FOURTH COLLOQUIUM ON MICROWAVE COMMUNICATION

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A LOCAL TIME-VARIABLE SYNTHESIS

SEPARATUM

AKADÉMIAI KIADÓ
PUBLISHING HOUSE OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST 1970

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"I've seen a dragon in the sky
(and I'm amused because he's made
of clouds, and I've played with him)" [1]

I. Introduction

Because of its many nuances, the theory of time-variable synthesis poses interesting challenges. For example, a conceptually simple scattering matrix technique [2] results in noncausal structures, block diagram methods [3, p. 30] require unnecessarily many active components where passive ones may suffice, while transform techniques [4, p. 273] are restricted to very limited classes of circuits. Thus, even though some state-variable ideas are also proving promising [5, p. 124], we look here at an operational method which for local time usage, avoids the major drawbacks of alternative techniques. Although this paper raises some interesting calculational problems it firmly lays the foundations for an operational calculus while also outlining sufficiency of previous necessary conditions for synthesis [6, p. 13].

Our idea is to introduce a calculus for time-variable differential polynomial operators, $P(p,t)$ with $p = d/dt$. This calculus rests upon abstract algebraic concepts, such as skew-field differential ring embeddings and Galois solution field extensions, allowing it to handle coefficient singularities which occur in physical networks, for example when gyrators vanish [6, p. 43]. This calculus is then applied to the synthesis of an n-port through a given scattering matrix expressed in terms of differential operators, $\underset{\sim}{s}(p,t)$. Indeed the scheme is to form another, essentially lossless, scattering matrix $\underset{\sim}{\Sigma}(p,t)$:

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{s} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix} \qquad (1)$$

$\underset{\sim}{\Sigma}$ is a bordered version of $\underset{\sim}{s}$, found by factoring two resistivity matrices, for which the adjoint is the inverse. When $\underset{\sim}{\Sigma}$ is also passive, called quasilossless, it can be synthesized by lossless interconnections (for example, time-variable gyrators and fixed capacitors) using available techniques [7, p. 251]. Termination of the final ports for $\underset{\sim}{\Sigma}$ in unit resistors yields $\underset{\sim}{s}$ at the input. As

+ This research was supported by the U.S. Air Force Office of Scientific Research under contract AF-AFOSR F44620-67-C-0001.

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yet, the factorizations used yield only passive components for local time, though with proper gyrator time variations global extensions are possible as will be discussed.

The primary algebraic source relied upon in the development is Rédei [8] with differential algebra concepts available in [9][10]. We assume some familiarity with n-port synthesis and take the Belevitch time-invariant synthesis as background [11, p. 288].

...

"What will occur, I ask myself,
if that is a true dragon
(that depth over there in the dark
is his eye which shines)." [1]

II. Differential Polynomials - Field Extensions

We start with available theories, investigating properties and extensions to quotient skew-fields, of polynomials in the derivative operator p.

Let $\delta[\ ]$ denote the degree and consider a polynomial of degree k, $\delta[P] = k$,

$$P(p,t) = a_k(t)p^k + \dots a_1(t)p + a_o(t) \qquad a_k \neq 0 \tag{2a}$$

where the $a_i(t)$ are assumed to lie in a differential field F, [9, p. 10]. Such a polynomial will be called a differential polynomial (over F) since algebraically p is assumed to operate through

$$pa(t) = a(t)p + a'(t) \ ; \quad a \in F \ , \quad a' \in F \tag{2b}$$

with the derivative a' uniquely determined by a. Here by a differential field is meant a field for which a mapping (derivative) into itself is defined satisfying $(ab)' = a'b + ab'$, $(b^{-1})' = -b'/b^2$; for example F may be rational functions in t with real coefficients or functions which are the ratio of functions holomorphic in a region. The constants are those elements a for which $a' = 0$. Any field can be made a differential field by taking $a' = 0$ for all a in the field, but this is trivial; we will generally interpret the derivative as time differentiation.

