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RICHARDS' THEOREM FOR MATRICES

by

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ABSTRACT

Extensions, complete in final form,
Conclude the theorem by Richards born.
Included scattering proof satisfactional,
Embraces matrices, skew and irrational.

CONTENTS

ILLUSTRATION

I. HISTORICAL INTRODUCTION

In 1947 Richards, [1, p. 779], published, concealed among other results, the theorem which now goes by his name. As is well-known, this was detected by Bott and Duffin, [3], and put to use in their transformerless synthesis. As is not quite so well known, Richards also called upon the theorem for use in a somewhat different theory; that of resistor, transmission-line synthesis, [4, p. 219]. The functions treated by Richards were the positive-real scalars whose singularities on the imaginary axis are poles.

Upon a study of Richards' transmission line theory in the 1961 Stanford Network Theory Seminar, an attempt was made to obtain an extension to n-ports. By applying Bayard's generalization of Richards' theorem, [5, p. 170], we did succeed in generalizing the theory, but, unfortunately, the results were not valid for the important case of nonreciprocal networks. This resulted from Bayard's theorem being restricted to (rational) symmetric, positive-real matrices.

Also, in 1961, Saito rediscovered Bayard's result, [6, p. 1033], but the proof, in Japanese, is about six times as lengthy and is valid only for the 2 x 2 case. Saito successfully applied this to a 2-port lossless line synthesis, [7]. Only a short time later, without proof, Hazony and Nain, in a letter requiring revision, essentially restated Bayard's result, [8], [9], [10].

Recent correspondence with Belevitch, pertaining to generalizations of the angle constraint for passive n-ports, [11], led to a reconsideration of the previous years seminar results, which in turn now allows Richards' theorem to be stated in its full generality. This result, which holds for non-symmetric matrices and allows other singularities than poles on the imaginary axis, is the object of the remainder of the report. From this a non-reciprocal, n-port, resistor-transmission-line synthesis has been developed, which will be discussed in an accompanying report.

II. THEOREM AND PROOF

Before proceeding to the theorem we introduce appropriate notations and definitions.

We let $p = \sigma + j\omega$; a superscript asterisk, $*$, denotes complex conjugation; a superscript tilde, $\sim$, denotes matrix transposition; A_H denotes the Hermitian part of the matrix $A = [a_{ij}]$, i.e., $2A_H = A + \tilde{A}^*$; A_{sy} denotes the symmetric part of A, i.e., $2A_{sy} = A + \tilde{A}$; A_{ss} denotes the skew-symmetric part of A, i.e., $2A_{ss} = A - \tilde{A}$; 1_n denotes the identity matrix of order n.

In the following we will always assume that k is a real, positive number, $k > 0$, and that $A(p)$ is a positive-real matrix. In this case $A(k)$ is real and we define

$$\overline{A}(p) = A(p) - A_{ss}(k) \tag{1}$$

which is also positive-real; physically this operation corresponds to a gyrator extraction. We have

$$\overline{A}(k) = A_{sy}(k) = \tilde{K}K \tag{2}$$

where $A_{sy}(k)$ can be factored by standard diagonalization techniques to obtain K, [12, p. 298]; recall that $A_{sy}(k)$ is positive semi-definite. In order to give the simplest possible proof, it is most expedient to work with normalized matrices. Since a theorem can only be given for non-singular A, no loss of generality results by assuming that K^{-1} exists. We then define

$$A_n(p) = \tilde{K}^{-1}\overline{A}(p)K^{-1} \tag{3}$$

in which case A_n is again positive-real and $A_n(k) = 1_n$. The main result, which requires $pA_n(p) - k1_n$ non-singular, is the following.

THEOREM:

If $A(p)$ is positive-real, then

$$A_{rn}(p) = [p1_n - kA_n(p)][pA_n(p) - k1_n]^{-1} \tag{4a}$$

is positive-real for any $k > 0$; A_n is defined by (1), (2), (3).

PROOF:

Without loss of generality, we interpret A_{rn} as an impedance matrix and first form its scattering matrix, S_{rn}. From (a-1) of the appendix,

$$S_{rn}(p) = (A_{rn} - 1_n)(A_{rn} + 1_n)^{-1} \tag{5a}$$

$$= -\left(\frac{p+k}{p-k}\right)[A_n(p) - 1_n][A_n(p) + 1_n]^{-1} \tag{5b}$$

Taking a clue from Bayard, [5, p. 170], we now show that S_{rn} satisfies the requirements given by Youla for scattering matrices, (see the appendix).

