there is no invariance in the case where $Z+\hat{Z}$ is identically singular, for (27) can then be satisfied at an arbitrary point. *A fortiori*, the number of Darlington extractions is not invariant in that case: as an example, a 2-port Darlington-section open-circuited at its output is a lossless 1-port and admits a pure Foster realization.

We now return to the case where $Z+\hat{Z}$ has normal rank n, assume that a section of degree one of type (a) has been extracted at p_0 and denote by Z_f the residual impedance matrix. This matrix always exists and is such that $Z_f + \hat{Z}_f$ has normal rank n: in the opposite case the n-port N_f would be locally lossless at arbitrary points, and so would be N by an immediate extension of a previous theorem. It has been proved in CNT (p. 348) that one has

$$2PR_f P = P(Z_f + \hat{Z}_f)\hat{P} = Z + \hat{Z} = 2R \tag{36}$$

where

$$P = (Z - Z_0)h\tilde{h}/(p - p_0) - 1_n. \tag{37}$$

By (37) P is clearly analytic in $\mathrm{Re}\,p > 0$, and it has been proved in CNT that so is P^{-1} (except for the point p_0). We need more, however, in this context. The poles of R_f^{-1} in $\mathrm{Re}\,p > 0$ are related to the poles of R^{-1} through the expression:

$$R_f^{-1} = TR^{-1}P - \frac{2h\tilde{h}}{p + p_0^*}P \tag{38}$$

where

$$T = \frac{h\tilde{h}}{p + p_0^*}(Z + \tilde{Z}_0) - 1_n \tag{39}$$

P and T are both non-singular in $\mathrm{Re}\,p > 0$ except for $p = p_0$ where T is singular and given by:

$$T(p_0) = -R_0^{-\frac{1}{2}}(1 - u\tilde{u})R_0^{-\frac{1}{2}} \tag{40}$$

and $P(p_0)$ is non-singular. We show that these facts are easy consequences of Schwarz' Lemma (see e.g. [**12**, p. 240]) if one maps the p-plane conformally on the unit circle by the transformation:

$$z = \frac{p - p_0}{p + p_0^*}. \tag{41}$$

It is obvious that only vectors proportional to h can annihilate T or P. This in turn will happen at some point $p \neq p_0$ only if respectively:

$$\tilde{h}(Z - Z_0)h = \zeta - \zeta_0 = p - p_0 \tag{42a}$$

$$\tilde{h}(Z + \tilde{Z}_0)h = \zeta + \zeta_0^* = p + p_0^* \tag{42b}$$

where

$$\zeta = \tilde{h}Zh. \tag{43}$$

The validity of (42a) implies that of (42b), and conversely, due to

$$\zeta_0 + \zeta_0^* = p_0 + p_0^* = 2\alpha_0 \tag{44}$$

by (43) and the normalizing condition (3) applied to h of (25). The equalities (42) then contradict Schwartz lemma [**12**, p. 240] requiring the strict inequality

$$\left|\frac{\zeta - \zeta_0}{\zeta + \zeta_0^*}\right| \cdot \left|\frac{p + p_0^*}{p - p_0}\right| < 1 \tag{45}$$

in $\mathrm{Re}\, p > 0$ for the positive function ζ, unless (45) is transformed into an equality holding for all p. In the latter case, however, (42) holds for all p. $Z + \hat{Z}$ is then identically singular, in contradiction with our hypothesis. At the point p_0 (45) reduces to:

$$\left| \frac{\mathrm{d}\zeta(p_0)}{\mathrm{d}p} \right| < 1 \tag{46}$$

while at p_0, P becomes:

$$P(p_0) = \frac{\mathrm{d}Z(p_0)}{\mathrm{d}p} h\tilde{h} - 1_n \tag{47}$$

$P(p_0)$ can only be singular if the left member of (46) is equal to one.

The degree of R_f^{-1} has thus to be equal to the degree of R^{-1} at any point $p \neq p_0$ in $\mathrm{Re}\, p > 0$. A degree reduction of one unit was already shown to occur at $p = p_0$, hence the extraction of an all-pass section leaves unchanged all degrees of transmission zeros except at p_0.

To continue the synthesis, one must solve for Z_f relations similar to (27), hence

$$\lim_{p \to p_a} (Z_f + \hat{Z}_f) g_a = 0; \quad \lim_{p \to p_a} g_a = h_a \neq 0 \tag{48}$$

at some p_a (possibly $= p_0$). We further show that the possible extraction points p_a and the associated vectors h_a are completely determined by the solution (27) for the original matrix, other than the one (p_0, h) already used. We thus consider another solution

$$\lim_{p \to p_a} (Z + \hat{Z}) g_b = 0; \quad \lim_{p \to p_a} g_b = h_b \neq 0 \tag{49}$$

with $h_b \neq h$ if $p_a = p_0$. Using (37) and the fact that $p_a \neq p_0$ or that $h_b \neq h$ in case $p_a = p_0$ we have that the choice

$$g_a = \hat{P} g_b \tag{50}$$

yields

$$h_a = T(p_a) h_b = \left[\frac{h\tilde{h}(Z_a + \hat{Z}_0)}{p_a + p_0^*} - 1_n \right] h_b \neq 0 \tag{51}$$

and satisfies the first condition in (48).