Using $p^i p^j = p^{i+j}$ we readily define the product P_1P_2 of two differential polynomials P_1 and P_2 from which we note, by observing the lead coefficient on using (2b), that $P_1P_2 = 0$ implies $P_1 = 0$ or $P_2 = 0$. Consequently, the set $\mathcal{P}$ of differential polynomials over F forms a noncommutative ring with no divisors of zero. Although this property is not sufficient for the formation of a quotient skew-field [12] we do have that for any nonzero $P_1, P_2 \in \mathcal{P}$ there exist nonzero $\bar{P}_1, \bar{P}_2, \hat{P}_1, \hat{P}_2 \in \mathcal{P}$ such that $\bar{P}_1P_1 = \bar{P}_2P_2$, $P_1\hat{P}_1 = P_2\hat{P}_2$, as is seen by equating coefficients. Because these equalities coupled with the absence of zero divisors are necessary and sufficient conditions for the creation of a right and a left inverse for every nonzero element of $\mathcal{P}$, we can extend $\mathcal{P}$ to a left quotient ring $\mathcal{Q}$ and a right quotient ring [13, p. 5][14, p. 487][15, p. 137] and these quotient rings are identical [16, p. 76]; $\mathcal{Q}$ is often called a skew-field or division algebra. Indeed the method of forming $\mathcal{Q}$ is constructive and calculations are carried out by cross multiplying denominators.

The reason for all this is that we can now work rigorously with quotients of two differential polynomials; for example, $(p+a)^{-1}(p+b) = (p+b+[a'-b']/[a-b]) \times (p+a+[a'-b']/[a-b])^{-1}$ for $a, b \in F$.

Of possible future interest for matrix decompositions is the fact that $\mathcal{P}$ is a Euclidean ring [8, p. 325]; that is, a Euclidean algorithm exists [14, p. 483] based upon either left or right hand division. Specifically, given $P_1, P_2 \in \mathcal{P}$, with $\delta[P_1] \geq \delta[P_2] > 0$, there exist differential polynomials $\bar{P}_1$ and P_3, having $\delta[\bar{P}_1] = \delta[P_1] - \delta[P_2]$ and $\delta[P_3] = \delta[P_2] - 1$ such that

$$P_1 = \bar{P}_1 P_2 + P_3 \qquad (3)$$

from which we deduce, for some k, the sequence : $P_2 = \bar{P}_2 P_3 + P_4, \ldots,$ $P_{k-2} = \bar{P}_{k-2} P_{k-1} + P_k$, $P_{k-1} = \bar{P}_{k-1} P_k$. The unique differential polynomial P_k is then the greatest common right divisor of P_1 and P_2. Using similar operations, elementary matrices $\underset{\sim}{P}$, $\underset{\sim}{Q}$ (having entries, and inverses with entries, in $\mathcal{P}$) can be found such that for any (square) matrix $\underset{\sim}{A}$ with entries in $\mathcal{P}$ a Smith form holds [15, p. 139]

$$\underset{\sim}{P}(p,t)\,\underset{\sim}{A}(p,t)\,\underset{\sim}{Q}(p,t) = \mathrm{diag}\,[a_1(p,t), \ldots, a_r(p,t), \underset{\sim}{0}] \qquad (4)$$

In (4) a_i is a left and right factor of a_j, and $\delta[a_i] < \delta[a_j]$ for $i < j$; the rank r is independent of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$. The result extends easily to nonsquare $\underset{\sim}{A}$ while $\delta[\underset{\sim}{A}] = \sum_{i=1}^{r} \delta[a_i]$ is usefully defined as the degree of $\underset{\sim}{A}$.

Of more immediate interest from the Euclidean algorithm is the consequence that as a Euclidean ring, $\mathcal{P}$ is a principal ideal ring with irreducible factor decomposition [8, p. 325] with, for what it is worth, all principal ideals generated by the elements of the normalizer of $\mathcal{P}$ (the normalizer is the set of polynomials P for which $P\mathcal{P} = \mathcal{P}P$; ideals are somewhat like divisors or factors).

The irreducible factor decomposition shows the possibility of factoring differential polynomials, though these factors are not in general unique unless $\mathcal{P}$ is commutative (the time-invariant case). Unfortunately the irreducible factor theorem is not constructive, so our next real problem is to determine the factors and their nature. Such requires an algebraic treatment of differential equations.

