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THE TIME-VARIABLE CALCULUS OF V. DOLEŽAL*

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Abstract: By simplifying the ideas of Doležal a promising method for the analysis of time-variable systems is presented. The method consists in converting describing equations for linear systems to integral (distributional kernel) form, and then inverting the result by the use of Volterra series.

I. Introduction

One of the problems in dealing with time-variable systems concerns the solution of terminal describing equations. Along this line V. Doležal has presented a method of solving linear integro-differential equations which should prove of some interest for both analysis and synthesis [1] [2] [3, pp. 115-187]. The method can be considered somewhat in the nature of an operational calculus for time-variable systems, though perhaps not the most appealing in this operational sense, and can be considered to rest upon the theory of distributions. The philosophy gives a theoretically appealing method but, without the use of a digital computer for which the method does seem imminently suitable, the computations are somewhat difficult since solutions are based upon infinite series solutions of integral equations. The class of equations which can be treated is much more general than the set of ordinary differential equations with variable coefficients since integral operators can be considered. Hence nondifferential systems and especially time-variable systems with delay can be handled.

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In the following we summarize, simplify, and extend to the widest class of distributions, the theory of Doležal. The presentation relies heavily on the theory of distributions [4] [5] and in particular the theory of distributional kernels [6], with which we assume the reader is somewhat familiar.

II. Distributional Preliminaries (Definitions and Notation)

The theory rests upon the vector valued theory of distributions in two variables. For this we proceed by reviewing the theory of L. Schwartz, [4] [5]. Since we will sometimes need results for both one and two variables, we actually will work with m variables and specialize m to one or two as needed.

Let R^m be the real m-dimensional vector space whose points $\underline{x}$ are defined by the m (real) coordinate variables $x_1, x_2, \ldots, x_m$. A complex valued n-vector function evaluated at the point $\underline{x}$ will be denoted by $\mathbf{f}(\underline{x})$. By the support of $\mathbf{f}$ is meant the closure of the set of points $\underline{x}$ for which $\mathbf{f}(\underline{x}) \neq \mathbf{0}$; thus in one dimension the (1-vector) unit step function

$$u(x_1) = \begin{cases} 1 & x_1 > 0 \\ 0 & x_1 < 0 \end{cases} \qquad (\text{II-1})$$

has support $x_1 \geq 0$. Key to the theory of Schwartz is the set of (n-vector) testing functions,* $\mathbf{\mathcal{D}}$. $\boldsymbol{\varphi}$ is, by definition, a testing function, $\boldsymbol{\varphi} \in \mathbf{\mathcal{D}}$, if $\boldsymbol{\varphi}$ is infinitely (continuously) differentiable (in all variables) and of compact support. Although these are not worked with much in calculations, it is of interest to know that testing functions exist; in the scalar (1-vector) case such are, [7, p. 106] [8, p. 2],

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*A wiggly underscore (for bold face) will mean n-vector or matrix quantities and will generally be omitted when only scalars, i.e., 1-vectors, are being explicitely referred to.

$$\varphi(x_1) = \exp[\frac{-1}{x_1^2}] \cdot \exp[\frac{1}{(1-x_1)^2}] u(x_1) \cdot u(1-x_1) \qquad \text{(II-2a)}$$

$$\varphi(x_1, x_2) = \exp[-(\frac{1}{1-r^2})] u(1-r); \; r = \sqrt{x_1^2 + x_2^2} \qquad \text{(II-2b)}$$

Now given any (locally summable) function $\underset{\sim}{f}$ the integral (a tilde denotes transpose)

$$\underbrace{\iint \cdots \int_{-\infty}^{\infty}}_{n} \tilde{\underset{\sim}{f}}(\underline{x}) \underset{\sim}{\varphi}(\underline{x}) d\underline{x} = \langle \underset{\sim}{T}_f, \underset{\sim}{\varphi} \rangle \qquad \text{(II-3)}$$

will always exist for every $\underset{\sim}{\varphi} \epsilon \underset{\sim}{\mathcal{D}}$; here $d\underset{\sim}{x} = dx_1 dx_2 \ldots dx_m$. $\langle \underset{\sim}{T}_f, \underset{\sim}{\varphi} \rangle$ is called the scalar product of the functional $\underset{\sim}{T}_f$ (defined by $\underset{\sim}{f}$) with the testing function $\underset{\sim}{\varphi}$. We note that $\underset{\sim}{f}$, through (II-3), defines the linear continuous functional $\underset{\sim}{T}_f$; that is given $\underset{\sim}{f}$, $\underset{\sim}{T} = \underset{\sim}{T}_f$ has the following properties for all $\underset{\sim}{\varphi}_i \epsilon \underset{\sim}{\mathcal{D}}$ and any complex constant scalar α:

1) $\langle \underset{\sim}{T}, \underset{\sim}{\varphi}_1 + \underset{\sim}{\varphi}_2 \rangle = \langle \underset{\sim}{T}, \underset{\sim}{\varphi}_1 \rangle + \langle \underset{\sim}{T}, \underset{\sim}{\varphi}_2 \rangle$
2) $\langle \underset{\sim}{T}, \alpha\underset{\sim}{\varphi}_1 \rangle = \alpha \langle \underset{\sim}{T}, \underset{\sim}{\varphi}_1 \rangle$
3) If all $\underset{\sim}{\varphi}_j$ have their support in a fixed compact subset of R^m, and, if they, as well as each of their derivatives, converge uniformly to zero, then the complex numbers $\langle \underset{\sim}{T}, \underset{\sim}{\varphi}_j \rangle$ converge to zero.

Any $\underset{\sim}{T}$ satisfying these three conditions, that is, any continuous linear functional, over $\underset{\sim}{\mathcal{D}}$ is called a (n-vector) distribution. If (as we do) we identify $\underset{\sim}{T}_f$ with $\underset{\sim}{f}$, writing $\underset{\sim}{T}_f = \underset{\sim}{f}$, we see that every (locally summable) function is a distribution; however, there are distributions which are not functions; such is the (1-vector) impulse "function," $\delta(\underline{x})$, defined by

$$\langle \delta, \varphi \rangle = \varphi(\underline{0}) \qquad \text{(II-4)}$$

We point out that not all normal "functions" are distributions, they must be locally summable; thus $\exp[1/x_1]$ is not a distribution. The space of distributions is the "topological dual" of $\underset{\sim}{\mathcal{D}}$ and is denoted by $\underset{\sim}{\mathcal{D}}'$.

That is $\underset{\sim}{T} \epsilon \underset{\sim}{\mathcal{D}}'$ if and only if $\underset{\sim}{T}$ is a linear continuous functional defined over the set of testing functions $\underset{\sim}{\mathcal{D}}$.

The use of distributions lies in the fact that they can be differentiated an infinite number of times with the resultant derivative still a distribution. In order that distributional properties agree with functional ones, (II-3) is used to define all properties of distributions. Thus, replacing $\underset{\sim}{f}$ by $\partial \underset{\sim}{f}/\partial x_i$ and integrating by parts, we are led to the distributional derivative of $\underset{\sim}{T}(x)$ with respect to any component x_i of $\underline{x}$

$$< \partial \underset{\sim}{T}/\partial x_i, \underset{\sim}{\varphi} > \;=\; < -\underset{\sim}{T}, \partial \underset{\sim}{\varphi}/\partial x_i > \tag{II-5}$$

Note that this shows why we wish $\underset{\sim}{\varphi}$ infinitely differentiable and why all derivative sequences of uniformly convergent $\underset{\sim}{\varphi}_j$ should also be uniformly convergent. As an example we have

$$\frac{\partial u(x_1)}{\partial x_1} = u' = \delta \tag{II-6}$$

In the m-variable 1-vector case it is convenient to work with the tensor product. Thus for two single variable distributions $T_1(x_1)$ and $T_2(x_2)$ the tensor product, $T(x_1,x_2) = T_1 \otimes T_2$, is defined by, [5, p. 109],

$$< T_1(x_1) \otimes T_2(x_2), \underset{\sim}{\varphi}(x_1,x_2) > \;=\; < \underset{\sim}{T}_1(x_1), < \underset{\sim}{T}_2(x_2), \underset{\sim}{\varphi}(x_1,x_2) > > \tag{II-7a}$$

