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TRANSFER FUNCTION SENSITIVITY FOR DISCRETE AND INTEGRATED CIRCUITS*

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Abstract—Structures consisting of uniform types of resistive and dielectric materials for the realization of tarnsfer functions are investigated. It is shown through the use of dimensional analysis that the transfer function sensitivity depends only upon the transfer function and not upon the realization configuration. An interpretation through the Bode plot shows that sharp cut-off leads to high sensitivity.

1. INTRODUCTION

In the design of circuits one of the points of consideration is that of sensitivity to parameter changes; for example, temperature changes in structures constructed of silicon are considered important. Previously it has been shown that to first order, reasonably designed integrated circuits have a transfer function sensitivity which depends only upon the material used and the transfer function obtained, not upon the realization configuration.[1, 2] (If both *n*-type and *p*-type resistive materials are present, this condition becomes slightly in error.) Here we show where, and that, the previously obtained results hold exactly; we also give an interpretation in terms of the Bode plot of the transfer function.

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2. PROBLEM STATEMENT AND DIMENSIONAL ANALYSIS RESULT

We consider first the situation where a transfer function $T(s)$, for the moment voltage-to-voltage or current-to-current, is synthesized using lumped or distributed, active or passive, components which are comprised of material of resistivity $\rho(x)$ and dielectric constant $\varepsilon(x)$. Here x is a parameter, such as temperature, which by some mechanism causes changes in ρ and ε. We then wish to explicitly evaluate the transfer function sensitivity

$$S_x{}^T = \frac{x}{T}\frac{\partial T}{\partial x}. \tag{1}$$

Note that $T(s)$ need not be rational while resistors, capacitors, distributed RC-lines and *ideal* voltage-to-voltage and current-to-current amplifiers fall into the category of components under consideration. (Whether there is practical sense in postulating ideal amplifiers may be thought debatable. However, as we argue in Section 5, we have great freedom in the R-C part of the model, frequently perhaps sufficient to model an actual amplifying device with an ideal amplifier and other allowable components. In any case, many of our results will still be true in a qualitative sense.)

Suppose that the correct values of ρ and ε occur at $x = x_0$ and that these are $\rho(x_0) = \rho_{\text{norm}}$ and

$\varepsilon(x_0) = \varepsilon_{norm}$. If $\rho \neq \rho_{norm}$ or $\varepsilon \neq \varepsilon_{norm}$ then generally $T(s)$ will not be precisely synthesized. Rather, we will obtain another transfer function; this transfer function—which is a mapping from the s-plane into the plane of complex numbers—may be considered as being parameterized by ρ and ε, or what is almost the same, by x. More generally we may regard it as a function of s, ρ and ε, where normally the latter two variables are given fixed values while s is permitted to vary. In any case we shall call this transfer function $\mathbf{T}(s;\ \rho,\varepsilon)$, or $T_x(s)$ if we wish to emphasize the x dependence. Evidently

$$T(s) = T_{x_0}(s) = \mathbf{T}(s;\ \rho_{norm}, \varepsilon_{norm}). \qquad (2)$$

Since

$$S_x{}^{T_x} = \frac{\rho}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial \rho}\cdot\frac{x}{\rho}\frac{\partial \rho}{\partial x} + \frac{\varepsilon}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial \varepsilon}\cdot\frac{x}{\varepsilon}\frac{\partial \varepsilon}{\partial x} \qquad (3a)$$

$$= S_\rho{}^{\mathbf{T}}\cdot S_x{}^{\rho} + S_\varepsilon{}^{\mathbf{T}}\cdot S_x{}^{\varepsilon} \qquad (3b)$$

our first goal is to evaluate $S_\rho{}^{\mathbf{T}}$ and $S_\varepsilon{}^{\mathbf{T}}$ for fixed values of $\rho = \rho_{norm}$ and $\varepsilon = \varepsilon_{norm}$, and then to study the variation of these quantities for various $s = j\omega$.