Toward this latter we note that the differential polynomials can operate on the differential field F, mapping F into itself; that is, we can introduce another multiplication, denoted $\bullet$, such that $P(p,t) \bullet a(t) \in F$ for $P \in \mathcal{P}$, $a \in F$. This pairing, $F|\mathcal{P}$, forms an operator structure [8, p. 223], and within the operator structure

$$P(p,t) \bullet u(t) = 0 \qquad (5)$$

represents a differential equation.

If the base field F has characteristic zero [17, p. 362] (meaning infinitely many distinct multiples of unity, as for the reals) and has an algebraically closed [17, p. 382] field of constants (meaning every polynomial with constant coefficients in F has a root in F) then there exists an extension field $\mathfrak{F}$, called a <u>Picard-Vessiot extension</u> [9, p. 21] such that:

a) F and $\mathfrak{F}$ have the same field of constants and

b) $\delta[P]$ linearly independent (over constants) solutions u_i of (5) lie in $\mathfrak{F}$.

In other words, we can assume that the original base field F is chosen as $\mathfrak{F}$ in which case not only the coefficients of P but all solutions of interest of $P \cdot u = 0$ lie in the field; by pairing complex solutions $\mathfrak{F}$ can be chosen real with F if desired. Unfortunately no constructive method for obtaining a Picard-Vessiot extension seems available; still the result is one of extreme power. We now assume $F = \mathfrak{F}$.

Now let $u(t)$ satisfy $P \cdot u = 0$, then by direct evaluation u also satisfies $[p - u'/u] \cdot u = 0$ and $u'/u \in \mathfrak{F}$ since $\mathfrak{F}$ is a differential field. By the irreducible factor decomposition we see that $(p - u'/u)$ is a right factor of P and that, in fact, when P is formed over $\mathfrak{F}$, all irreducible nonunit factors are of degree one. Thus, for $P \in \mathcal{P}$ of degree k we get

$$P(p,t) = a_k(t)p^k + \ldots + a_1(t)p + a_o(t) \ ; \quad a_i \in \mathfrak{F} \tag{6a}$$

$$= \alpha_o(t)[p - \alpha_k(t)] \ldots [p - \alpha_1(t)] \ ; \quad \alpha_i \in \mathfrak{F} \tag{6b}$$

For convenience we will call the α_i <u>roots</u> of P. Of numerical interest is the fact that if u satisfies $[p - a] \cdot u = 0$ then

$$u(t) = u(t_o) \exp\left[- \int_{t_o}^{t} a(\tau) d\tau\right] \tag{7}$$

Thus, elements of the extension field $\mathfrak{F}$ can be exponentials of integrals of elements of F. But, indeed, such integrals need not exist in the classical sense; they are obtained by Galois type extensions [9, p. 23] of F. The integrals and exponentials of integrals may then be in $\mathfrak{F}$ but need not complete $\mathfrak{F}$, as shown by $P = p^2 - t$ [9, p. 44], though we believe Volterra compositions of such will do the job [6, p. 7].

At this point it is appropriate to introduce the differential polynomial, P^a, <u>adjoint</u> to P of (6a)

$$P^a(p,t) = (-1)^k p^k a_k(t) + \ldots + p^2 a_2(t) - p a_1(t) + a_o(t) \tag{8}$$

For a self-adjoint $P \in \mathcal{P}$, extending the constructive method of Frobenius [18, p. 193], we can obtain a factorization $P = P^a = P_1{}^a P_1$, $P_1 \in \mathcal{P}$ (with base field extended to $\mathfrak{F}$). The actual form, which indicates the construction, obtained by Frobenius is

$$P = P^a = [c_1(-p)c_2(-p) \ldots c_\ell(-p)] \ c_o [p c_\ell p \ldots p c_2 p c_1], \quad 2\ell = k \tag{9a}$$

where $c_1 = 1/u_1$ with u_1 as before. Thus $P = P_1^a P_1$ has

$$P_1 = \sqrt{c_o}\, pc_\ell \ldots c_2 pc_1 = \sqrt{a_k}\;[p - \alpha_\ell]\ldots[p + c'_1/c_1] \;. \qquad (9b)$$

$P_1 \in \mathfrak{P}$ requires $\sqrt{c_o}$, $\sqrt{a_k} \in \mathfrak{F}$ which will be true if F is algebraically closed or, more important for passive synthesis, if P is a nonnegative differential operator, this concept being defined as follows for a more general matrix case.