This is uniquely determined by using "degenerate" testing functions, that is ones of the form $\varphi(x_1,x_2) = \varphi_1(x_1)\cdot\varphi_2(x_2)$, [5, p. 109],

$$< T_1 \otimes T_2, \varphi_1 \cdot \varphi_2 > \;=\; < T_1, \varphi_1 > \cdot < T_2, \varphi_2 > \tag{II-7b}$$

For example we can define a two variable unit step function by

$$u(\underline{x}) = u(x_1) \otimes u(x_2) \tag{II-8a}$$

Then by (II-6)

$$\frac{\partial u(\underline{x})}{\partial x_1} = \delta(x_1) \otimes u(x_2) \qquad \text{(II-8b)}$$

$$\frac{\partial^2 u(\underline{x})}{\partial x_1 \partial x_2} = \delta(x_1) \otimes \delta(x_2) = \delta(\underline{x}) \qquad \text{(II-8c)}$$

A matter of interest is the support of a distribution. For this a distribution $\underset{\sim}{T}$ is said to be zero in a set $\Omega \subset R^m$ if $< \underset{\sim}{T}, \underset{\sim}{\varphi} > = 0$ whenever the testing function $\underset{\sim}{\varphi}$ has its support (i.e., is nonzero) in Ω. The support of $\underset{\sim}{T}$ is then the closure of the set of points in R^m for which $\underset{\sim}{T} \neq \underset{\sim}{0}$; $u(\underline{x})$ in (II-8a) has support $x_1 \geq 0$, $x_2 \geq 0$; $\delta(\underline{x})$ in (II-8c) has support $\underline{x} = 0$.

A special notion of supplementary interest here is the convolution, $S*T$. For 1-vectors this can be defined through the tensor product by [5, p. 11]

$$< S*T(\underline{x}), \varphi(\underline{x}) > = < S(\underline{x}) \otimes T(\underline{y}), \varphi(\underline{x}+\underline{y}) > \qquad \text{(II-9a)}$$

$$= < S(\underline{x}), < T(\underline{y}), \varphi(\underline{x}+\underline{y}) > > \qquad \text{(II-9b)}$$

Thus, for $S \in \mathcal{D}'$,

$$S*\delta = S \qquad \text{(II-9c)}$$

In working with one and two variable distributions we will normally let

$$x_1 = t, \quad x_2 = \tau$$

in the following. Further to distinguish the one and two variable spaces we will sometimes insert subscripts, and for example, write $\underset{\sim}{\mathcal{D}}_t$, $\underset{\sim}{\mathcal{D}}_{t,\tau}$ $\underset{\sim}{\mathcal{D}}'_t$, $\underset{\sim}{\mathcal{D}}'_{t,\tau}$. Some other spaces will be of interest. The important set of $n \times n$ matrices, $\underset{\sim}{T}(t,\tau)$, whose entries are distributions which have their support in the half-plate $t > \tau$ will be denoted by $\underset{\sim}{\mathcal{D}}'_{t \triangleright \tau}$. The set of infinitely differentiable functions (not necessarily of compact support) will be denoted by $\mathcal{E}$ (with appropriate subscripts, as used on $\mathcal{D}$), for example $\exp[t]\in\mathcal{E}_t$. The set of distributions of compact support is denoted by $\mathcal{E}'$, for instance $\delta(t)\in\mathcal{E}'_t$, $u(t)\notin\mathcal{E}'_t$. When the dimensions are the same $\mathcal{D}\subset\mathcal{E}$, $\mathcal{D}\subset\mathcal{E}'\subset\mathcal{D}'$. Of considerable interest is the set of $n\times n$ matrices, $G(t,\tau)$, of functions of the form

$$G(t,\tau) = g(t,\tau)u(t-\tau);\; g(t,\tau)\in\mathcal{E} \tag{II-10}$$

These will be denoted by $\mathcal{E}_{t>\tau}$; we have then $\mathcal{E}_{t>\tau}\subset\mathcal{D}_{t>\tau}$. Also, the set of n-vector or $n\times n$ matrix distributions in one variable with support $t > c > -\infty$ (bounded on the left), c = arbitrary constant, will be denoted by $\mathcal{D}'_+$; thus, in the scalar case $u(t)\in\mathcal{D}'_+$ and $\delta(t)\in\mathcal{D}'_+$ but $\exp[t]\notin\mathcal{D}'_+$.

By a distributional kernel will be meant any $n\times n$ matrix of distribution; $T(t,\tau)$, in two variables; that is T is a distributional kernel if $T\in\mathcal{D}'_{t,\tau}$. Then given a distributional kernel, it defines a linear continuous mapping of (n-vectors in) $\mathcal{D}_\tau$ into (n-vectors in) $\mathcal{D}'_t$, [6, p. 221]. We can denote this mapping by the notation* (the operation denoted is simply called composition)

$$[T\cdot\psi] = [T(t,\tau)\cdot\psi(\tau)] = \int_{-\infty}^{\infty} T(t,\tau)\psi(\tau)\,d\tau \tag{II-11}$$

which is somewhat a (nonscalar) scalar product over the variable τ. However, note the difference between $[T\cdot\psi]$ and the scalar product $< T, \varphi >$; $[T\cdot\psi]$ is an n-vector distribution of one variable while $< T, \varphi >$ is a complex scalar number. In fact for, [6, p. 221], $\varphi\in\mathcal{D}_{t,\tau}$, $\varphi_1\in\mathcal{D}_t$, $\varphi_2\in\mathcal{D}_\tau$, and

$$\varphi(t,\tau) = \varphi_1(t)\varphi_2(\tau) \tag{II-12a}$$

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*Because we will have frequent use for differentiation of $[T\cdot\psi]$, we use this bracket notation of Doležal for composition. However $[T\cdot\psi]$ has the identical meaning of the notation $T\cdot\psi$ of Schwarz, and used commonly in previous works [9].

we have

$$< T,\ \varphi > \ = \ < [T\cdot\varphi_2],\ \varphi_1 > \qquad \text{(II-12b)}$$

where the scalar product on the left is in two variables and that on the right is in one. As an example, let, with φ_2 as in (II-12a),

$$T(t,\tau) \ = \ u(t) \otimes \delta(\tau) \qquad \text{(II-13a)}$$

$$[T\cdot\varphi_2] \ = \ u(t)\varphi_2(0) \qquad \text{(II-13b)}$$

$$< [T\cdot\varphi_2],\ \varphi_1 > \ = \ \varphi_2(0)\int_0^\infty \varphi_1(t)dt \ = \ < T(t,\tau),\ \varphi_1(t)\varphi_2(\tau) > \text{(II-13c)}$$

We recall that a knowledge of the scalar product $< T(t,\tau),\ \varphi_1(t)\varphi_2(\tau) >$ for all φ_1, φ_2 in $\mathcal{D}$ is sufficient to determine $< T(t,\tau),\ \varphi(t,\tau)>$ for all $\varphi\in\mathcal{D}$ [4, p. 108] and hence sufficient to define T.

For many $\mathbf{T}\in\mathcal{D}'_{\mathbf{t},\tau}$ it is possible to form $[\mathbf{T}\cdot\boldsymbol{\psi}]$ for other $\boldsymbol{\psi}$ than those in $\mathcal{D}_{\boldsymbol{\tau}}$, this is easily seen to be possible for any $\boldsymbol{\psi}\in\mathcal{D}'_{\boldsymbol{\tau}}$ if $\mathbf{T}\in\mathcal{D}_{\mathbf{t},\tau}$. A particular case of interest is when $\mathbf{T}\in\mathcal{E}_{\mathbf{t}>\tau}$, then $[\mathbf{T}\cdot\boldsymbol{\psi}]$ can be formed for any $\boldsymbol{\psi}\in\mathcal{D}'_{\mathbf{+}}$.

