A Passive Synthesis for Time-Invariant Transfer Functions

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Abstract—A passive transfer-function synthesis based upon state-space techniques is presented. The method rests upon the formation of a coupling admittance that, when synthesized by resistors and gyrators, is to be loaded by capacitors whose voltages form the state. By the use of a Lyapunov transformation, the coupling admittance is made positive real, while further transformations allow internal dissipation to be moved to the source or the load. A general class of configurations applicable to integrated circuits and using only grounded gyrators, resistors, and a minimal number of capacitors is obtained. The minimum number of resistors for the structure is also obtained. The technique illustrates how state-variable theory can be used to obtain results not yet available through other methods.

I. Introduction

In [1], a procedure for time-variable minimum-reactance passive synthesis of a "stable" impulse response matrix was given based on a new-state-equation technique for imbedding the impulse-response matrix in a passive driving-point impulse-response matrix. This method avoided increasing the dynamic order of the system realizations and resulted as a modification of [2]. An extension to unstable systems was obtained in [3]. The existence of passive realizations for stable time-invariant systems was also shown by a different approach in [4].

The general time-invariant problem is considered in detail here for a stable rational transfer-function matrix $\mathbf{T}(p)$. Our major aim is to extend the techniques of [1]–[3] to provide a broad class of passive realizations for a given stable but otherwise general $\mathbf{T}(p)$. This allows the systematic introduction of other physically desirable constraints. In particular, a passive realization with a minimum number of reactances and resistors is obtained and it is shown how resistors can be extracted at the input and output ports. Unlike many other syntheses that have been considered from the state equation point of view, the results obtained here are not merely a reformulation of classical frequency-domain techniques. Indeed the state-variable usage allows us to generalize by newer methods the classical result of Belevitch ([5], [6], p. 307), in which a minimal-reactance scalar transfer-function synthesis can be obtained; it provides a circuit consideration of such concepts as stability, controllability, and observability. Background and the general theory and position of state-space techniques in network theory can be found in [7].

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II. Preliminaries

We first recall several facts pertinent to the intended synthesis.

Given a transfer function $n \times m$ matrix $\mathbf{T}(p)$ that is rational with real coefficients (called *real-rational*) and that has $\mathbf{T}(\infty) = \mathbf{D}$, a finite constant matrix, there exist real constant matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$, such that ($\mathbf{1}_k$ is the $k \times k$ identity, $\mathcal{L}[\ ]$ is the Laplace transform)

$$\mathbf{T}(p) = \mathbf{D} + \mathbf{C}[p\mathbf{1}_k - \mathbf{A}]^{-1}\mathbf{B}, \qquad \mathcal{L}[\mathbf{y}] = \mathbf{T}(p)\mathcal{L}[\mathbf{u}] \tag{1a}$$

results from the state equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1b}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{1c}$$

Here $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is called a *system realization* of $\mathbf{T}(p)$; $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are the input and output m- and n-vectors, respectively, while $\mathbf{x}(t)$ is the k-vector state, all evaluated at time t. In fact, k can be chosen, as we will often assume, equal to its smallest possible value [8], which is also the degree δ of $\mathbf{T}(p)$ [9], [10]. By way of terminology we will call $T(p)$ *strictly stable* if all its poles lie in $\operatorname{Re} p < 0$ and *stable* if all its poles lie in $\operatorname{Re} p < 0$ or are simple on $\operatorname{Re} p = 0$. In the minimal case ($k = \delta$) strict stability is equivalent of course to $\mathbf{A}$ having all its eigenvalues with negative real parts (negative real parts or simple imaginary parts for stability).

Thus, when we assume constant $\mathbf{A}$ matrices of minimal size $k = \delta$, the stability of $\mathbf{T}(p)$ is independent of the $\mathbf{A}$ matrix chosen, while all possible minimal system realizations $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ result from a given one $(\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0, \mathbf{D}_0)$ preserving $\mathbf{T}(p)$ by letting $\mathbf{P}$ range through all nonsingular $\delta \times \delta$ constant matrices in the following relationships [8], p. 17,

$$\begin{aligned} \mathbf{A} &= \mathbf{P}^{-1}\mathbf{A}_0\mathbf{P}, \\ \mathbf{B} &= \mathbf{P}^{-1}\mathbf{B}_0, \\ \mathbf{C} &= \mathbf{C}_0\mathbf{P}, \\ \mathbf{D} &= \mathbf{D}_0. \end{aligned} \tag{2}$$

Equations (2), which result from the change of state $\mathbf{x}_0 = \mathbf{P}\mathbf{x}$, also hold in the nonminimal case where, however, not all possible system realizations are obtained [11].