First we note that the adjoint $\underset{\sim}{Q}^a$ of a matrix $\underset{\sim}{Q}$ with entries in the extension skew-field $\mathcal{L}$ of $\mathcal{P}$ is formed by transposing, denoted by a superscript tilde $\tilde{}$, and taking the adjoint of its differential polynomials as defined by (8). Then a self-adjoint $n \times n$ matrix $\underset{\sim}{Q}$ is called <u>nonnegative</u>, written $Q \geq 0$, if for all n-vectors $\underset{\sim}{x}(t)$ having all entries infinitely differentiable and of compact support (the L. Schwartz distributional testing function space $\mathcal{D}$ [19, p. 21]), the following integral <u>exists</u> and is nonnegative

$$\int_{-\infty}^{\infty} \tilde{\underset{\sim}{x}}(t)\; [\underset{\sim}{Q}(p,t)\cdot \underset{\sim}{x}(t)]\; dt \geq 0 \qquad (10)$$

At (10) we have really introduced an operator structure on $\mathcal{D}$; since this is mathematically little explored we begin to tread on thin ice! Note though that $-p^2 -1 = (1/\sin t)(-p)[\sin^2 t]p(1/\sin t)$ is not nonnegative while $-p^2 + 1 = e^{-t}(-p)e^{2t}pe^{-t}$ is; too $\underset{\sim}{x}(t)/\sin t$ does not have integrable entries for all $x_i \in \mathcal{D}$. Actually when $P = P^a \in \mathcal{P}$ is written as [18, p. 191]

$$P = (-p)^\ell \beta_k p^\ell + (-p)^{\ell-1}\beta_{k-2}p^{\ell-1} + \ldots + (-p)\beta_1 p + \beta_o \qquad (11)$$

then $P \geq 0$ if and only if $\beta_i(t) \geq 0$ for all i and t [20].

...

"I have become sombre, taken by doubt
but that it is a game:
We are fearful from youth,
it is time to commence seriousness."[1]

III. <u>Gauss Factorization</u>

With the above preliminaries we can obtain a useful, spectral type, Gauss factorization upon which the synthesis is based. By way of further notation we use $\underset{\sim}{1}_m$ for the $m \times m$ identity matrix.

Consider as given a self-adjoint, nonnegative matrix $\underset{\sim}{Q}(p,t)$ of differential polynomials. On partitioning $\underset{\sim}{Q} = \underset{\sim}{Q}^a$ with q_{11} a scalar matrix, we can write

$$\underset{\sim}{Q}(p,t) = \begin{bmatrix} q_{11} & \underset{\sim}{q}_{12} \\ \underset{\sim}{q}_{12}^a & \underset{\sim}{q}_{22} \end{bmatrix} \qquad (12a)$$

$$= \begin{bmatrix} q_{11} & 0 \\ \underset{\sim}{q}_{12}^a & \underset{\sim}{1}_{n-1} \end{bmatrix} \begin{bmatrix} q_{11}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{Q}_2 \end{bmatrix} \begin{bmatrix} q_{11} & \underset{\sim}{q}_{12} \\ \underset{\sim}{0} & \underset{\sim}{1}_{n-1} \end{bmatrix} \;;\; \underset{\sim}{Q}_2 = \underset{\sim}{q}_{22} - \underset{\sim}{q}_{12}^a\, q_{11}^{-1}\, \underset{\sim}{q}_{12} \qquad (12b)$$

where $q_{11} \neq 0$ is assumed, if necessary by the use of a permutation of rows and columns. Now $q_{11} = q_{11}^{a} > 0$ and, hence by (9a), $q_{11} = q_1 q_1^{a}$. Further, by the quotient ring conditions, we can find a differential polynomial Q_1 and a differ-polynomial matrix $\underline{Q}_{12}$ such that $q_{11}^{-1} \underline{Q}_{12} = \underline{Q}_{12} Q_1^{-1}$. Thus, from (12b) with a superscript $-a$ denoting the adjoint's inverse,

$$\underline{Q}_2 = Q_1^{-a} [\underline{Q}_1^{a} \underline{Q}_{22} Q_1 - \underline{Q}_{12}^{a} \underline{Q}_{12}] Q_1^{-1} \geq 0 \qquad (12c)$$