A supplementary result, needed for calculations, is the procedure for testing the scalar $\delta(t-\tau)$ as a distribution in two variables. As preliminary steps we have

$$< u(t-\tau),\ \varphi(t,\tau) > \ = \int_{-\infty}^{\infty}\int_{-\infty}^{t}\varphi(t,\tau)d\tau dt \qquad \text{(II-14a)}$$

$$= \int_{-\infty}^{\infty}\int_{\tau}^{\infty}\varphi(t,\tau)dtd\tau \qquad \text{(II-14b)}$$

$$< u(t-\tau),\ -\frac{\partial\varphi(t,\tau)}{\partial t} > \ = \int_{-\infty}^{\infty}\int_{\tau}^{\infty} -\frac{\partial\varphi(t,\tau)}{\partial t}dtd\tau \qquad \text{(II-14c)}$$

$$= \int_{-\infty}^{\infty} \varphi(\tau,\tau)d\tau \qquad \text{(II-14d)}$$

Thus

$$< \delta(t-\tau), \varphi(t,\tau) > \; = \; < u(t-\tau), \frac{\partial\varphi(t,\tau)}{\partial\tau} > \qquad \text{(II-14e)}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t} \frac{\partial\varphi(t,\tau)}{\partial\tau} d\tau dt \; = \int_{-\infty}^{\infty} \varphi(t,t)dt \qquad \text{(II-14f)}$$

$$= \; < 1, \varphi(t,t) > \qquad \text{(II-14g)}$$

Also, of interest is the definition of the product of $\alpha\epsilon\mathcal{E}$ by $T\epsilon\mathcal{D}'$ through

$$< \alpha T, \varphi > \; = \; < T, \alpha\varphi > \qquad \text{(II-15)}$$

For instance

$$< \alpha(t,\tau)\delta(t,\tau), \varphi(t,\tau) > \; = \; < \delta(t,\tau), \alpha(t,\tau)\varphi(t,\tau) > \qquad \text{(II-16a)}$$

$$= \; \alpha(0,0)\varphi(0,0) \qquad \text{(II-16b)}$$

or

$$\alpha(t,\tau)\delta(t,\tau) \; = \; \alpha(0,0)\delta(t,\tau) \qquad \text{(II-16c)}$$

With this short review of the needed distributional background we can turn to the operators of Doležal.

III. Operations in $\mathcal{E}_{t>\tau}$; Operators

The kernels of most interest in systems analysis are those in $\mathcal{E}_{t>\tau}$, that is, those of the form

$$\mathbf{G}(t,\tau) = \mathbf{g}(t,\tau)u(t-\tau) \qquad \text{(III-1)}$$

with $\mathbf{g}$ infinitely differentiable, i.e., $\mathbf{g}(t,\tau)\epsilon\boldsymbol{\mathcal{S}}_{t,\tau}$. Consequently we define specific operations on such kernels and give special notation to certain of them.

For $\mathbf{G}_1$, $\mathbf{G}_2\epsilon\boldsymbol{\mathcal{S}}_{t>\tau}$ the <u>Volterra composition</u>, $\mathbf{G}_1 o \mathbf{G}_2$, is defined by

$$\mathbf{G}_1 o \mathbf{G}_2(t,\tau) = \int_{-\infty}^{\infty} \mathbf{G}_1(t,\xi)\mathbf{G}_2(\xi,\tau)d\xi \qquad \text{(III-2a)}$$

$$= \{ \int_{\tau}^{t} \mathbf{g}_1(t,\xi)\mathbf{g}_2(\xi,\tau)d\xi\}u(t-\tau) \qquad \text{(III-2b)}$$

As seen by (III-2b), $\mathbf{G}_1 o \mathbf{G}_2\epsilon\boldsymbol{\mathcal{S}}_{t>\tau}$, and consequently $\boldsymbol{\mathcal{S}}_{t>\tau}$ forms a <u>ring</u>, $\boldsymbol{\mathcal{S}}_{t>\tau}(o,+)$, under the operations of o and $+$, [10, p. 346]. However, this ring is not commutative even in the 1-vector case since in general $G_1 o G_2 \neq G_2 o G_1$ as is seen by letting

$$G_1(t,\tau) = e^{t}u(t-\tau) \qquad \text{(III-3a)}$$

$$G_2(t,\tau) = u(t-\tau) \qquad \text{(III-3b)}$$

giving

$$G_1 o G_2(t,\tau) = [t-\tau]e^{t}u(t-\tau) \qquad \text{(III-3c)}$$

$$G_2 o G_1(t,\tau) = [e^{t}-e^{\tau}]u(t-\tau) \neq G_1 o G_2 \qquad \text{(III-3d)}$$

Further there is no unit element, $\mathbf{G}_u$, $\mathbf{G}_u\epsilon\boldsymbol{\mathcal{S}}_{t>\tau}$, such that $\mathbf{G}_u o \mathbf{G} = \mathbf{G} o \mathbf{G}_u = \mathbf{G}$ for all $\mathbf{G}\epsilon\boldsymbol{\mathcal{S}}_{t>\tau}$, since, if there were, (III-2b) would require for the specific choice $G(t,\tau) = u(t-\tau)$ that

$$1 = \int_{\tau}^{t} g_u(t,\xi)d\xi$$

This is impossible, as is seen by letting $t = \tau$. Of course such a $\underset{\sim}{G}_u$ corresponds to an $n \times n$ identity matrix of impulses $\delta(t-\tau)\underset{\sim}{1}_n$ which are not functions but distributions. In summary $\underset{\sim}{\mathcal{E}}_{t>\tau}(o,+)$ forms a non-commutative ring with no unit element.

It appears that in the scalar case $G_1 o G_2 = 0$ if and only if $G_1 = 0$ or $G_2 = 0$; this is certainly the case for degenerate kernels, that is when g_1 and g_2 are of the form $\sum_{i=1}^{m} h_i(t)k_i(\tau)$, m finite. Such a result would be of interest for an alternate development (to be described elsewhere) but is not of major importance for the ideas of Doležal.

Some elements of importance in $\underset{\sim}{\mathcal{E}}_{t>\tau}$ are defined by (the $n \times n$ kernel functions)

$$\underset{\sim}{U}_{-\lambda}(t,\tau) = \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} u(t-\tau)\underset{\sim}{1}_n, \ \lambda \geq 1 \qquad \text{(III-4)}$$

where Γ is the Gamma function, $\underset{\sim}{1}_n$ is the $n \times n$ identity matrix, and λ can be any real number not less than 1. Here we can actually take $\underset{\sim}{U}_{-\lambda}$ equal to the associated $\underset{\sim}{g}$ in equation (III-1), if so desired. It is convenient to define, for any $\underset{\sim}{\psi} \in \underset{\sim}{\mathcal{D}}'_+$ {recall (II-11) for the meaning of $[\cdot]$}

$$\underset{\sim}{\psi}^{(-\lambda)}(t) = [\underset{\sim}{U}_{-\lambda}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)], \ \lambda \geq 1 \qquad \text{(III-5a)}$$

In all cases (III-5a) actually reduces to a convolution, and, when λ is a positive integer, $\underset{\sim}{\psi}^{(-\lambda)}$ is the λth integral of $\underset{\sim}{\psi}$. Since every derivative of a distribution exists, (III-5a) allows $\underset{\sim}{\psi}^{(k)}$ to be defined for every (positive or negative) integer k whenever $\underset{\sim}{\psi} \in \underset{\sim}{\mathcal{D}}'_+$, and then

$$(\underset{\sim}{\psi}^{(m)})^{(n)} = \underset{\sim}{\psi}^{(m+n)} \qquad \text{(III-5b)}$$

for all integers m and n.