1. S_{rn} is analytic in Re $p > 0$. This results from the fact that the singularity at $p = k$ vanishes, as is seen by factoring $p-k$ from the Taylor series expansion of $A_n(p) - 1_n$ about $p = k$. Note that $A_n(p) - 1_n$ and $[A_n(p)+1_n]^{-1}$ are analytic in Re $p > 0$, since $A_n(p)$ is positive-real.
2. $S^*_{rn}(p) = S_{rn}(p^*)$, since this holds for $A(p)$, in Re $p > 0$.
3. $1_n - \tilde{S}^*_{rn}(j\omega)S_{rn}(j\omega)$ is positive semi-definite for almost all ω. To see this, form

$$S_n(p) = [A_n(p) - 1_n][A_n(p) + 1_n]^{-1} \tag{6}$$

Then

$$S_{rn}(p) = -\left(\frac{p+k}{p-k}\right) S_n(p) \tag{7}$$

and the desired Hermitian form becomes

$$1_n - \tilde{S}^*_{rn}(j\omega)S_{rn}(j\omega) = 1_n - \tilde{S}^*_n(j\omega)S_n(j\omega) \tag{8}$$

by virtue of the fact that the all-pass factor multiplying S_n in (7) has a magnitude of unity for $p = j\omega$. But the right of (8) is positive semi-definite for almost all ω, since A is positive-real.

4. $\sup e^{-|p|^{\alpha}} ||S_{rn}(p)|| < \infty$ in Re $p > 0$ for some $\alpha < 1$, since this holds for S_n and we note that $\left(\frac{p+k}{p-k}\right)$ is bounded outside a neighborhood of $p = k$, and in a neighborhood of $p = k$, S_{rn} is analytic. Q.E.D.

Besides A_{rn}, several other forms satisfy the theorem. For instance, the use of (a-2) shows that

$$A'_{rn}(p) = [pA_n(p)-k1_n]^{-1}[p1_n-kA_n(p)] \tag{4b}$$

is positive-real. By taking inverses, which are assumed to exist, this is also true of

$$A^{-1}_{rn}(p) = [pA_n(p)-k1_n][p1_n-kA_n(p)]^{-1} \tag{4c}$$

$$A'^{-1}_{rn}(p) = [p1_n-kA_n(p)]^{-1}[pA_n(p)-k1_n] \tag{4d}$$

By inserting (3) in (4a) we also have

$$A_{rn}(p) = \tilde{K}^{-1}[p\overline{A}(k)-k\overline{A}(p)][p\overline{A}(p)-k\overline{A}(k)]^{-1}\tilde{K} \tag{4e}$$

which upon denormalization yields

$$A_r(p) = \tilde{K}A_{rn}(p)K = [p\overline{A}(k)-k\overline{A}(p)][p\overline{A}(p)-k\overline{A}(k)]^{-1}\overline{A}(k) \tag{4f}$$

A_r is usually easier to work with than A_{rn}, since the given matrices appear directly. Similar manipulations yield

$$A'_r(p) = \tilde{K}A'_{rn}(p)K = \overline{A}(k)[p\overline{A}(p)-k\overline{A}(k)]^{-1}[p\overline{A}(k)-k\overline{A}(p)] \tag{4g}$$

which reduces to the form treated by Bayard in the rational, symmetric case. Other forms result by similar arguments on (4c), (4d) and upon inverting (4f), etc.

Two examples illustrate the theorem.

EXAMPLE 1:

Consider the positive-real scalar, $A(p) = \sqrt{p}$. Using (4f)

$$A_r(p) = \sqrt{k}\left[\frac{p\sqrt{k}-k\sqrt{p}}{p\sqrt{p}-k\sqrt{k}}\right]$$

By simple manipulations, this is seen to be positive-real, since

$$A_r^{-1}(p) = p^{-1/2} + k^{-1}p^{1/2} + k^{-1/2}$$

Note that A(p) is not rational or meromorphic, but has a branch point at p = 0. Further, p-k doesn't "cancel" in $A_r(p)$.

EXAMPLE 2:

Let k = 3 and assume

$$A(p) = \begin{bmatrix} 4 & 8\left(\frac{p-1}{p+1}\right) \\ 0 & 4 \end{bmatrix}$$

which is positive-real, and rational. Then, using (1),

$$A_{ss}(k) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},\ \bar{A}(p) = \begin{bmatrix} 4 & 2\left(\frac{3p-5}{p+1}\right) \\ 2 & 4 \end{bmatrix},\ \bar{A}(k) = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

and from (4f)

$$A_r(p) = \left\{2(p-3)\begin{bmatrix} 2 & \left(\frac{p-5}{p+1}\right) \\ 1 & 2 \end{bmatrix}\right\}\left\{\frac{p+1}{2(p^2-9)}\begin{bmatrix} 2 & -\left(\frac{3p+1}{p+1}\right) \\ -1 & 2 \end{bmatrix}\right\}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}$$

which is clearly positive-real. Note that $p-3$ cancels, as expected, as well as $p+3$, even though $\overline{A}(3) \neq -\overline{A}(-3)$.