We also define the transition matrix $\mathbf{\Phi}(t)$ as that $k \times k$ matrix satisfying

$$\frac{d\mathbf{\Phi}(t)}{dt} = \mathbf{A}\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbf{1}_k. \tag{3}$$

The properties of systems exhibiting these equations have been intensively investigated in the theory of linear dynamical systems. We will not attempt here to give a rigorous definition of a time-invariant continous dynamical system but simply refer to the literature [9], pp. 4–10, [10], p. 502, where the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ in (1) are viewed, respectively, as the "state transition map," the "input-state map," and the "state-output map" of a dynamical system whose state space is a vector space of dimension k. The time-dependent vector $\mathbf{x}(t)$ describes the evolution of the state in the state space. We shall denote any *dynamical system* by $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Now the class of dynamical systems $(\mathbf{A}, \mathbf{B}, \cdot)$ is said to be completely controllable, if for every state $\mathbf{a}$ existing at $t = 0$, there is a control law $\mathbf{u}(t)$, for $t > 0$, such that $\mathbf{a}$ can be brought to (and remain thereafter at) zero in a finite time interval.

Likewise, the class of dynamical systems $(\mathbf{A}, \cdot, \mathbf{C})$ is said to be completely observable if the class of dual systems $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}, \cdot)$ is completely controllable.

For a strictly stable transfer function $\mathbf{T}(p)$, the minimal $\mathbf{A}$ possess the property that given any $\delta \times \delta$ real constant symmetric matrix $\mathbf{W}$, there exists a unique symmetric $\mathbf{K}$ satisfying the equation ($\sim$ denoting transposition)

$$\mathbf{A}\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} = -\mathbf{W}. \tag{5a}$$

We shall call this latter a "Lyapunov equation," [12], p. 82. As is easily checked by direct substitution, on using (3), this $\mathbf{K}$ is given by the well-known expression

$$\mathbf{K} = \int_0^\infty \mathbf{\Phi}(\tau)\mathbf{W}\tilde{\mathbf{\Phi}}(\tau)\, d\tau. \tag{5b}$$

In particular, if $\mathbf{W}$ is chosen positive semi-definite, written for convenience in any factored form $\mathbf{W} = \mathbf{L}_0\tilde{\mathbf{L}}_0$ as can easily be done [13], p. 107, [14], then $\mathbf{K}$ will be positive definite if and only if the dynamical systems represented by $(\mathbf{A}, \mathbf{L}_0, \cdot)$ are completely controllable.

The if statement is proven by contradiction; suppose there does exist a constant nonzero state $\mathbf{a}$ such that $\mathbf{K}\mathbf{a} = \mathbf{0}$. The particular form of $\mathbf{K}$ then implies $\tilde{\mathbf{a}}\mathbf{\Phi}(t)\mathbf{L}_0 = \mathbf{0}$ everywhere on $[0, \infty]$, and since this expression is an analytic function of t, it holds everywhere on $(-\infty, \infty)$. Since we assume complete controllability, we can find a T and a control law $\mathbf{u}(t)$ such that

$$\mathbf{a} = -\int_0^T \mathbf{\Phi}(-\tau)\mathbf{L}_0\mathbf{u}(\tau)\, d\tau.$$

If we multiply this expression on the left by $\tilde{\mathbf{a}}$ and remark that the integrand is then everywhere zero, we obtain $\tilde{\mathbf{a}}\mathbf{a} = 0$ or $\mathbf{a} = \mathbf{0}$, contradicting our hypothesis.

Conversely, if the dynamical system $(\mathbf{A}, \mathbf{L}_0, \cdot)$ is not completely controllable, then there exists an $\mathbf{a}$ outside the range of

$$\int_0^t \mathbf{\Phi}(-\tau)\mathbf{L}_0\mathbf{u}(\tau)\, d\tau$$

for all t. This implies that

$$\int_0^t \mathbf{\Phi}(-\tau)\mathbf{L}_0\tilde{\mathbf{L}}_0\tilde{\mathbf{\Phi}}(-\tau)\, d\tau$$

is singular for each t, for if it were not, then $\mathbf{u}(\tau) = \tilde{\mathbf{L}}_0\tilde{\mathbf{\Phi}}(-\tau)\mathbf{a}$ would be a possible control law leading from $\mathbf{a}$ to $\mathbf{0}$. It follows that $\tilde{\mathbf{a}}\mathbf{\Phi}(-\tau)\mathbf{L}_0 = \mathbf{0}$ on $[0, t]$ for any t and by analytic continuation on $(-\infty, \infty)$. Thus, $\mathbf{K}\mathbf{a} = \mathbf{0}$, which shows the converse statement.