An extension of this idea of defining the generalized integral or derivative is important and vital to the theory. For any $\underset{\sim}{G}$ of the form $\underset{\sim}{g}u$, as in (III-1), i.e., $\underset{\sim}{G}\in\underset{\sim}{\mathcal{E}}_{t>\tau}$, and any $\underset{\sim}{\psi}\in\underset{\sim}{\mathcal{D}}'_{+}$, $[\underset{\sim}{G}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)]$ is again a distribution in $\underset{\sim}{\mathcal{D}}'_{+}$. To see this note that the following scalar product is well-defined for $G\in\mathcal{E}_{t>\tau}$ and can be expressed as, see (II-12b),

$$< G(t,\tau),\ \varphi_1(t)\psi(\tau) > \ = \ < [G(t,\tau)\cdot\psi(\tau)],\ \varphi_1(t) > \qquad \text{(III-6)}$$

If the support of ψ is $\tau > c$ then the left side of this is zero for φ_1 if support $t < c$, and, thus, the right side must be also, which shows, considering matrix component by component, that $[\underset{\sim}{G}\cdot\underset{\sim}{\psi}]\in\underset{\sim}{\mathcal{D}}'_{+}$ when $\underset{\sim}{\psi}$ is. Consequently $[\underset{\sim}{G}\underset{\sim}{\psi}]^{(k)}$ can be formed, by differentiating when $K \geq 0$ or by using (III-5a) when $k < 0$, for any integer k. By (III-6) and use of the differentiation definition, (II-5), on other $\underset{\sim}{\varphi}$ than those in $\underset{\sim}{\mathcal{D}}$ we easily obtain

$$[\frac{\partial^k \underset{\sim}{G}(t,\tau)}{\partial t^k}\cdot\underset{\sim}{\psi}(\tau)] \ = \ [\underset{\sim}{G}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)]^{(k)} \qquad \text{(III-7)}$$

where $\underset{\sim}{G}\in\underset{\sim}{\mathcal{E}}_{t>\tau}$, $\underset{\sim}{\psi}\in\underset{\sim}{\mathcal{D}}'_{+}$ and k is a positive integer.

The point to notice here is that the space $\underset{\sim}{\mathcal{E}}_{t>\tau}$ defines an important class of operators, $\underset{\sim}{A}[\ ]$, which map $\underset{\sim}{\mathcal{D}}'_{+}$ into $\underset{\sim}{\mathcal{D}}'_{+}$,

$$\underset{\sim}{A}[\underset{\sim}{\psi}] \ = \ [\underset{\sim}{G}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)]^{(k)} \qquad \text{(III-8)}$$

These operators $\underset{\sim}{A}[\ ]$ form the core of the theory and could justly be called Doležal operators. The Doležal operators are then seen to be characterised by the n×n distributional kernels $\partial^k \underset{\sim}{g}(t,\tau)u(t-\tau)/\partial t^k$ which have $\underset{\sim}{g}$ infinitely differential.

As examples of such operators we mention the following four derivative and integral 1-vector operators:

$$A[\psi] = \psi(t) = [u(t-\tau)\cdot\psi(\tau)]^{(1)} = [\delta(t-\tau)\cdot\psi(\tau)] = \delta * \psi \qquad \text{(III-9a)}$$

$$A[\psi] = \alpha'(t)\psi^{(-1)}(t) + \alpha(t)\psi(t) = [\alpha(t)u(t-\tau)\cdot\psi(\tau)]^{(1)} \qquad \text{(III-9b)}$$

$$A[\psi] = \psi^{(k)}(t) = [u(t-\tau)\cdot\psi(\tau)]^{(k+1)};\quad k > 0 \qquad \text{(III-9c)}$$

$$A[\psi] = \psi^{(-k)}(t) = [U_{-k}(t,\tau)\cdot\psi(t)], \quad k > 0 \qquad \text{(III-9d)}$$

As a consequence of (III-9) we can represent time-variable differential equations in the (general) form of Doležal operators, (III-8); most integro-differential equations also have this Dolezal operator representation as will be seen in section VI.

By induction, see Appendix 1, we get the useful result for all integer $k > 0$,

$$[G(t,\tau)\cdot\psi(\tau)]^{(k)} = \sum_{i=0}^{k-1}\left\{\left(\frac{\partial^i g(t,\tau)}{\partial t^i}\right)\Big|_{\tau=t}\psi(t)\right\}^{(k-i-1)} + \left[\frac{\partial^k g(t,\tau)}{\partial t^k}u(t-\tau)\cdot\psi(\tau)\right] \qquad \text{(III-10)}$$

which is valid for $G \in \mathfrak{G}_{t>\tau}$, $\psi \in \mathcal{D}'_+$. By observing this result in conjunction with the definition of composition, (II-11), we see that the class of operators under consideration are indeed generalized integro-differential operators.

As an example of (III-10) let $k = 2$ and

$$G(t,\tau) = (\tau e^t - te^\tau)u(t-\tau)$$

then

$$g(t,t) = 0; \ \frac{\partial g(t,\tau)}{\partial t}\Big|_{\tau=t} = (\tau e^t - e^\tau)\Big|_{\tau=t} = (t-1)e^t; \ \frac{\partial^2 g(t,\tau)}{\partial t^2} = \tau e^t$$

From the integral notation of (II-11), (III-10) can be written here as

$$\frac{d^2}{dt^2}\left\{\int_{-\infty}^{\infty}(\tau e^t - te^\tau)u(t-\tau)\psi(\tau)d\tau\right\} = (t-1)e^t\psi(t) + \int_{-\infty}^{\infty}\tau e^t u(t-\tau)\psi(\tau)d\tau$$

In this section we have introduced the major concepts and symbols which allow us to express system descriptions in Doležal operator form. These operators are equivalent to a special class of distributional kernels, our real interest in them lying in means of inversion for which the concept of order is useful.

IV. Order of an Operator

A Notion of considerable importance for the inversion of operators is that of order. Several definitions are possible, but the following one seems most useful.

Let c be a fixed real constant ($c = -\infty$ is allowed), then by $\underset{\sim}{D}'_{[c,\infty)}$ is meant the set of n-vector distributions in one variable with support in the interval $[c,\infty)$; for instance $e^t u(t-1)$ is in $\mathcal{D}'_{[1,\infty)} \subset \mathcal{D}'_+$ but not in $\mathcal{D}'_{[2,\infty)}$. By definition $\underset{\sim}{\psi} \in \underset{\sim}{\mathcal{D}}'$ is said to have <u>order</u> n over $[c,\infty)$, denoted by $r_c(\underset{\sim}{\psi})$, $n = r_c(\underset{\sim}{\psi})$, if n is the smallest integer such that $\underset{\sim}{\psi}^{(-n-1)}$ is a locally integrable function, over $[c,\infty)$. For instance $r_o(\delta) = 0$, $r_1(\delta) = -\infty$, $r_o(u) = -1$, $r_o(\delta') = 1$;

$$\psi(t) = \sum_{i=1}^{\infty} \delta(t-i)$$ has no order. In the definition, closure on the left at c is rather important.

The above defines the order of an n-vector distribution in $\underset{\sim}{D}'_{[c,\infty)}$; the <u>order</u> of a kernel matrix $\underset{\sim}{G} \in \underset{\sim}{\mathcal{E}}_{t>\tau}$ is defined differently. If $\underset{\sim}{G} \in \underset{\sim}{\mathcal{E}}_{t>\tau}$ and if for <u>all</u> $t \in [c,\infty)$ [writing det for determinant]

$$\det\left[\frac{\partial^q \underset{\sim}{g}(t,\tau)}{\partial \tau^q}\Big|_{\tau=t}\right] \neq 0; \ \infty > q \geq 0 \qquad \text{(IV-1a)}$$

$$\frac{\partial^i \underset{\sim}{g}(t,\tau)}{\partial \tau^i}\Big|_{\tau=t} = 0; \ i = 0, 1, \ldots, q-1 \quad (\text{if } q > 0) \qquad \text{(IV-1b)}$$

then we define the order of $\underset{\sim}{G}$ on $[c,\infty)$ as

$$r_c(\underset{\sim}{G}) = -q \qquad \text{(IV-1c)}$$

Note that the order of a kernel $\underset{\sim}{G}$ is always nonpositive when it exists. However, since (IV-1a) may be violated for some t the order need not exist.