III. PROPERTIES

In the important rational case several useful properties result.

COROLLARY 1:

If $A(p)$ is rational, then the scalar term $p-k$ cancels between the two factors of $A_{rn}(p)$.

PROOF:

By the rationality, $p-k$ clearly factors every term of $[p1_n - kA_n(p)]$ and $[pA_n(p) - k1_n]$, since these matrices vanish at $p=k$. Q.E.D.

COROLLARY 2:

If $A(p)$ is rational with $A(k) = -\tilde{A}(-k)$, then the scalar term $p+k$ cancels between the two factors of $A_{rn}(p)$.

PROOF:

Both $[p1_n - kA_n(p)]$ and $[pA_n(p) - k1_n]$ vanish at $p = -k$ if $A(k) = -\tilde{A}(-k)$. The result follows by rationality. Q.E.D.

As seen by Example 1, $p-k$ need not completely cancel, if A isn't rational. As seen by Example 2, $p+k$ can cancel even when $A(k) \neq -\tilde{A}(-k)$, in contrast to the result in the scalar case, [1, p. 779]. By observing (7) it is clear that no other cancellations than those of $p-k$ and $p+k$ can occur in A_{rn}. When A is rational, we can use McMillan's degree, $\delta(A)$, to tell how effective the cancellation is, [13, p. 580]. By McMillan's properties, we know that $\delta(A_{rn}) = \delta(S_{rn})$, and from (7), (3) and (1), $\delta(S_{rn}) \leq \delta(S_n) = \delta(A_n) = \delta(A)$, since $p-k$ cancels into the numerators of S_n. Here equality holds if $p+k$ doesn't cancel with any terms of S_n. In case the conditions of Corollary 2 hold, $p+k$ cancels with every term of S_n and we obtain $\delta(S_{rn}) = \delta(S_n)-n$ in this case, since no other cancellation can occur. Thus

$$\delta(A_{rn}) \leq \delta(A) \tag{9a}$$

$$\delta(A_{rn}) = \delta(A)-n \quad \text{if } A(k) = -\tilde{A}(-k) \tag{9b}$$

These relations give a rigorous way of showing that a synthesis method will terminate; they hold for all other matrices in (4). We point out that if $A(p)$ is a lossless impedance matrix, that is

$A(p) = -\tilde{A}(-p)$, it must at least have degree n if $\bar{A}(p)$ is non-singular, which is consistent with (9b).

If we interpret A(p) as an impedance matrix, we can give a physical realization of A(p) in terms of $A_{rn}(p)$, even in the irrational case. Solving (4a) we have

$$A_n(p) = [k1_n + pA_{rn}(p)]^{-1}[kA_{rn}(p) + p1_n] \tag{10a}$$

$$= [A_{rn}^{-1}(p) + (p/k)1_n]^{-1} + [(k/p)1_n + A_{rn}(p)]^{-1} \tag{10b}$$

$$= \{[A_{rn}(p) + (p/k)1_n]^{-1} + [(k/p)1_n + A_{rn}^{-1}(p)]^{-1}\}^{-1} \tag{10c}$$

where we have assumed A_{rn} non-singular (if it is singular, it can be transformed to the direct sum of zeros and a non-singular matrix and similar calculations made). Using (1), (3) and (10) the realization of Figure 1 results.