Continuing the iteration, after interchanging factors with Q_1^{-1} and absorbing any permutations in the outer matrices finally gives a <u>Gauss factorization</u>

$$\underline{Q} = [\underline{M}^{a} \underline{1}_{\rho,n} \underline{L}^{-a}] [\underline{L}^{-1} \underline{1}_{\rho,n} \underline{M}] = \underline{q}^{a} \underline{q} \qquad (13)$$

where $\underline{1}_{\rho,n}$ is the $\rho \times n$ zero matrix with its first ρ columns replaced by $\underline{1}_{\rho}$; ρ is the rank of $\underline{Q}$, and $\underline{M}$ and $\underline{L}$ are polynomial with $\underline{L}$ diagonal.

We comment that bacause of the nonuniqueness of irreducible factors in $\mathcal{P}$, various properties for $\underline{L}$ are possible; briefly we investigate a "Hurwitz" property. First we note that an ordering can be introduced in $\mathfrak{F}$ [8, pp. 568-582] from which it is possible to isolate positive and negative quantities. Next we illustrate how entries in $\underline{L}$ can have their roots positive; for if not we make an interchange with entries in $\underline{L}^{a}$. This interchange is justified by considering

$$(p - a)(p + b) = (p + c)(p - d) \qquad (14a)$$

with all roots positive in $\mathfrak{F}$. Multiplying and equating coefficients, the equality of (14a) gives

$$d' + (b - a)d + d^2 = b' + ab, \qquad c = d + (b - a) \qquad (14b)$$

Using the substitution $d = y'/y$ [21, p. 295], the left of (14b) reduces to

$$y'' + (b - a)\, y' - (b' + ab)\, y = 0 \qquad (14c)$$

for which, via the theory associated with (5), we know a solution exists. Given then a, b we find c, d which "numerically" will be positive through a proper initial condition choice (here our ice is actually thicker, though still thin!)

...

"'But that's not possible, that's not possible'
said the dragon taking the word to the top of the mountain.
'For all pass, and I too, you'll see
I'm defeated; there is no hope.'
(that's what the dragon said, and his eyes cried tears of rain
and his crest romantically fumed)." [1]

IV. Synthesis

With such details now in hand synthesis ideas become rather straightforward to develop, at least in theory.

We assume as given a general (differential equation) description

$$\underline{a}(p,t)\underline{v}^r(t) = \underline{b}(p,t)\underline{v}^i(t) \qquad (15b)$$

with $a_{ij}, b_{ij} \in \mathcal{P}$, $v_j^i \in D$. $\underline{v}^i$ and $\underline{v}^r$ are incident and reflected n-vectors from which an n x n <u>scattering matrix</u>

$$\underline{s}(p,t) = \underline{a}^{-1}(p,t)\,\underline{b}(p,t) \qquad (15b)$$

is defined. The base field F and scattering matrix $\underline{s}$ are taken such that $\underline{s}$ satisfies the necessary passivity constraints [6, p. 13]. In essence these are that $\underline{s}$ maps square-integrable $\underline{v}^i$ causally into square-integrable $\underline{v}^r$ and that the following resistivity matrix $\underline{R}(p,t)$ is nonnegative:

$$\underline{R} = \underline{1}_n - \underline{s}^a\,\underline{s} \geq 0 \qquad (16a)$$

Observation on the equality of eigenvalues of $\underline{s}\underline{s}^a$ with $\underline{s}^a\underline{s}$ shows that also

$$\underline{R}_{II} = \underline{1}_n - \underline{s}\,\underline{s}^a \geq 0 \qquad (16b)$$