The justification for (IV-1c) stems from the fact that, whenever $r_c(\underset{\sim}{G})$ and $r_c(\underset{\sim}{\psi})$ exist, then

$$r_c([\underset{\sim}{G}\cdot\underset{\sim}{\psi}]) = r_c(\underset{\sim}{G}) + r_c(\underset{\sim}{\psi}) \qquad \text{(IV-1d)}$$

To see (IV-1d), we can form

$$\underset{\sim}{\psi} = \underset{\sim}{\Psi}^{(q+1)} \qquad \text{(IV-2)}$$

where $\underset{\sim}{\Psi}$ can be formed using (III-5). Then, as shown in Appendix 2,

$$[\underset{\sim}{G}\cdot\underset{\sim}{\psi}] = \sum_{i=0}^{q} (-1)^i \left(\frac{\partial^i g(t,\tau)}{\partial\tau^i}\Big|_{\tau=t}\right)\underset{\sim}{\Psi}^{(q-1)}(t) + (-1)^{q+1}\left[\frac{\partial^{q+1} g(t,\tau)}{\partial\tau^{q+1}}u(t-\tau)\cdot\underset{\sim}{\Psi}(\tau)\right] \qquad \text{(IV-3a)}$$

$$= (-1)^q \frac{\partial^q g(t,\tau)}{\partial\tau^q}\Big|_{\tau=t}\underset{\sim}{\Psi}(t) + (-1)^{q+1}\left[\frac{\partial^{q+1} g(t,\tau)}{\partial\tau^{q+1}}u(t-\tau)\cdot\underset{\sim}{\Psi}(\tau)\right] \qquad \text{(IV-3b)}$$

The order of $[\underset{\sim}{G}\cdot\underset{\sim}{\psi}]$ is then equal to that of $\underset{\sim}{\Psi}$, the term on the very right of (IV-3b) having order less than that of $\underset{\sim}{\Psi}$ (involving an "integral" of $\underset{\sim}{\Psi}$). Thus $r_c([\underset{\sim}{G}\cdot\underset{\sim}{\psi}]) = r_c(\underset{\sim}{\Psi}) = r_c(\underset{\sim}{\psi})-q$ which is (IV-1d). As examples we have:

1) $G(t,\tau) = (e^{2t}-e^{2\tau})u(t-\tau)$

   $g(t,t) = 0$

   $\frac{\partial g(t,\tau)}{\partial\tau}\Big|_{\tau=t} = -e^{2t}$

   giving $r_c(G) = -1$ for any finite c.

2) $G(t,\tau) = (1+e^{-\tau})u(t\ \tau)$

   $g(t,t) = 1+e^{-t}$

   giving $r_c(G) = 0$ for any finite c.

3) $G(t,\tau) = t\tau u(t-\tau)$

   $g(t,t) = t^2$

   giving $r_c(G) = 0$ for any $c > 0$. Note that in this case the order does not exist if $c \leq 0$.

The <u>order of a Doležal operator</u> A[ ]

$$\underset{\sim}{A}[\underset{\sim}{\psi}] = [\underset{\sim}{G}\cdot\underset{\sim}{\psi}]^{(k)} \qquad \text{(III-8)}$$

is defined as

$$r_c(\underset{\sim}{A}[\ ]) = k + r_c(\underset{\sim}{G}) = k - q \tag{IV-4}$$

An operator $\underset{\sim}{A}[\ ]$ is then said to possess <u>finite order</u>, on $[c,\infty)$, if $r_c(\underset{\sim}{G})$ exists.

The equation which will be of interest for inversion of operators is the expansion of $[\underset{\sim}{G}\cdot\underset{\sim}{\psi}]^{(k)}$ is terms of $\underset{\sim}{g}$ given in (III-10). Since derivatives with respect to t, in place of τ, occur there, the following equation which is shown in a straightforward manner, is important, [2, p. 212].

$$\frac{\partial^k \underset{\sim}{g}(t,\tau)}{\partial t^k}\Big|_{\tau=t} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\frac{\partial^i \underset{\sim}{g}(t,\tau)}{\partial \tau^i}\Big|_{\tau=t}\right)^{(k-i)} \tag{IV-5}$$

Here $\binom{k}{i} = \frac{k!}{(k-i)!i!}$.

Letting $k = q$ shows immediately that the derivatives with respect to τ, in (IV-1), can be replaced by derivatives with respect to t; that is, $r_c(\underset{\sim}{G}) = -q$ if and only if, for all $t\epsilon[c,\infty)$,

$$\det\left[\frac{\partial^q \underset{\sim}{g}(t,\tau)}{\partial t^q}\Big|_{\tau=t}\right] \neq 0;\ \infty > q \geq 0 \tag{IV-6a}$$

$$\frac{\partial^i \underset{\sim}{g}(t,\tau)}{\partial t^i}\Big|_{\tau=t} = \underset{\sim}{0};\quad i = 0, 1, \ldots, q-1 \tag{IV-6b}$$

At this point we have all background tools for the inversion of Doležal operators. Because the order of $\underset{\sim}{G}$ may not exist over some intervals, different inversion results obtain depending upon the interval under consideration. But if $\underset{\sim}{G}$ has finite order over $[c,\infty)$ then we are most interested in $\underset{\sim}{\psi}\in\underset{\sim}{\mathcal{D}}'[c,\infty)$ for $[\underset{\sim}{G}\cdot\underset{\sim}{\psi}]$.

V. <u>Inversion</u>

If over $[c,\infty)$ an operator $\underset{\sim}{A}[\ ]$ defined by

$$\underset{\sim}{A}[\psi] = [\underset{\sim}{G}\cdot\underset{\sim}{\psi}]^{(k)} \quad \text{(III-8)}$$

has finite order, that is $r_c(\underset{\sim}{A}[\ ])$ defined and finite, then, for $\underset{\sim}{\psi}\epsilon\underset{\sim}{\mathcal{D}}'[c,\infty)$, $\underset{\sim}{A}[\ ]$ can be inverted by the process to now be given. That is, for $\underset{\sim}{\psi}\epsilon\underset{\sim}{\mathcal{D}}'[c,\infty)$ we will find $\underset{\sim}{A}^{-1}[\ ]$ such that

$$\underset{\sim}{A}^{-1}[\underset{\sim}{A}[\underset{\sim}{\psi}]] = \underset{\sim}{A}[\underset{\sim}{A}^{-1}[\underset{\sim}{\psi}]] = \underset{\sim}{\psi} \quad \text{(V-1)}$$

Let $r_c(\underset{\sim}{G}) = -q$, $\infty > q \geq 0$, then for all $t\epsilon[c,\infty)$ by (III-10) and (IV-6a)

$$[\underset{\sim}{G}\cdot\underset{\sim}{\psi}]^{(q+1)} = (\frac{\partial^q \underset{\sim}{g}(t,\tau)}{\partial t^q}\Big|_{\tau=t})\underset{\sim}{\psi}(t) + [\frac{\partial^{q+1}\underset{\sim}{g}(t,\tau)}{\partial t^{q+1}}u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)] \quad \text{(V-2)}$$

Since the first coefficient on the left is never singular for any $t\epsilon[c,\infty)$ we can invert it. Thus define

$$\underset{\sim}{a}(t) = \frac{\partial^q \underset{\sim}{g}(t,\tau)}{\partial t^q}\Big|_{\tau=t} \quad \text{(V-3a)}$$

$$\bar{\underset{\sim}{g}}(t,\tau) = \underset{\sim}{a}^{-1}(t)\,\frac{\partial^{q+1}\underset{\sim}{g}(t,\tau)}{\partial t^{q+1}} \quad \text{(V-3b)}$$

$$\bar{\underset{\sim}{G}}(t,\tau) = \bar{\underset{\sim}{g}}(t,\tau)u(t-\tau) \quad \text{(V-3c)}$$

We note that, since $\underset{\sim}{a}(t)$ may have a zero determinant for $t < c$, $\bar{\underset{\sim}{G}}$ may not lie in $\underset{\sim}{\mathcal{E}}_{t>\tau}$. To cure this and guarantee $\bar{\underset{\sim}{G}}\epsilon\underset{\sim}{\mathcal{E}}_{t>\tau}$, we can take $\underset{\sim}{a}(t)$ for $t < c$ to be any nonsingular infinitely differentiable extension of $\underset{\sim}{a}(t)$ for $t > c$. This will be done where necessary, but since $\underset{\sim}{\psi}\epsilon\underset{\sim}{\mathcal{D}}'[c,\infty)$ this extension will usually be immaterial, as we are then only concerned with $t\epsilon[c,\infty)$. With the definition of (V-3), (III-8) becomes

$$\boxed{\underset{\sim}{A}[\psi] = (\underset{\sim}{a}(t)(\underset{\sim}{\psi}(t) + [\bar{\underset{\sim}{G}}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)]))^{(k-1-q)}} \quad \text{(V-4)}$$