For the actual synthesis we will be interested in the structure of Fig. 1 where unit capacitors load a "constant" $(m + n + k)$-port described by a constant admittance matrix $\mathbf{Y}_c$; partitioning the admittance as the ports, we have for the solid portion of the figure

$$\begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} & \mathbf{y}_{13} \\ \mathbf{y}_{21} & \mathbf{y}_{22} & \mathbf{y}_{23} \\ \mathbf{y}_{31} & \mathbf{y}_{32} & \mathbf{y}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix}. \tag{6a}$$

Considering the capacitor-loaded structure with, at this point, inputs as the $(m + n)$ left-hand port voltages and outputs as the corresponding currents, then since $\mathbf{i}_3 = -\dot{\mathbf{v}}_3$, we can write from (6a) [15],

$$\dot{\mathbf{v}}_3 = -\mathbf{y}_{33}\mathbf{v}_3 - [\mathbf{y}_{31}, \mathbf{y}_{32}] \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \tag{6b}$$

$$\begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{13} \\ \mathbf{y}_{23} \end{bmatrix} \mathbf{v}_3 + \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}. \tag{6c}$$

Since these equations are analogous to the state equations of (1b), (1c), we are able to associate a constant system realization, and hence a dynamical system, with a constant coupling admittance. Since we will desire a passive synthesis we will wish $\mathbf{Y}_c$ to be positive real [6], p. 96. Of course $\mathbf{Y}$ is said to be positive real, or passive, if and only if its symmetric part is constant or if for all real constant n-vectors x has

$$\tilde{\mathbf{x}}\mathbf{Y}\mathbf{x} = \tfrac{1}{2}\tilde{\mathbf{x}}[\mathbf{Y} + \tilde{\mathbf{Y}}]\mathbf{x} \geq 0.$$

III. Transfer-Function Synthesis

Let it be desired to synthesize a given short-circuit (voltage to current), strictly stable, transfer admittance function $\mathbf{T}(p)$, $\mathcal{L}[\mathbf{i}_2] = \mathbf{T}(p)\mathcal{L}[\mathbf{v}_1]$, with which a dynamical system and system realization is associated through

$$\dot{\mathbf{x}}_0 = \mathbf{A}_0\mathbf{x}_0 + \mathbf{B}_0\mathbf{v}_1 \tag{7a}$$

$$\mathbf{i}_2 = \mathbf{C}_0\mathbf{x}_0 + \mathbf{D}\mathbf{v}_1 \tag{7b}$$

$$\mathbf{T}(p) = \mathbf{D} + \mathbf{C}_0[p\mathbf{1}_k - \mathbf{A}_0]^{-1}\mathbf{B}_0. \tag{7c}$$

In this section we will imbed these coefficient matrices, after a transformation, in a coupling admittance to

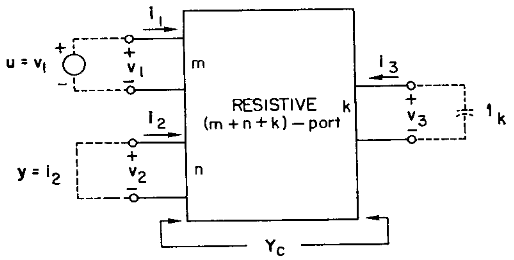

Fig. 1. Basic imbedding structure.

yield a passive circuit that is close to being lossless, this in the form of Fig. 1 (including the dotted portion).

To introduce another degree of freedom we consider a transformation on the state, as at (2), while also incorporating the output short-circuit load condition; thus

$$\mathbf{x}_0 = \mathbf{P}\mathbf{v}_3 \tag{8a}$$

$$\mathbf{v}_2 = \mathbf{0}. \tag{8b}$$

Then on ignoring $\mathbf{i}_1$ in (6c) and identifying (6b), (6c) with (7a), (7b), we have

$$\dot{\mathbf{v}}_3 = \mathbf{P}^{-1}\mathbf{A}_0\mathbf{P}\mathbf{v}_3 + \mathbf{P}^{-1}\mathbf{B}\mathbf{v}_1 = -\mathbf{y}_{33}\mathbf{v}_3 - \mathbf{y}_{31}\mathbf{v}_1 \tag{8c}$$

$$\mathbf{i}_2 = \mathbf{C}_0\mathbf{P}\mathbf{v}_3 + \mathbf{D}\mathbf{v}_1 = \mathbf{y}_{23}\mathbf{v}_3 + \mathbf{y}_{21}\mathbf{v}_1. \tag{8d}$$

This shows that we have specified the (2, 1), (2, 3), (3, 1), and (3, 3) terms of $\mathbf{Y}_c$ in (6a); we choose the remainder to introduce as much skew symmetry as possible. Entries belonging to a skew-symmetric principal sub-matrix result in a lossless structure, so that our choice will produce minimal resistivity networks. Thus

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{y}_{11} & -\tilde{\mathbf{D}} & \tilde{\mathbf{B}}_0\tilde{\mathbf{P}}^{-1} \\ \mathbf{D} & \mathbf{y}_{22} & \mathbf{C}_0\mathbf{P} \\ -\mathbf{P}^{-1}\mathbf{B}_0 & -\tilde{\mathbf{P}}\tilde{\mathbf{C}}_0 & -\mathbf{P}^{-1}\mathbf{A}_0\mathbf{P} \end{bmatrix}, \quad \begin{matrix} \mathbf{y}_{11} = -\tilde{\mathbf{y}}_{11} \\ \mathbf{y}_{22} = -\tilde{\mathbf{y}}_{22} \end{matrix}. \tag{9}$$

Here the skew-symmetric matrices $\mathbf{y}_{11}$ and $\mathbf{y}_{22}$ may be chosen at will since their respective ports are short circuited. Their value therefore has no importance, and for simplicity we shall here take them to be zero.