Returning to (1) the primary (quasilossless) condition for applying available synthesis techniques using only lossless components is that [7, p. 256]

$$\underline{\Sigma}^a\underline{\Sigma} = \underline{\Sigma}\,\underline{\Sigma}^a = \underline{1}_{n+r} \qquad (17)$$

where r, the number of resistors in which to terminate, is the number of rows and columns by which the given $\underline{s}$ is augmented. Multiplying and recording, the crucial entries of (16) are

$$\underline{R} = \underline{\Sigma}_{21}^a\,\underline{\Sigma}_{21}\;,\quad \underline{R}_{II} = \underline{\Sigma}_{12}\,\underline{\Sigma}_{12}^a\;,\quad \underline{\Sigma}_{22}\,\underline{\Sigma}_{12}^a = -\underline{\Sigma}_{21}\underline{s}^a \qquad (18)$$

Our philosophy is to factor the two left resistivity matrices for $\underline{\Sigma}_{21}$ and $\underline{\Sigma}_{12}$, hence the reason for the Gauss factorization, and then solve for $\underline{\Sigma}_{22}$ from the right term on introduction of a pseudo-inverse $\underline{\Sigma}_{12}^{(-a)}$ for $\underline{\Sigma}_{12}^a$. First we reduce all quantities to differential polynomials through

$$\underline{s} = \underline{K}\,g^{-1} = g_2^{-1}\,\underline{K}_2 \quad ; \quad K_{ij},\ K_{2ij},\ g,\ g_2 \in \mathcal{P} \qquad (19a)$$

We have, from (12) with ρ = rank $\underline{R}$ = rank $\underline{R}_{II}$ = r :

$$\underline{R} = g^{-a}[g^a\,g\underline{1}_n - \underline{K}^a\,\underline{K}]\,g^{-1} \qquad \underline{R}_{II} = g_2^{-1}[g_2 g_2^a\underline{1}_n - \underline{K}_2\,\underline{K}_2^a]\,g_2^{-a} \qquad (19b)$$

$$= g^{-a}\underline{M}^a\tilde{\underline{1}}_{\rho,n}\underline{L}^{-a}\underline{L}^{-1}\underline{1}_{\rho,n}\,\underline{M}g^{-1} \qquad = g_2^{-1}\underline{M}_2\tilde{\underline{1}}_{\rho,n}\underline{L}_2^{-1}\underline{L}_2^{-a}\underline{1}_{\rho,n}\underline{M}_2^a\,g_2^{-a} \qquad (19c)$$

or finally

$$\underline{\Sigma}_{12} = g_2^{-1}\underline{M}_2\tilde{\underline{1}}_{\rho,n}\underline{L}_2^{-1} \quad ; \quad \underline{\Sigma}_{12}^{(-1)} = \underline{L}_2\underline{1}_{\rho,n}\underline{M}_2^{-1}g_2 \qquad (19d)$$

$$\underline{\Sigma}_{21} = \underline{L}^{-1}\,\underline{1}_{\rho,n}\underline{M}g^{-1} \quad ; \quad \underline{\Sigma}_{22} = -\underline{\Sigma}_{21}\underline{s}^a\underline{\Sigma}_{12}^{(-a)} = \qquad (19e)$$

$$-\underline{L}^{-1}\underline{1}_{\rho,n}\underline{M}g^{-1}\underline{K}_2\underline{M}_2^{-a}\tilde{\underline{1}}_{\rho,n}\underline{L}_2^a \qquad (19f)$$

The $\underline{\Sigma}$ formed from these last terms via the bordering of $\underline{s}$ in (1) automatically satisfies the quasilossless condition of (17). Likewise any ratio of differential operators can be represented by a causal map (as well as a noncausal one, if ever desired). The further required square-integrable map property appears obtainable by the choice of $\underline{L}$ and $\underline{L}_2$ having positive roots. However, this positivity is in $\mathfrak{J}$; numerically it only holds locally and hence different interchanges between $\underline{L}$ and $\underline{L}^a$ are needed for different local times.

Finally $\underline{\Sigma}$ is synthesized as a quasilossless $(n+\rho)$ -port [7, p. 251], using unit capacitors and time-variable-gyrators (all grounded, if desired) and terminated in $r = \rho$ unit resistors (also grounded, if desired) at the final ports to yield $\underline{s}$ at the input n ports. Since the structure remains fixed, and ρ is constant, one can simply place all time-variation in gyrators which can switch from one local solution to another.