The idea of decomposing $\underset{\sim}{A}$ in this manner is the most important one of the theory. This operator can be inverted in two steps with the use of some results on integral equations. First we write (V-4) as the product of two operators

$$\underset{\sim}{A}_1[\underset{\sim}{\zeta}] = \underset{\sim}{\zeta}^{(k-1-q)} \tag{V-5a}$$

$$\underset{\sim}{A}_2[\underset{\sim}{\xi}] = \underset{\sim}{a}(t)(\underset{\sim}{\xi}(t) + [\bar{\underset{\sim}{G}}(t,\tau)\cdot\underset{\sim}{\xi}(\tau)]) \tag{V-5b}$$

such that (V-4) becomes

$$\underset{\sim}{A}[\underset{\sim}{\psi}] = \underset{\sim}{A}_1[\underset{\sim}{A}_2[\underset{\sim}{\psi}]] = \underset{\sim}{A}_1\cdot\underset{\sim}{A}_2[\underset{\sim}{\psi}] \tag{V-5c}$$

which serves to define the product notation, $\underset{\sim}{A}_1\cdot A_2[\ ]$. $\underset{\sim}{A}_1[\ ]$ is easily inverted for any (positive or negative) integer k-1-q [using (III-5)]

$$\boxed{\underset{\sim}{A}_1^{-1}[\underset{\sim}{\zeta}] = \underset{\sim}{\zeta}^{(q+1-k)}} \tag{V-6}$$

The inverse for $\underset{\sim}{A}_2[\ ]$ is found by noting that (V-5b) defines an integral operator. Thus we find an $\underset{\sim}{H}(t,\tau)\in\underset{\sim}{\mathcal{S}}_{t>\tau}$ such that [$\underset{\sim}{0}$ is of course the n×n zero matrix]

$$\underset{\sim}{H} + \bar{\underset{\sim}{G}} + \underset{\sim}{H}\circ\bar{\underset{\sim}{G}} = \underset{\sim}{H} + \bar{\underset{\sim}{G}} + \bar{\underset{\sim}{G}}\circ\underset{\sim}{H} = \underset{\sim}{0} \tag{V-7a}$$

for all $t\in[c,\infty)$; an $\underset{\sim}{H}$ can always be found from, [11, p. 45],

$$\underset{\sim}{H} = \sum_{i=1}^{\infty}(-1)^i(\bar{\underset{\sim}{G}})^i \tag{V-7b}$$

where $(\bar{\underset{\sim}{G}})^i$ represents the ith Volterra composition, formed by repeated Volterra composition from (III-2). $\underset{\sim}{A}_2^{-1}[\ ]$ can then be given as

$$\boxed{\underset{\sim}{A}_2^{-1}[\underset{\sim}{\xi}] = \underset{\sim}{a}^{-1}(t)\underset{\sim}{\xi}(t) + [\underset{\sim}{a}^{-1}(t)\underset{\sim}{H}(t,\tau)\cdot\underset{\sim}{\xi}(\tau)]} \tag{V-8}$$

This is seen to be the inverse of $\underset{\sim}{A}_2[\ ]$ by the following reasoning. We have

$$\underset{\sim}{A}_2^{-1}[\underset{\sim}{A}_2[\underset{\sim}{\xi}]] = \underset{\sim}{a}^{-1}(t)\{\underset{\sim}{a}(t)(\underset{\sim}{\xi}(t) + [\bar{\underset{\sim}{G}}(t,\tau)\cdot\underset{\sim}{\xi}(\tau)])\} + \qquad \text{(V-9a)}$$

$$[\underset{\sim}{a}^{-1}(\tau)\underset{\sim}{H}(t,\tau)\cdot\{\underset{\sim}{a}(\tau)(\underset{\sim}{\xi}(\tau) + [\bar{\underset{\sim}{G}}(\tau,\eta)\cdot\underset{\sim}{\xi}(\eta)]\}]$$

$$= \underset{\sim}{\xi}(t) + [\bar{\underset{\sim}{G}}(t,\tau)\cdot\underset{\sim}{\xi}(\tau)] + [\underset{\sim}{H}(t,\tau)\cdot\underset{\sim}{\xi}(\tau)] + [\underset{\sim}{H}(t,\tau)\cdot[\bar{\underset{\sim}{G}}(\tau,\eta)\cdot\underset{\sim}{\xi}(\eta)]] \qquad \text{(V-9b)}$$

Now the last term on the right is simply $[(\underset{\sim}{H}o\bar{\underset{\sim}{G}})\cdot\underset{\sim}{\xi}]$, which is easily seen for $\underset{\sim}{\xi}\in\underset{\sim}{\mathcal{D}}$ by using the integral of (II-11) (for other $\underset{\sim}{\xi}$, we use $< [\underset{\sim}{T}(t,\tau)\cdot\underset{\sim}{\xi}(\tau)], \underset{\sim}{\varphi}_1(t) > = < \underset{\sim}{\xi}(\tau), [\tilde{\underset{\sim}{T}}(t,\tau)\cdot\underset{\sim}{\varphi}_1(t)] >$). Using this fact, combining all square brackets in (V-9b) and applying (V-7a) gives $\underset{\sim}{A}_2^{-1}[\underset{\sim}{A}_2[\underset{\sim}{\xi}]] = \underset{\sim}{\xi}$. Similarly one determines that $\underset{\sim}{A}_2\cdot\underset{\sim}{A}_2^{-1}[\ ]$ is also the identity operator.

Consequently, if $\underset{\sim}{A}[\ ]$ has $r_c(\underset{\sim}{A}[\ ]) = k-q$ finite, $\underset{\sim}{A}^{-1}[\ ]$ is defined for every $\underset{\sim}{\psi}\in\underset{\sim}{\mathcal{D}}'[c,\infty)$ by

$$\boxed{\underset{\sim}{A}^{-1}[\underset{\sim}{\psi}] = \underset{\sim}{a}^{-1}(t)\{\underset{\sim}{\psi}\}^{(q+1-k)} + [\underset{\sim}{a}^{-1}(t)\underset{\sim}{H}(t,\tau)\cdot\{\underset{\sim}{\psi}\}^{(q+1-k)}]} \qquad \text{(V-10)}$$

Equation (V-10) is the main result of the theory. It gives an explicit means of finding the inverse of a Doležal operator and hence a means of finding inverse kernels $\underset{\sim}{T}^{(-1)}$ to given kernels $\underset{\sim}{T}$, $\underset{\sim}{T}^{(-1)}o\underset{\sim}{T} = \delta\underset{\sim}{1}_n$, as needed, for example, for the synthesis of time-variable networks [9]. As a very simple example, let

$$G(t,\tau) = e^{\tau}u(t-\tau)$$

then on any interval $[c,\infty)$ this has order zero, i.e., $r_c(G) = 0$. If

$$A[\psi] = [G\cdot\psi]^{(2)} = \frac{d^2\{\int_{-\infty}^{\infty} e^{\tau}u(t-\tau)\psi(\tau)d\tau\}}{dt^2}$$

then we can write, by (V-3),

$$a(t) = g(t,t) = e^t$$

$$\bar{g}(t,\tau) = e^{-t}\frac{\partial g(t,\tau)}{\partial t} = 0$$

which gives $\bar{G} = 0$ and thus by (V-7b), $H = 0$. Consequently, by (V-4),

$$A[\psi] = [e^{\tau}u(t-\tau)\cdot\psi(\tau)]^{(2)}, \; k = 2$$

$$= \{e^{t}\psi(t)\}^{(1)}, \; k-1-q = 1$$

$$= e^{t}\psi^{(1)}(t) + e^{t}\psi(t)$$

which is intuitively checked by differentiating the unit step function under the (symbolic) integral used to define $A[\;]$. By (V-5)

$$A_1[\zeta] = \zeta^{(1)}$$

$$A_2[\xi] = e^{t}\xi$$

By (V-6) and (V-8)

$$A_1^{-1}[\zeta] = \zeta^{(-1)}$$

$$A_2^{-1}[\zeta] = e^{-t}\xi$$

which gives, equation (V-10) with (III-5a) and $H = 0$,

$$A^{-1}[\psi] = e^{-t}\psi^{(-1)}(t) = e^{-t}[u(t-\tau)\cdot\psi(\tau)]$$

$$= e^{-t}\int_{-\infty}^{t} \psi(\tau)d\tau$$

The most important use for the presented theory seems to lie in the area of finding distributional kernels (impulse responses) for physical systems described by integr-differential input-output relationships. These are next considered.