It should be noted that, except for the arbitrariness through $\mathbf{P}$, (9) gives the only possible $\mathbf{Y}_c$ such that resistive elements are allowed only as loading the capacitive ports. We still have to require passivity, which requires (≤ 0 for a matrix denotes negative semidefiniteness)

$$\mathbf{P}^{-1}\mathbf{A}_0\mathbf{P} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_0\tilde{\mathbf{P}}^{-1} \leq 0.$$

That is, for $\mathbf{Y}_c$ to be positive real, the symmetric part of $\mathbf{P}^{-1}\mathbf{A}_0\mathbf{P}$ must be negative semidefinite. Since $\mathbf{P}$ is constrained to be nonsingular, we can equivalently require

$$\mathbf{A}_0\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}_0 = -\mathbf{L}_0\tilde{\mathbf{L}}_0 \tag{10a}$$

$$\mathbf{P}\tilde{\mathbf{P}} = \mathbf{K} \tag{10b}$$

where we have to choose an adequate $\mathbf{L}_0$ such that $\mathbf{K}$ will be symmetric and positive definite, in which case (10b) can be immediately solved [13]. The theory of Lyapunov equations mentioned in Section II shows that (10a) can be solved if and only if we require $(\mathbf{A}_0, \mathbf{L}_0\ \cdot)$ to be completely controllable, since $\mathbf{T}(p)$ was supposed to be stable. Once $\mathbf{L}_0$ is chosen as a $k \times l$ matrix (in the most convenient way for instance with minimal column dimension), then $\mathbf{K}$ can easily be determined and decomposed to yield $\mathbf{P}$, according to (10b).

If we define, as in (2),

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{A}_0\mathbf{P}, \qquad \mathbf{B} = \mathbf{P}^{-1}\mathbf{B}_0, \qquad \mathbf{C} = \mathbf{C}_0\mathbf{P}, \tag{11a}$$
$$\mathbf{D} = \mathbf{D}, \qquad \mathbf{L} = \frac{1}{\sqrt{2}}\mathbf{P}^{-1}\mathbf{L}_0$$

then ($\mathbf{0}_m$ is the $m \times n$ zero matrix)

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{0}_m & -\tilde{\mathbf{D}} & \tilde{\mathbf{B}} \\ \mathbf{D} & \mathbf{0}_n & \mathbf{C} \\ -\mathbf{B} & -\tilde{\mathbf{C}} & -\mathbf{A} - \mathbf{L}\tilde{\mathbf{L}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_m & 0 & 0 \\ 0 & \mathbf{0}_n & 0 \\ 0 & 0 & \mathbf{L}\tilde{\mathbf{L}} \end{bmatrix} \tag{11b}$$

which is realized in Fig. 2 with $\mathbf{R}_1 = \mathbf{0}_m$, $\mathbf{R}_2 = \mathbf{0}_n$ (i.e., short circuits), in which case the gyrator conductances are as follows: $\mathbf{G} = \tilde{\mathbf{L}}$, $\mathbf{M} = -\mathbf{B}$, $\mathbf{N} = \mathbf{D}$, $\mathbf{Q} = \mathbf{C}$, $\mathbf{S} = \mathbf{A}_{sk}$ (where the skew-symmetric part of $\mathbf{A}$ is defined as $2\mathbf{A}_{sk} = \mathbf{A} - \tilde{\mathbf{A}}$). It should be noted that all gyrators, resistors, and capacitors are grounded. If a minimal-size state space is chosen, $k = \delta$, then the circuit uses a minimum number of capacitors. While if $\mathbf{L}_0$ is chosen with minimal rank l, then the circuit uses the minimum number of resistors for the given k, assuming that the coupling admittance exists, since any other choice for $\mathbf{Y}_c$ necessarily has a symmetric part of higher rank. We should mention that this minimum size for l is known as the largest number of Jordan blocks for any one eigenvalue of $\mathbf{A}_0$, though a proof of this fact is algebraically extensive. In particular this means that if all eigenvalues have only one Jordan block (i.e., if $\mathbf{A}_0$ is cyclic) then only one resistor is required.

IV. Dissipation Extractions

Instead of concentrating all dissipation at the capacitor ports, we may prefer either to remove it as load resistance or as generator impedance or both. Circuits and conditions to perform this are investigated in this section.