At this point we conduct a dimensional analysis on $\mathbf{T}(s;\ \rho,\varepsilon)$. For fixed s, ρ and ε, this transfer function is a dimensionless number—in particular the ratio of two voltages or two currents. Now from dimensional analysis[3] the function $s^a\rho^b\varepsilon^c$ must be dimensionless; we determine $a = b = c$ since

$$[s] = [\text{time}]^{-1},\ [\rho] = [\text{resistance}][\text{length}],$$
$$[\varepsilon] = [\text{capacitance}][\text{length}]^{-1} \qquad (4a)$$

or

$$[s\rho\varepsilon] = [\text{resistance}][\text{capacitance}][\text{time}]^{-1} = [\text{constant}]. \qquad (4b)$$

We conclude that $\mathbf{T}$ is in fact a function of the product of s, ρ and ε, i.e.

$$\mathbf{T}(s;\ \rho,\varepsilon) = \hat{T}(s\rho\varepsilon) \qquad (5)$$

for some function $\hat{T}$ of a single variable. As a consequence, by elementary manipulation we have then

$$\frac{\rho}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial \rho} = \frac{\varepsilon}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial \varepsilon} = \frac{s}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial s} \qquad (6a)$$

or, also setting $s = j\omega$,

$$S_\rho{}^{\mathbf{T}} = S_\varepsilon{}^{\mathbf{T}} = S_s{}^{\mathbf{T}} = S_\omega{}^{\mathbf{T}} = \frac{\partial \ln \mathbf{T}(j\omega;\ \rho,\varepsilon)}{\partial \ln \omega} \qquad (6b)$$

Finally, from equation (3b):

$$S_x{}^{\mathbf{T}(j\omega)} = S_\omega{}^{\mathbf{T}(j\omega;\ \rho_{norm},\ \varepsilon_{norm})}\,[S_x{}^{\rho} + S_x{}^{\varepsilon}] \qquad (7a)$$

$$= \frac{\mathrm{d} \ln T(j\omega)}{\mathrm{d} \ln \omega}\,[S_x{}^{\rho} + S_x{}^{\varepsilon}]. \qquad (7b)$$

3. INTERPRETATION

The interpretation of the last result is as follows. The quantities

$$\left.\frac{\rho}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial \rho}\right|_{\rho_{norm},\ \varepsilon_{norm}}$$

and

$$\left.\frac{\varepsilon}{\mathbf{T}}\frac{\partial \mathbf{T}}{\partial \varepsilon}\right|_{\rho_{norm},\ \varepsilon_{norm}}$$

are the (percentage) sensitivity coefficients of the transfer function $T(s)$ with respect to resistivity and permittivity at the normal values of resistivity and permittivity. These quantities are equal, being given by the logarithmic derivative

$$\frac{\mathrm{d} \ln T(j\omega)}{\mathrm{d} \ln \omega}.$$

This latter quantity has an interesting interpretation in terms of Bode diagrams which, it will be recalled,[4] plot $20 \log_{10} |\ T(j\omega)\ |$ and $\arg T(j\omega)$ vs. $\log_{10} \omega$. Now, since $\ln x = \ln 10 \log_{10} x$,

$$\frac{\mathrm{d} \ln T(j\omega)}{\mathrm{d} \ln \omega} = \frac{\mathrm{d} \log_{10} T(j\omega)}{\mathrm{d} \log_{10} \omega} =$$

$$= \frac{1}{20}\frac{\mathrm{d}\, 20 \log_{10} |\ T(j\omega)\ |}{\mathrm{d} \log_{10} \omega} + j\frac{\mathrm{d} \arg T(j\omega)}{\mathrm{d} \log_{10} \omega}. \qquad (8)$$

We conclude that fast variations in either the magnitude or phase Bode plot will cause relatively high values in the sensitivity, as shown by equations (7b) and (8).

4. SIMPLE EXAMPLES

As an illustration of the result we note that the single pole voltage–voltage transfer function $1/(s+1)$ has near $\omega = 0$ sensitivities which are approximately zero. Well above the breakpoint they are approximately $(-1+j0)$ times the sum of resistivity and permittivity sensitivities. In the vicinity of the breakpoint, the coefficient of these material sensitivities has a real and an imaginary part between zero and -1.

As a second example, consider

$$T(s) = \frac{s}{s^2+0{\cdot}1s+1} \tag{9a}$$

The magnitude Bode diagram suggests that relatively high sensitivity will be encountered at about the 3 dB points, while the phase Bode diagram suggests high sensitivity at $\omega = 1$. Indeed we have

$$\frac{\mathrm{d}\ln T(j\omega)}{\mathrm{d}\ln\omega} = \frac{1+\omega^2}{(1-\omega^2)+0{\cdot}1j\omega} \tag{9b}$$

The magnitude of this function peaks at $\omega = 1$ where the functional value is actually $-20j$.