In the last theorem we have assumed that the first extraction (at p_0) was of the Darlington type ($\mathrm{Re}\, p_0 > 0$). We now prove that the theorem remains true if the point of the first (but not of the second) extraction is of the Brune or Foster type. We thus take $p_0 = j\omega_0$ but keep $\mathrm{Re}\, p_a > 0$. The theorem is trivial in the Foster case, since $Z_f + \hat{Z}_f = Z + \hat{Z}$. The Brune case can be considered as the limit of the Darlington case for $\alpha_0 = 0$ [CNT, sec. 11.28] and one has $R_0 h = 0$, so that ζ_0 in (42a) reduces to $j\tilde{h} X_0 h$ and is imaginary. The function $\eta = \zeta - \zeta_0$ is a positive function of p, and also of $q = p - j\omega_0$, and (32) gives $\eta = q$ in contradiction with Schwarz' lemma requiring $|\arg \eta| < |\arg q|$ in $\mathrm{Re}\, p > 0$. The same modification applies to (42b).

VIII. Symmetric Factorization of a Bounded Matrix

In the synthesis of reciprocal n-ports, where S is symmetric, the factorization (1) must be continued by

$$S_1 = S_2' S_3 \tag{52}$$

so as to obtain $S = S_2' S_3 S_2$ with S_3 symmetric. We assume that the criterion of Section III has been satisfied for the first factorization so that v has been determined by (8), where A stands for $\hat{S}$, and investigate under what conditions the second factorization (where u, hence S_2, is now fixed) produces an additional degree reduction. If we rewrite (52) as $\hat{S}_2' S_1 = S_3$, the second condition is $\deg S_2 \hat{S}_1' < \deg \hat{S}_1'$, identical to (7) but with A now replaced by $\hat{S}_1'$.

By (8), the second condition is

$$\lim \hat{S}_1' v_a = u; \quad \lim v_a = 0. \tag{53}$$

In (53), $\hat{S}_1'$ is $\hat{S}' S_2'$ by (1), hence $\hat{S} S_2'$, for S was assumed symmetric. Since $\hat{S}$ has been noted A in the first condition, one has $\hat{S}_1' = A S_2'$ and the first relation (53) becomes

$$\lim A S_2' v_a = u. \tag{54}$$

By comparison with (8), (54) is satisfied if $S_2' v_a = v$, hence if $v_a = \hat{S}_2' v$, or

$$v_a = \left(1_n + \frac{2\alpha_0 u^* u'}{p - p_0}\right) v. \tag{55}$$

The second condition (53), forces $u'v$ in (49) to tend to zero at least as ε^2, with $\varepsilon = p - p_0$. Since v is of the form $\varepsilon x_0 + \varepsilon^2 x_1 + \ldots$ mentioned in Section IV, the condition is finally

$$u' x_0 = 0 \tag{56}$$

and must be added to (21).

At a multiple pole ($m > 1$), (55) can always be satisfied, for it is sufficient to choose $x_0 = 0$ in (21). At a simple pole ($m = 1$), (21) reduces to the single relation $K_0 x_0 = u$, and its combination with (56) requires

$$x_0' K_0 x_0 = 0 \tag{57}$$

so that the symmetric residue matrix K_0 of $\hat{S}$ at p_0 must be indefinite. The last condition was obtained by Oono–Yasuura (**6**) but the proof given in [CNT, sec. 12.27] is erroneous.

In addition to degree reduction, one has to consider boundedness. For the first factorization, condition (4) makes S_1 bounded with S. Since S_3 is symmetric, (52) is equivalent to $S_1' = S_3 S_2$. By analogy with (1), S_3 will be bounded with S_1 if and only if $S_1'(p_0) u = 0$, hence $u' S_1(p_0) = 0$. By (1), this becomes $u' S(p_0) \hat{S}_2(p_0) = 0$, hence

$$u' S(p_0) + 2\alpha_0 \left[\lim_{p \to p_0} \frac{u' S(p)}{p - p_0}\right] u\tilde{u} = 0.$$

By the transpose of (4) and l'Hopital's rule, this becomes

$$u' \frac{\mathrm{d}S}{\mathrm{d}p}\bigg|_{p_0} u\tilde{u} = 0$$

and, since $\tilde{u} \neq 0$,

$$u' \frac{\mathrm{d}S}{\mathrm{d}p}\bigg|_{p_0} u = 0. \tag{58}$$

IX. Discussion

In the foregoing we have presented the theory needed to guarantee that degree reduction will occur when a scattering matrix is synthesized in cascade form by its factorization. The condition for degree reduction through extraction of a right-half plane factor with a pole at $-p_0^*$ is simply that p_0 is a pole of $G = (Z+\hat{Z})^{-1}$; numerical values needed for the extraction can be deduced from (25) (27) while (7) (8), or equivalently (14) (15), are intermediate for justification of the extraction.