...

"'But there is hope, there is'
have said the thousands of blades of grass in the garden
the mountain and lake, and the band of swallows in the wind." [1]

IV. Discussion

In this paper we have developed a calculus for differential operators and applied it toward the synthesis of finite time-variable n-ports through the scattering matrix. Of these, the calculus, which is rather rigorously complete, is probably more significant since it applies to many situations in general systems theory. In essence the calculus generalizes and rigorizes that discussed by Saeks [22].

The synthesis presented can best be considered as an existence theorem since it rests upon Picard-Vessiot solutions for the factorizations, and, as yet, no construction method for these solutions is available. The synthesis is, however, practical in that only a finite number of capacitors, and a minimum number of resistors occur in conjunction with time-variable gyrators, all of which can be grounded for integrated constructions [23].

One could ask, though, why such abstractness is needed. First, the use of differential polynomials seems the most natural and physical way to proceed. Second, the algebraic method appears as the most rigorous way of obtaining meaningful results. Third, the theory is beautiful, combining algebra with analysis with synthesis. Fourth, the ideas were inspired by an Hungarian source and a masterful work [8].

As with new approaches, the development leaves considerable to continue upon - there is material to investigate with regard to choices of $\underline{L}$, at (12), and methods of finding roots originally need further treatment. Extensions to distributed and nonlinear domains appear possible within the area of differential algebra [10].

Much research between function spaces, as $\mathcal{D}$, and algebraic ones, as $\mathfrak{F}$, seems needed.

...

"If you see a dragon, companion, soul who lives,
let us struggle to the last shred of our fear,
for I would tell it thus: So much am I
even ready to have fear, as it is fear
which gives birth to hope,
according to the law." [1]

...

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VI. Appendix (Synthesis Example)

Consider

$$s(p,t) = [p + g^2 - g'/g]^{-1}[p + \tfrac{3}{5}g^2 - g'/g] \quad , \quad g \in F$$

then

$$R = 1 - s^a s = [-p + g^2 + g'/g]^{-1}[\tfrac{16}{25}g^4][p + g^2 + g'/g]^{-1}$$

$$R_{II} = 1 - ss^a = [p + g^2 - g'/g]^{-1}[\tfrac{16}{25}g^4][-p + g^2 - g'/g]^{-1}$$

choose

$$\Sigma_{21} = -\tfrac{4}{5}g^2[p + g^2 + g'/g]^{-1} = [p + g - g'/g]^{-1}[-\tfrac{4}{5}g^2]$$

$$\Sigma_{12} = [p + g^2 - g'/g]^{-1}[\tfrac{4}{5}g^2] \quad \text{with} \quad \Sigma_{12}^{(-a)} = [-p + g^2 - g'/g]/[\tfrac{4}{5}g^2]$$

$$\Sigma_{22} = -\Sigma_{21}s^a\Sigma_{12}^{(-a)} = [p + g^2 - g'/g]^{-1}[p - \tfrac{3}{5}g^2 - g'/g]$$

Thus

$$\underline{\Sigma} = \frac{1}{p + g^2 - \frac{g'}{g}} \begin{bmatrix} p + \frac{3}{5}g^2 - \frac{g'}{g} & +\frac{4}{5}g^2 \\ -\frac{4}{5}g^2 & p - \frac{3}{5}g^2 - \frac{g'}{g} \end{bmatrix}$$

Passive synthesis is immediate from

$$\underline{Z} = (\underline{1}_2 - \underline{\Sigma})^{-1}(\underline{1}_2 + \underline{\Sigma}) = \begin{bmatrix} \frac{5}{g^2}(p - g') & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{g}p\frac{\sqrt{5}}{g} & 2 \\ -2 & 0 \end{bmatrix}$$

and is seen to use one time variable gyrator of gyration conductance $g(t)/\sqrt{5}$ loaded in a unit capacitor,this in series with a fixed gyrator to be loaded in a unit resistor. Note the singular behavior whe. $g(t) = t$.