5. GENERALIZATIONS AND SPECIFICATIONS

Hitherto we have only considered dimensionless transfer functions $T(s)$. Nevertheless, we may be interested in transfer functions with other dimensions, as ohms or $(\text{ohms})^{-1}$, for example. An extension of the earlier dimensional analysis argument shows that when $T(s)$ has the dimension of ohms, we must have

$$\mathbf{T}(s;\rho,\varepsilon) = K\rho T_0(s\rho\varepsilon) \tag{10a}$$

for some function $T_0(\cdot)$ and some factor K which is independent of s, ρ and ε with dimension $[\text{length}]^{-1}$. It is then easily checked that

$$\frac{\rho}{\mathbf{T}}\frac{\partial\mathbf{T}}{\partial\rho} = 1+\frac{\varepsilon}{\mathbf{T}}\frac{\partial\mathbf{T}}{\partial\varepsilon} = 1+\frac{\partial\ln\mathbf{T}(j\omega;\rho,\varepsilon)}{\partial\ln\omega} \tag{10b}$$

Consequently, in this case the previous formula holds for dielectric sensitivity while the formula for resistivity sensitivity simply increases the old value by unity. The Bode diagram interpretation is still valid.

When $T(s)$ has the dimension $[\text{ohm}]^{-1}$, the same argument shows that the sole change is to decrease the value of resistivity sensitivity by unity over the dimensionless case.

It may well occur that changes in the parameter x cause inverse changes in resistivity and dielectric constant. Indeed if it is the case that $\rho(x)\varepsilon(x) =$ constant, then

$$S_x{}^{\rho} = -S_x{}^{\varepsilon}. \tag{11}$$

Hence, using equation (7b), we see that $S_x{}^T = 0$ and $T(s)$ is unaltered by changes of x in the dimensionless transfer function case.

If several types of resistive and capacitive materials are present we can write, again by dimensional analysis, $\mathbf{T} = \mathbf{T}(s\rho_1\varepsilon_1, \ldots, s\rho_n\varepsilon_n)$ and hence

$$\frac{s}{\mathbf{T}}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \sum_{i=1}^{n}\frac{\rho_i}{\mathbf{T}}\frac{\partial\mathbf{T}}{\partial\rho_i} = \sum_{i=1}^{n}\frac{\varepsilon_i}{\mathbf{T}}\frac{\partial\mathbf{T}}{\partial\varepsilon_i}. \tag{12a}$$

Too

$$S_x{}^{T(s)} = \sum_{i=1}^{n}\left[S_{\rho_i}{}^{\mathbf{T}}\,.\,S_x{}^{\rho_i}+S_{\varepsilon_i}{}^{\mathbf{T}}\,.\,S_x{}^{\varepsilon_i}\right] \tag{12b}$$

Further, *if all resistivity sensitivities are identical,* $S_x{}^{\rho_i} = S_x{}^{\rho}$, and *all dielectric constant sensitivities are the same,* $S_x{}^{\varepsilon_i} = S_x{}^{\varepsilon}$, then we obtain directly from equations (12), for dimensionless transfer functions,

$$S_x{}^{T(s)} = S_s{}^{T(s)}[S_x{}^{\rho}+S_x{}^{\varepsilon}]. \tag{13}$$

In essence, this is the integrated circuit case where $n = 4$ and $\rho_1,\rho_2,\rho_3,\rho_4$ could be respectively emitter, base, collector and isolation resistivities.

6. DISCUSSION

We have shown that if a circuit is constructed of one type of resistive material and one type of capacitive material, then rigorously the transfer function with respect to any parameter depends only upon resistive and capacitive material sensitivities with respect to this parameter, as well as the logarithmic derivative of the transfer function with regard to frequency, equation (7b). Thus the sensitivity is independent of the circuit configuration with its important frequency dependence being determined from the magnitude and phase Bode plots.

What is more important practically is that the result simply extends to several kinds of resistive capacitive materials. This extension, equation (13), shows, for example, precisely that *the temperature sensitivity of monolithic integrated circuit transfer function realizations is independent of the realization method.* That is, for a given circuit designed in completely integrated form it is immaterial for temperature sensitivity behaviour whether it is realized through gyrator circuits or state-variable configurations, etc.

In contrast, we note that the results will not in general be valid for circuits other than the sort described, for example, for lumped circuits containing L, C and R elements. Suppose all resistors increase by 1 per cent, while the L and C elements remain constant; it does not then follow that all quantities with the dimension of ohms increase by 1 per cent [e.g. $\sqrt{L/C}$ for any inductance and capacitance remains unchanged, though it has the dimension of ohms]. This has the corollary that to achieve, for example, a realization of a second order transfer function with high Q, we are doomed to having much higher sensitivity in an integrated circuit realization than in a passive circuit realization (which checks out to have quite low sensitivity) unless we can assure that ρ and ε variations cancel, or unless we can introduce a component into the circuit which is neither resistive, capacitive or dimensionless in its operation. Such a component would need to obey some law of physics which is not commonly made use of in integrated circuits.

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