For the actual factorization synthesis of a given matrix, one proceeds through the following sequence of steps.

(1) If S is given with respect to a normalizing impedance Z_1, convert to the impedance matrix

$$Z = 2R_1^{\frac{1}{2}}(1_n - S)^{-1}R_1^{\frac{1}{2}} - \tilde{Z}_1 \tag{59}$$

[found from (24): recall that $2R_1 = Z_1 + \tilde{Z}_1$].

(2) Determine the zeros of transmission $p_0 = \alpha_0 + j\omega_0$. These are $j\omega$ axis poles of Z as well as $\mathrm{Re}\, p \geqslant 0$ poles of $G = (Z+\hat{Z})^{-1}$ [or equivalently, $\mathrm{Re}\, p \geqslant 0$ poles of $(1_n - S)(1_n - \hat{S}S)^{-1}(1_n - \hat{S})$]. A cascade section is to be extracted for each transmission zero.

(3) For $\alpha_0 > 0$ extractions, choose a u for (2). This occurs by first choosing an h_1 from the fact that

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_1 \end{bmatrix}$$

is a linear combination of the columns of the Hankel matrix, (6), for G at p_0. All h_1 are found in this manner and are such that there exists a $g(p)$ satisfying $h_1 = \lim_{p \to p_0} g(p)$ and $\lim_{p\to p_0} Rg = 0$ (this makes sense since p_0 is a pole of $G = 2R^{-1}$).

Given h_1 one chooses the normalizing impedance Z_0 such that $Z(p_0)h_1 = Z_0 h_1$; for simplicity, and without loss of generality, one can choose $Z_0 = Z(p_0)$. A knowledge of Z_0 fixes the Hermitian part R_0,

$$2R_0 = Z_0 + \tilde{Z}_0,$$

and then one forms $u_1 = \alpha_0^{-\frac{1}{2}} R_0^{\frac{1}{2}} h_1$ according to (25) (in which h is h_1 except for a scale factor). The normalization of (3) is obtained by choosing

$$u = u_1/\| u_1 \| = u_1/(\tilde{u}_1 u_1)^{\frac{1}{2}}. \tag{60}$$

Note that $S(p_0)u = 0$ as required but that any u satisfying $S(p_0)u = 0$ is not satisfactory; one must proceed by obtaining proper u for degree reduction by the steps given here. As discussed after (23), a convenient choice for u_1 is given by $u_1 = K_0 x_{m-1}$ with x_{m-1} arbitrary vector for which $u_1 \neq 0$.

(4) The matrix $S = R_0^{-\frac{1}{2}}(Z - Z_0)(Z + \tilde{Z}_0)^{-1} R_0^{\frac{1}{2}}$ is then factored into $S = S_1 S_2$, when $\alpha_0 > 0$, where

$$S_2 = 1_n - \frac{2\alpha_0 u\tilde{u}}{p + p_0^*} = U\left[\frac{p - p_0}{p + p_0^*} + 1_{n-1}\right]\tilde{U} \tag{61}$$

(the right hand form, which exists for some U, shows why p_0 is a zero of transmission). One then repeats on

$$S_1 = SS_2^{-1} = S\hat{S}_2. \tag{62}$$

Section VIII discusses a further factorization when S is symmetric, while if real factors are desired from originally real matrices a factorization for p_0^* is next carried out.

(5) For $\alpha_0 = 0$, Brune or Foster extractions are made.

(6) The process is continued until it terminates which occurs when all zeros of transmission have been extracted. The zeros of transmission remaining after a given extraction are invariant to the extraction, and as many sections as the degree of the original S occur. The number, and the nature, of the sections is fixed by the set of zeros of transmission, as with the scalar case.

(7) Physical realizations of each zero of transmission section follow from classical n-port synthesis techniques; the sections are connected in cascade by realizing the coupling scattering matrix, when $\alpha_0 > 0$,

$$\Sigma = \begin{bmatrix} 0_n & 1_n \\ S_2 & 0_n \end{bmatrix} \tag{63}$$

which is loaded in a realization for S_1. A circuit realizing Σ need only use one reactive element and one gyrator [CNT, pp. 336, 311], though complex transformers may be required before conjugate sections are combined. Further discussions on the circuit realization of real and symmetric sections are given in CNT, chapters 10 and 11.

In summary, except for the further restrictions developed here on the choice of u, the theory is as developed in [(**3**), chapter 11] and abstracted in [(**13**), p. 286].

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