VI. Applications to Time-Variable Integro-Differential Equations

The previous ideas can be applied to systems described by linear equations of the form*

$$\underset{\sim}{A}[\underset{\sim}{y}] = \underset{\sim}{a}_n \underset{\sim}{y}^{(n)} + \underset{\sim}{a}_{n-1} \underset{\sim}{y}^{(n-1)} + \ldots + \underset{\sim}{a}_o \underset{\sim}{y} + [\underset{\sim}{G}_1 \cdot \underset{\sim}{y}] \qquad \text{(VI-1)}$$

$$= \underset{\sim}{B}[\underset{\sim}{x}] = \underset{\sim}{b}_m \underset{\sim}{x}^{(m)} + \underset{\sim}{b}_{m-1} \underset{\sim}{x}^{(m-1)} + \ldots + \underset{\sim}{b}_o \underset{\sim}{x} + [\underset{\sim}{G}_2\ \underset{\sim}{x}]$$

where the $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_i$ are $n \times n$ matrices which vary with time; $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are input and output n-vectors. This application is made by rewriting $\underset{\sim}{A}[\ ]$ as

$$\underset{\sim}{A}[\underset{\sim}{y}] = [\underset{\sim}{G}_y \cdot \underset{\sim}{y}]^{(n+1)} = \underset{\sim}{a}_n \underset{\sim}{y}^{(n)} + \underset{\sim}{a}_{n-1} \underset{\sim}{y}^{(n-1)} + \ldots + \underset{\sim}{a}_o \underset{\sim}{y} + [\underset{\sim}{G}_1 \cdot \underset{\sim}{y}] \qquad \text{(VI-2)}$$

through the use of (III-9). For this one uses the readily established fact that, [2, p. 211],

$$\underset{\sim}{a}_j(t) \underset{\sim}{y}^{(j)}(t) = \sum_{i=0}^{j} (-1)^i (\underset{\sim}{a}_j^{(i)} \underset{\sim}{y})^{(j-1)} \qquad \text{(VI-3)}$$

to rewrite (VI-2) as

$$\underset{\sim}{A}[\underset{\sim}{y}] = \sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^i (\underset{\sim}{a}_j^{(i)} \underset{\sim}{y})^{(j-i)} + [\underset{\sim}{G}_1 \cdot \underset{\sim}{y}] \qquad \text{(VI-4a)}$$

$$= \sum_{k=0}^{n} (\underset{\sim}{c}_k \underset{\sim}{y})^{(k)} + [\underset{\sim}{G}_1 \cdot \underset{\sim}{y}] \qquad \text{(VI-4b)}$$

which serves to define the $\underset{\sim}{c}_k$, for instance

$$\underset{\sim}{c}_n = \underset{\sim}{a}_n$$

$$\underset{\sim}{c}_{n-1} = -\underset{\sim}{a}_n^{(1)} + \underset{\sim}{a}_{n-1}$$

----

*Here we use m and n as the highest orders of differentiation. No confusion with their previous uses as the number of variables or size of matrices should result since these latter are never mentioned beyond the first paragraph of this section.

and in fact for any k

$$\underset{\sim}{c}_k = \sum_{\ell=k}^{n} (-1)^{\ell-k} \underset{\sim}{a}_\ell^{(\ell-k)} \qquad \text{(VI-4c)}$$

Using (III-9c), which for reference is

$$\underset{\sim}{\psi}^{(k)} = [u(t-\tau)\underset{\sim}{1}_n \cdot \underset{\sim}{\psi}(\tau)]^{(k+1)} \qquad \text{(III-9c)}$$

(VI-4b) becomes

$$\underset{\sim}{A}[\underset{\sim}{y}] = \sum_{k=0}^{n} [u(t-\tau)\underset{\sim}{1}_n \cdot \underset{\sim}{c}_k(\tau)\underset{\sim}{y}(\tau)]^{(k+1)} + [\underset{\sim}{G}_1 \cdot \underset{\sim}{y}] \qquad \text{(VI-5a)}$$

Using (III-5a) we can then write, since $uu = u$,

$$\underset{\sim}{A}[\underset{\sim}{y}] = \sum_{k=0}^{n} [\underset{\sim}{c}_k(\tau)\underset{\sim}{U}_{k-n-1}(t,\tau) \cdot \underset{\sim}{y}(\tau)]^{(n+1)} + [\underset{\sim}{U}_{-n-1}(t,\xi) \circ \underset{\sim}{G}_1(\underset{\sim}{\xi},\tau) \cdot y(\tau)]^{(n+1)} \qquad \text{(VI-5b)}$$

$$= [\underset{\sim}{G}_y \cdot \underset{\sim}{y}]^{(n+1)}$$

Clearly, by comparing (VI-2) with (VI-5b),

$$\underset{\sim}{G}_y(t,\tau) = \sum_{k=0}^{n} \underset{\sim}{c}_k(\tau)\underset{\sim}{U}_{k-n-1}(t,\tau) + \underset{\sim}{U}_{-n-1}(t,\xi) \circ \underset{\sim}{G}_1(\xi,\tau) \qquad \text{(VI-6)}$$

Intuitively the above process amounts to realizing that derivative operators are equivalent to impulsive integral operators. But impulses are themselves derivatives of unit step functions. Consequently, we converted to impulsive operators, expressed as derivatives of unit step functions, at (VI-5a). The n+1 times integration through $\underset{\sim}{U}_{-n-1}$ is used at (VI-5b) to combine terms into the derivative of one integral operator, $\underset{\sim}{G}_y$. Since

$$\underset{\sim}{G}_y(t,\tau) = \underset{\sim}{a}_n(\tau)u(t-\tau) + \sum_{k=0}^{n-1} \underset{\sim}{c}_k(\tau)\frac{(t-\tau)^{n-k}}{\Gamma(n-k+1)}u(t-\tau) + \underset{\sim}{U}_{-n-1}(t_1,\xi) \circ \underset{\sim}{G}_1(\xi,\tau)$$

it is seen that $\underset{\sim}{G}_y(t,\tau)$ has order -1 if $\underset{\sim}{a}_n(\tau)$ is nonsingular for all τ. Consequently, from (VI-4) and (VI-2) we see that

$$r_c(\underset{\sim}{A}[\ ]) = n \quad \text{if} \quad \det a_n(t) \neq 0 \quad \text{for all} \quad t \in [c,\infty) \qquad \text{(VI-7)}$$

Therefore, in this order n case, (VI-1) can be solved by using the inversion of section V.

The operator $\underset{\sim}{B}[\ ]$ can be converted to the same form i.e., $\underset{\sim}{B}[\underset{\sim}{x}] = [\underset{\sim}{G}_x \cdot x]^{(m+1)}$ and then (VI-1) is

$$\underset{\sim}{A}[\underset{\sim}{y}] = [\underset{\sim}{G}_y \cdot \underset{\sim}{y}]^{(n+1)} = \underset{\sim}{B}[\underset{\sim}{x}] = [\underset{\sim}{G}_x \cdot \underset{\sim}{x}]^{(m+1)} \qquad \text{(VI-8)}$$

giving

$$\underset{\sim}{y} = \underset{\sim}{A}^{-1}[[\underset{\sim}{G}_x \cdot \underset{\sim}{x}]^{(m+1)}] = \underset{\sim}{A}^{-1}[B[\underset{\sim}{x}]] \qquad \text{(VI-9)}$$

Unfortunately the computations for obtaining $\underset{\sim}{A}^{-1}$ are usually prohibitive, since exact evaluation of $\underset{\sim}{H}$ of (V-7a) is determined by using an infinite number of integrations and summations. However, it appears that the method should be very suitable for finding approximate solutions by means of a computer.