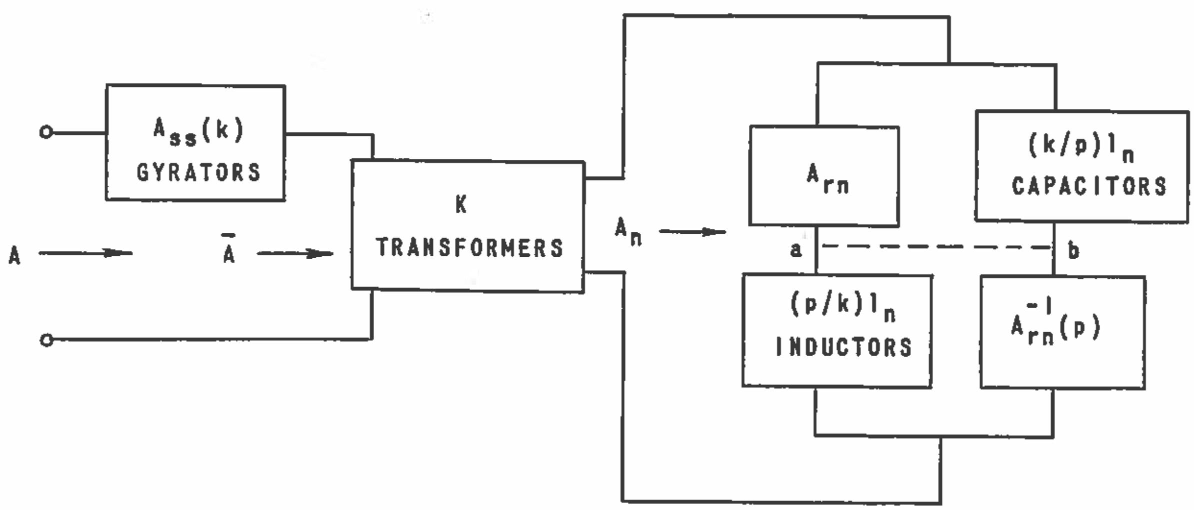

FIG. 1. STRUCTURE FOR IMPEDANCE A, USING RICHARDS' THEOREM. ALL BLOCKS ARE n-PORTS.

In Figure 1 points a and b are open for (10b) and closed for (10c). A network can be inserted between a and b, as with one-ports, such that a realization using fewer inductors and capacitors can be obtained, [14].

By substituting (4a) for A_{rn} in (10b) and simplifying terms, (10a) can be rewritten as

$$A_n(p) = k\left[\frac{kA_n(p)-pl_n}{k^2-p^2}\right] + p\left[\frac{kl_n-pA_n(p)}{k^2-p^2}\right] \tag{10d}$$

with each term on the right individually positive-real. This is essentially the form quoted by Hazony and Nain in the symmetric, rational case, [8].

If A is symmetric, then A_{rn} is symmetric, as is seen by writing (4a) as

$$A_{rn}(p) = p[pA_n(p)-kl_n]^{-1} - k[pl_n-kA_n^{-1}(p)]^{-1} \tag{11}$$

in which all terms on the right are symmetric. By observing all the other equations (4), it is clear that they are also symmetric. In the symmetric case the $A_{ss}(k)$ term in Figure 1 is replaced by short circuits. Note, however, that in general, A_{rn} will not be symmetric as is shown by Example 2.

IV. CONCLUSIONS

The theorem of section II extends that of Richards' to cover the most general kinds of positive-real functions and matrices. The result is not limited to functions whose singularities on the $j\omega$ axis are poles, as is Richards', or to symmetric matrices, as is Bayard's.

In case A is rational, then cancellations of p-k and p+k, and only of these, occur; but cancellation of p+k can occur in other situations than listed in Corollary 2, if A is not a scalar, as shown by Example 2. Again in the rational case, the degree can't increase, and, when it decreases, Figure 1 can be used for an impedance synthesis. Since $A(p) = -\tilde{A}(-p)$ if A is the impedance matrix of a lossless network, this will always yield a lossless synthesis. However, this method of synthesis (lossless or not) is somewhat impractical since an excessive number of elements is required. This is dramatically illustrated in the non-lossless case by carrying out the synthesis of Figure 1 for Example 2; three gyrators, two inductors, two capacitors and two resistors would be needed even though $\delta(A) = 1$. Since the input impedance matrix of lossless transmission lines in terms of the load takes the form of (10a), this does yield a practical distributed parameter cascade synthesis, as discussed in a companion report.

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APPENDIX: POSITIVE-REAL MATRICES

Here positive-real matrices are defined, and Youla's equivalent in terms of the scattering matrix, used in the proof of section II, is stated.

An n x n matrix A(p) is called positive-real if, [15, p. 122],

1) A(p) is analytic in Re $p > 0$

2) $A^*(p) = A(p^*)$ in Re $p > 0$

3) $A_H(p)$ is positive semi-definite in Re $p > 0$

When they exist, the immittance matrices of a passive network must necessarily be positive-real, [15, p. 122]. However, if an immittance matrix is positive-real, it need not correspond to a physical network, as examples show, unless it is rational, in which case synthesis methods apply, [16, pp. 113-169].

If A is an impedance matrix, A = Z, then the scattering matrix can be defined by

$$S = (A-1_n)(A+1_n)^{-1} \tag{a-1}$$

$$= (A+1_n)^{-1}(A-1_n) \tag{a-2}$$

In terms of S, we know that A is positive-real if and only if, [15, pp. 116, 123],

1) S(p) is analytic in Re $p > 0$

2) $S^*(p) = S(p^*)$ in Re $p > 0$

3) $1_n - \tilde{S}^*(j\omega)S(j\omega)$ is positive semi-definite for almost all (real) ω.

4) $\sup e^{-|p|^\alpha}||S(p)|| < \infty$ in Re $p > 0$ for some $\alpha < 1$ where

$$||S|| = \sum_{i=1}^{n}\sum_{j=1}^{n} |s_{ij}| \text{ and } S(j\omega) = \lim_{\sigma\to 0^+} S(\sigma+j\omega).$$

Note that for rational S, condition 3 shows that S is analytic on Re $p = 0$, and condition 4 is automatically satisfied.

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