A. Extraction to the Load

If we put a constant symmetric impedance $\mathbf{R}_2$ at the output ports as is done in Fig. 2, with $\mathbf{R}_1 = \mathbf{0}_m$, then $\mathbf{v}_2 = -\mathbf{R}_2\mathbf{i}_2$ can be eliminated in (6) to yield

$$\dot{\mathbf{v}}_3 = [-\mathbf{y}_{33} + \mathbf{y}_{32}\mathbf{R}_2\mathbf{y}_{23}]\mathbf{v}_3 + [-\mathbf{y}_{31} + \mathbf{y}_{32}\mathbf{R}_2\mathbf{y}_{21}]\mathbf{v}_1 \tag{12a}$$

$$\mathbf{i}_2 = \mathbf{y}_{23}\mathbf{v}_3 + \mathbf{y}_{21}\mathbf{v}_1 \tag{12b}$$

where, again, we choose $\mathbf{y}_{11}$ and $\mathbf{y}_{22}$ zero, judging their influence to be uninteresting.

As we did in Section III, we again introduce a state transformation (8a), and identify (12) and (1). At the same time we require the coupling admittance to be

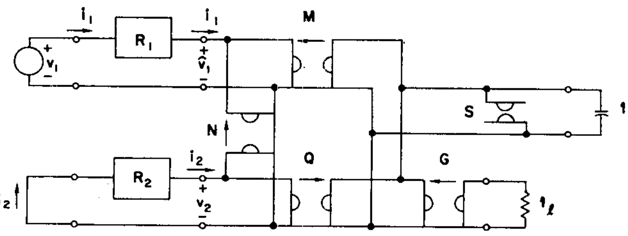

Fig. 2. General form of the obtained realizations.

lossless except for loading at the capacitance ports. This produces

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{0}_m & -\tilde{\mathbf{D}} & \tilde{\mathbf{B}} + \tilde{\mathbf{D}}\tilde{\mathbf{R}}_2\mathbf{C} \\ \mathbf{D} & \mathbf{0}_n & \mathbf{C} \\ -\mathbf{B} - \tilde{\mathbf{C}}\mathbf{R}_2\mathbf{D} & -\tilde{\mathbf{C}} & -\mathbf{A} - \tilde{\mathbf{C}}\mathbf{R}_2\mathbf{C} \end{bmatrix} \qquad (12c)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are as in (11a).

The passivity condition now becomes

$$\tilde{\mathbf{P}}^{-1}\mathbf{P}^{-1}\mathbf{A}_0 + \tilde{\mathbf{A}}_0\tilde{\mathbf{P}}^{-1}\mathbf{P}^{-1} + 2\tilde{\mathbf{C}}_0\mathbf{R}_2\mathbf{C}_0 \le 0, \qquad (12d)$$

which is equivalent to

$$\mathbf{K}\mathbf{A}_0 + \tilde{\mathbf{A}}_0\mathbf{K} = -2\tilde{\mathbf{C}}_0\mathbf{R}_2\mathbf{C}_0 - \mathbf{L}_0\tilde{\mathbf{L}}_0. \qquad (12e)$$

Then $\mathbf{L}_0$ should be chosen so that (12e) produces a positive definite symmetric $\mathbf{K} = \tilde{\mathbf{P}}^{-1}\mathbf{P}^{-1}$. The theory of Section II provides the following interesting conclusions.

1) Equation (12e) can always be solved regardless of $\mathbf{R}_2$ (even active loads could be dealt with!) by judicious choice of $\mathbf{L}_0$. Considering all possible $\mathbf{L}_0$ we can find one of minimal dimension and this will provide a circuit with minimal internal dissipation ($\mathbf{G} = \mathbf{L}_0$ for Fig. 2).

2) Writing $\mathbf{R}_2 = \tilde{\mathbf{R}}_{20}\mathbf{R}_{20}$, if $(\mathbf{A}_0, \cdot, \mathbf{R}_{20}\mathbf{C}_0)$ is completely observable, which is for instance the case if $(\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0)$ is minimal and $\mathbf{R}_2$ nonsingular; then (12e) can be solved for a positive definite $\mathbf{K}$ with $\mathbf{L}_0 = \mathbf{0}$. Thus, in this case, we are able to realize the transfer function as a lossless coupling structure loaded with $\mathbf{R}_2$. Fig. 2 again gives a realization but now with $\mathbf{R}_1 = \mathbf{0}_m$, $\mathbf{M} = -\mathbf{B} - \tilde{\mathbf{C}}\mathbf{R}_2\mathbf{D}$, $\mathbf{N} = \mathbf{D}$, $\mathbf{Q} = \mathbf{C}$, $\mathbf{S} = \mathbf{A}_{sk}$, $\mathbf{G} = \mathbf{0}_\delta$.

B. Extraction at the Input Port

We now introduce a source-impedance matrix $\mathbf{R}_1$, as in Fig. 2, while requiring $\mathbf{R}_2 = \mathbf{0}_n$ and try to realize our transfer function, with $\mathbf{y}_{11}$ and $\mathbf{y}_{22}$ zero in the coupling admittance. We proceed again through somewhat the same steps described in the previous synthesis. The situation here is mainly dual to the situation of case A with controllability replacing observability.

Again we require $\mathbf{R}_1$ to be constant and symmetric and now use the equation $\mathbf{v}_1 = \mathbf{R}_1\mathbf{i}_1 + \hat{\mathbf{v}}_1$ in (6) to obtain

$$\dot{\mathbf{v}}_3 = [-\mathbf{y}_{33} + \mathbf{y}_{31}\mathbf{R}_1\mathbf{y}_{13}]\mathbf{v}_3 + [-\mathbf{y}_{31}, \mathbf{y}_{31}\mathbf{R}_1\mathbf{y}_{12} - \mathbf{y}_{32}]\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \qquad (13a)$$

$$\begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{13} \\ \mathbf{y}_{23} - \mathbf{y}_{21}\mathbf{R}_1\mathbf{y}_{13} \end{bmatrix}\mathbf{v}_3 + \begin{bmatrix} \mathbf{0}_m & \mathbf{y}_{12} \\ \mathbf{y}_{21} & -\mathbf{y}_{21}\mathbf{R}_1\mathbf{y}_{12} \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}. \qquad (13b)$$

Next we introduce a state transformation $\mathbf{x}_0 = \mathbf{P}\mathbf{v}_3$, (8a), while requiring $\mathbf{v}_2 = \mathbf{0}$. Requiring $\mathbf{Y}_c$ to be lossless except perhaps at the capacitor ports produces

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{0}_m & -\tilde{\mathbf{D}} & \tilde{\mathbf{B}} \\ \mathbf{D} & \mathbf{0}_n & \mathbf{C} + \mathbf{D}\mathbf{R}_1\tilde{\mathbf{B}} \\ -\mathbf{B} & -[\tilde{\mathbf{C}} + \mathbf{B}\mathbf{R}_1\tilde{\mathbf{D}}] & -[\mathbf{A} + \mathbf{B}\mathbf{R}_1\tilde{\mathbf{B}}] \end{bmatrix}. \qquad (13c)$$

The passivity condition now becomes

$$\mathbf{A}_0\mathbf{P}\tilde{\mathbf{P}} + \mathbf{P}\tilde{\mathbf{P}}\tilde{\mathbf{A}}_0 + 2\mathbf{B}_0\mathbf{R}_1\tilde{\mathbf{B}}_0 \le 0, \qquad (13d)$$

which is equivalent to

$$\mathbf{A}_0\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}_0 = -\mathbf{B}_0\mathbf{R}_1\tilde{\mathbf{B}}_0 + \mathbf{L}_0\tilde{\mathbf{L}}_0. \qquad (13e)$$

Again $\mathbf{L}_0$ is to be chosen so that (13e) produces a positive definite symmetric $\mathbf{K} = \mathbf{P}\tilde{\mathbf{P}}$. The theory in Section II now allows us to conclude the following.

1) Equation (13e) can always be solved regardless of $\mathbf{R}_1$ (active source-impedances can be dealt with), provided $\mathbf{L}_0$ is well chosen. From all possible $\mathbf{L}_0$ we can find one of minimal dimension, and this will provide a circuit with minimal internal dissipation.

2) If we write $\mathbf{R}_1 = \tilde{\mathbf{R}}_{10}\mathbf{R}_{10}$ and if $(\mathbf{A}_0, \mathbf{B}_0\mathbf{R}_{10}, \cdot)$ is completely controllable, which is the case if $(\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0)$ is minimal and $\mathbf{R}_1$ nonsingular, then (13a) has a solution with $\mathbf{L}_0 = \mathbf{0}$. Thus, in this case we are able to realize the transfer function as a lossless coupling structure fed by a generator with internal impedance $\mathbf{R}_1$. Fig. 2 gives a realization with $\mathbf{R}_2 = \mathbf{0}_n$, $\mathbf{G} = \mathbf{0}_\delta$, $\mathbf{M} = -\mathbf{B}$, $\mathbf{N} = \mathbf{D}$, $\mathbf{Q} = \mathbf{C} + \mathbf{D}\mathbf{R}_1\tilde{\mathbf{B}}$, $\mathbf{S} = \mathbf{A}_{sk}$.

C. Extraction at Both Ports

This case is a generalization of the two previous cases but results in great algebraic difficulties if it is treated with no restriction on the transfer function. To avoid these complications we ask $\mathbf{D}_0 = \mathbf{0}_n$; consequently we will be able to set $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{21}$, and $\mathbf{y}_{22}$ at zero. This allows us mainly to rule out the nondynamical $\mathbf{D}$ part of the system. We begin as in case B and obtain (13) with $\mathbf{y}_{12} = \tilde{\mathbf{y}}_{21} = \mathbf{0}$. Instead of asking for $\mathbf{v}_2 = \mathbf{0}$ we now ask that $\mathbf{v}_2 = -\mathbf{R}_2\mathbf{i}_2$. Again we require $\mathbf{Y}_c$ to be lossless except perhaps at the capacitor ports. These requirements give

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{0}_m & \mathbf{0} & \tilde{\mathbf{B}} \\ \mathbf{0} & \mathbf{0}_n & \mathbf{C} \\ -\mathbf{B} & -\tilde{\mathbf{C}} & -[\mathbf{A} + \mathbf{B}\mathbf{R}_1\tilde{\mathbf{B}} + \tilde{\mathbf{C}}\mathbf{R}_2\mathbf{C}] \end{bmatrix}. \qquad (14a)$$

Proceeding next as in part A, we are led to

$$\mathbf{A}_0\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}_0 = -2\mathbf{B}_0\mathbf{R}_1\tilde{\mathbf{B}}_0 - 2\mathbf{K}\tilde{\mathbf{C}}_0\mathbf{R}_2\mathbf{C}_0\mathbf{K} - \mathbf{L}_0\tilde{\mathbf{L}}_0. \qquad (14b)$$

If we try again to obtain a lossless coupling structure, then we have to solve a quadratic matrix equation in $\mathbf{K} = \mathbf{P}\tilde{\mathbf{P}}$, (14b), with $\mathbf{L}_0 = \mathbf{0}$. Solutions for $\mathbf{K}$ can be found in [16] and [17], although no general criteria on $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{R}_1$, and $\mathbf{R}_2$ seem to be available at present.

We are again able to draw some interesting conclusions.

1) When (14b) can be solved for $\mathbf{K}$, then the circuit so obtained is again given by Fig. 2. In this case $\mathbf{M} = -\mathbf{B}$, $\mathbf{N} = \mathbf{0}$, $\mathbf{Q} = \mathbf{C}$, $\mathbf{S} = \mathbf{A}_{sk}$, $\mathbf{G} = \mathbf{0}_k$.

2) If $\mathbf{R}_1$ (or $\mathbf{R}_2$) is given then there always exists a suitably small nonsingular $\mathbf{R}_2$ (or $\mathbf{R}_1$) such that (14b) has a solution. This can be shown through an iterative procedure based on (5b).

3) If, given $\mathbf{R}_1$ and $\mathbf{R}_2$ a symmetric solution $\mathbf{K}$ has been found (for instance by use of the method described in [16]), then this solution will be positive definite if either $(\mathbf{A}_0, \ \mathbf{B}_0\mathbf{R}_{10}, \ \cdot)$ is completely controllable or $(\mathbf{A}_0, \ \cdot \ , \mathbf{C}_0\mathbf{R}_{20})$ is completely observable. This follows from the theory in Section II.

V. Example

To clarify the procedure, we will consider at this point the simple example

$$\frac{\mathcal{L}[i_2]}{\mathcal{L}[v_1]} = \frac{\frac{1}{2}}{p^2 + \sqrt{2}\,p + 1}, \tag{15a}$$

which corresponds to a second-order Butterworth response.

A corresponding minimal system realization can be read directly from (15a) as

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} \qquad \mathbf{B}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_0 = [\tfrac{1}{2} \ \ 0] \qquad D = 0.$$

This system realization already has the form required in the synthesis of Section III. It can be directly realized as in Fig. 3 with

$$R_1 = R_2 = 0, \qquad M_{21} = -1,$$

$$Q_{11} = \tfrac{1}{2}, \qquad S_{21} = 1, \qquad G_{22}^2 = 2.$$

Suppose we wish to extract a unit resistor at the input port. Then, solving (13e) gives, with $\mathbf{L}_0 = \mathbf{0}$

$$\mathbf{K} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{15b}$$

so that the resulting coupling admittance now becomes, from (13c) and $2^{1/4} = 1.189$, $2^{-5/4} = 0.420$,

$$\mathbf{Y}_c = \begin{bmatrix} 0 & 0 & 0 & 1.189 \\ 0 & 0 & 0.420 & 0 \\ 0 & -0.420 & 0 & -1 \\ -1.189 & 0 & 1 & 0 \end{bmatrix}. \tag{15c}$$

This is now lossless, and is realized in Fig. 3 with

$$R_1 = 1, \qquad R_2 = 0, \qquad M_{21} = -1.189,$$

$$Q_{11} = 0.420, \qquad S_{12} = 1, \qquad G_{22} = 0.$$

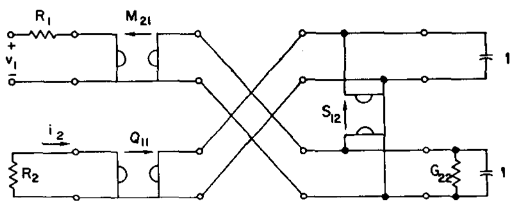

Fig. 3. Realization of $\mathcal{L}[i_2]/\mathcal{L}[v_1] = \frac{1}{2}/p^2 + \sqrt{2}\,p + 1$.

It should be remarked that in this case the extraction is almost trival since it corresponds to pulling the resistor G_{22} out through the gyrator M_{21} in cascade with the input port. Therefore, $\mathbf{K}$ becomes diagonal and merely scales the gyrator values so as to produce the value 1 for the resistor R_1.

More interesting is the extraction of a resistor at the output port. This cannot be achieved by a similar physical procedure. We need to solve (12e) with $\mathbf{L}_0 = \mathbf{0}$ (to obtain losslessness), this yields

$$\mathbf{K} = \begin{bmatrix} \frac{3}{4\sqrt{2}} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2^{-5/4} & 2^{-3/4} \\ 0 & 2^{-5/4} \end{bmatrix} \begin{bmatrix} 2^{-5/4} & 0 \\ 2^{-3/4} & 2^{-5/4} \end{bmatrix}$$

or

$$\mathbf{Y}_c = \begin{bmatrix} 0 & 0 & 0 & 0.420 \\ 0 & 0 & +1.189 & 0 \\ 0 & -1.189 & 0 & -1 \\ -0.420 & 0 & 1 & 0 \end{bmatrix}. \tag{15d}$$

The values on Fig. 3 now become

$$R_1 = 0, \qquad R_2 = 1, \qquad M_{21} = -0.420,$$

$$Q_{11} = 1.189, \qquad S_{12} = 1, \qquad G_{22} = 0.$$

Since $D = 0$, extraction of unit resistors at both ports is possible if (14b) can be solved with $\mathbf{L}_0 = \mathbf{0}$ to obtain losslessness. The only solution in this case gives

$$\mathbf{K} = \begin{bmatrix} 2\sqrt{2} & -2 \\ -2 & 2\sqrt{2} \end{bmatrix} = 2^{1/4}\begin{bmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{bmatrix} \cdot 2^{1/4}\begin{bmatrix} \sqrt{2} & -1 \\ 0 & 1 \end{bmatrix}$$

yielding

$$\mathbf{Y}_c = \begin{bmatrix} 0 & 0 & 0 & 0.841 \\ 0 & 0 & 0.841 & 0 \\ 0 & -0.841 & 0 & -0.707 \\ -0.841 & 0 & 0.707 & 0 \end{bmatrix}. \tag{15e}$$

In Fig. 3 we have to introduce

$$R_1 = R_2 = 1, \qquad -M_{21} = Q_{11} = 0.841,$$

$$S_{12} = 0.707, \qquad G_{22} = 0.$$

By using gyrator replacements the different circuits that we obtain are seen to be equivalent to the LC-tee obtained through classical synthesis. All gyrators are grounded, and the several resistor extractions can easily be carried out because the dynamical system we started with was minimal and the transfer function properly scaled (to yield a solution to the quadratic equation in **K**).

VI. Discussion

Using the ideas of previous syntheses [1]–[4], we have given a passive synthesis of strictly stable transfer functions using only time-invariant components. Although the treatment directly covered only voltage-to-current transfer functions, the remaining electrical ones are clear, some by duality, others by loading the outputs on either gyrators or resistances.

In essence we have been able to obtain the synthesis found by inserting proper entries into (8c) to yield an almost skew-symmetric coupling admittance $\mathbf{Y}_c$ that is positive real. Other insertions would yield other types of structures. Referring to Section III it is worth recognizing that the $\mathbf{X}_i$ are arbitrary in

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \\ \mathbf{D} & \mathbf{X}_4 & -\mathbf{C} \\ \mathbf{B} & \mathbf{X}_5 & -\mathbf{A} \end{bmatrix} \tag{16}$$

so that if one allows other, more lavish, distributions of resistances over the circuit, many other configurations can be obtained. From the point of view of minimality, those presented here have most practical importance. Too, by proper use of gyrators at all ports, $\mathbf{Y}_c$ can be converted to an hybrid matrix, for example an impedance, and we then see that the theory is equivalent to that using hybrid coupling matrices. Because of the generality of a theory based upon hybrid matrices [18], this being equivalent to a theory based upon scattering matrices [19], we get a feeling for the generality of the presented results.

Because of the desire to extract resistors, we have considered only the strictly stable case. If simple eigenvalues of A_0 on the imaginary axis are present, the theory of Section III holds, but care in the choice of L_0 is required [20], p. 27. Consequently, since the freedom in choice of W in (5a) is restricted, we are not able to solve (12e) and (13e), for the removal of resistors to external ports, except in special cases.

The structure of Fig. 2 is of interest for integrated-circuit design of given voltage-transfer functions. First, because it relies upon grounded gyrators whose integration possibilities are well established [21], [22] and whose commercial availability appears not too far off [23]. Second, a passive structure offers Lyapunov stability [24], while also giving promising sensitivity advantages [25], [26]. Because also a minimum number of capacitors are used, all of which are grounded, the structure offers real practical interest in the integrated-circuit field.

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