As a simple example to illustrate the difficulties, consider the voltage transucer network of Fig. 1. Then

$$\frac{r_1 dcv_2}{dt} + (1+\frac{r_1}{r_2})v_2 = v_1$$

or, with $v_1 = x$ and $v_2 = y$,

$$(e^t y)^{(1)} + y = \frac{1}{2}x$$

giving

$$\begin{aligned} A[y] &= (e^t y(t))^{(1)} + y(t) \\ &= [e^\tau U_{-1}(t,\tau)\cdot y(\tau)]^{(2)} + [U_{-2}(t,\tau)\cdot y(\tau)]^{(2)} \\ &= [\{e^\tau U_{-1}(t,\tau) + U_{-2}(t,\tau)\}\cdot y(\tau)]^{(2)} \end{aligned}$$

Therefore, with $k = 2$ and $A[y] = [G_y \cdot y]^{(2)}$,

$$G_y(t,\tau) = e^{\tau}u(t-\tau) + (t-\tau)u(t-\tau)$$

$$= \{e^{\tau}+(t-\tau)\}u(t-\tau)$$

giving

$$g_y(t,\tau) = e^{\tau}+(t-\tau); \frac{\partial g_y(t,\tau)}{\partial t} = 1$$

Thus

$$g_y(t,t) = e^{t}$$

and $q = 0$ or $r_c(G_y) = -1$ for any finite c.

By equation (V-2) we write

$$[G_y \cdot y]^{(0+1)} = e^{t}y+[u(t-\tau)\cdot y(\tau)]$$

giving, by (V-3)

$$a(t) = e^{+t}$$

$$\bar{g}(t,\tau) = e^{-t}; \bar{G}(t,\tau) = e^{-t}u(t-\tau)$$

and

$$A[y] = \{e^{+t}(y(t) + [e^{-t}u(t-\tau)\cdot y(\tau)])\}^{(2-1-0)}$$

$$= \{e^{+t}(y(t) + [e^{-t}u(t-\tau)\cdot y(\tau)])\}^{(1)}$$

Then

$$A_1[\zeta] = \zeta^{(1)}$$

$$A_2[\xi] = e^{+t}(\xi(t) + [e^{-t}u(t-\tau)\cdot x(\tau)])$$

$$A_1^{-1}[\zeta] = \zeta^{(-1)} = [U_{-1}(t,\tau)\cdot\zeta] = [u(t-\tau)\cdot\zeta(\tau)] = \int_{-\infty}^{t}\zeta(\tau)d\tau$$

To find H for $A_2^{-1}[\ ]$ we must form

$$H(t,\tau) = -\bar{G}(t,\tau) + \bar{G}o\bar{G} - \bar{G}o\bar{G}o\bar{G} + \ldots$$

But, on recalling $\bar{G}(t,\tau) = e^{-t}u(t-\tau)$

$$\bar{G}o\bar{G}(t,\tau) = (\int_{\tau}^{t} e^{-t}e^{-\xi}d\xi)u(t-\tau)$$

$$= e^{-t}[e^{-\tau}-e^{-t}]u(t-\tau)$$

$$= \{e^{-(t+\tau)}-e^{-2t}\}u(t-\tau)$$

$$\bar{G}o\bar{G}o\bar{G} = (\int_{\tau}^{t} e^{-t}[e^{-(\xi+\tau)}-e^{-2\xi}]d\xi)u(t-\tau)$$

$$= e^{-t}[e^{-\tau}\{e^{-\tau}-e^{-t}\}+ \frac{1}{2}\{e^{-2t}-e^{-2\tau}\}]u(t-\tau)$$

$$= \{e^{-(t+2\tau)}-e^{-(\tau+2t)}+ \frac{1}{2}e^{-3t}-\frac{1}{2}e^{-(t+2\tau)}\}u(t-\tau)$$

etc., i.e.,

$$H(t,\tau) = \{-e^{-t}+e^{-(t+\tau)}-e^{-2t}+ \frac{1}{2}e^{-(t+2\tau)}-e^{-(\tau+2t)}+ \frac{1}{2}e^{-3t}+\ldots\}u(t-\tau)$$

$$= h(t,\tau)u(t-\tau)$$

which in general is quite difficult to explicitely evaluate. Then, by (V-8) $A_2^{-1}[\xi] = e^{-t}\xi(t) + [e^{-t}H(t,\tau)\cdot\xi(t)]$ by (V-10), we finally then arrive at

$$A^{-1}[\frac{x}{2}] = e^{-t}\,\frac{x}{2}^{(-1)} + [e^{-t}H(t,\tau)\cdot[\frac{x}{2}]^{(-1)}]$$

$$= e^{-t}\int_{-\infty}^{t}\frac{x}{2}(\tau)d\tau + \int_{-\infty}^{t} e^{-t}h(t,\tau)\{\int_{-\infty}^{\tau}\frac{x}{2}(\sigma)d\sigma\}d\tau$$

VII. Summary of the Concepts of Doležal Applied to Systems Equations

Given a system of integro-differential equations

$$\underset{\sim}{A}[\underset{\sim}{y}] = \underset{\sim}{a}_n \underset{\sim}{y}^{(n)} + \ldots + \underset{\sim}{a}_o \underset{\sim}{y} + \int_{-\infty}^{t} \underset{\sim}{g}_1(t,\tau)\underset{\sim}{y}(\tau)d\tau \qquad \text{(VI-1)}$$

$$= \underset{\sim}{b}_m \underset{\sim}{y}^{(m)} + \ldots + \underset{\sim}{b}_o \underset{\sim}{x} + \int_{-\infty}^{t} \underset{\sim}{g}_2(t,\tau)\underset{\sim}{x}(\tau)d\tau = \underset{\sim}{B}[\underset{\sim}{x}]$$

where $\underset{\sim}{x}$ is a known input vector and the $\underset{\sim}{a}$'s and $\underset{\sim}{b}$'s are time varying matrices, the method of this paper solves this system by an operational means. Since $\underset{\sim}{y}$, the output vector, is the unknown, we really are only interested in inverting the operator

$$\underset{\sim}{A}[\underset{\sim}{y}] = \underset{\sim}{a}_n \underset{\sim}{y}^{(n)} + \ldots + \underset{\sim}{a}_o \underset{\sim}{y} + \int_{-\infty}^{t} \underset{\sim}{g}_1(t,\tau)\underset{\sim}{y}(\tau)d\tau \qquad \text{(VI-2)}$$

By converting the derivatives to impulse terms $\underset{\sim}{A}[\;]$ is converted to one integral operator (precisely in eqs. (VI-2) through (VI-6)). By $(n+1)$ integrations the impulses are turned into unit step functions in a new "kernel" $\underset{\sim}{G}_y(t,\tau) = \underset{\sim}{g}_y(t,\tau)u(t-\tau)$; one then differentiates $(n+1)$ times to cancel out the integrations. The result is the representation

$$\underset{\sim}{A}[\underset{\sim}{y}] = \{\int_{-\infty}^{t} \underset{\sim}{g}_y(t,\tau)\underset{\sim}{y}(\tau)d\tau\}^{(n+1)} = [\underset{\sim}{G}_y \cdot \underset{\sim}{y}]^{(n+1)} \qquad \text{(VI-5b)}$$

If the leading coefficient matrix, $\underset{\sim}{a}_n(t)$, is never singular in an interval of interest, $c < t\ \infty$ for some c, then $\underset{\sim}{A}[\underset{\sim}{y}]$ is a special case of an operator of finite order, in fact of order n. It can then be inverted, at least for those (distributional) $\underset{\sim}{y}$ which are zero for $t < c$. For the inversion one uses the theory of section V, now applied to this special case. One writes

$$A[y] = [G_y \cdot y]^{(n+1)} \qquad \text{(VI-5b)}$$

with

$$[G_y \cdot y]^{(1)} = a_n(t)y(t) + \int_{-\infty}^{t} \frac{\partial g_y(t,\tau)}{\partial t} y(\tau)d\tau \qquad \text{(V-2)}$$

$$= a_n(t)\{y(t) + \int_{-\infty}^{t} a_n^{-1}(t)\frac{\partial g_y(t,\tau)}{\partial t} y(\tau)d\tau\}$$

Writing

$$a(t) = a_n(t) \qquad \text{(V-3a)}$$

$$\bar{g}_y(t,\tau) = a_n^{-1}(t)\frac{\partial g_y(t,\tau)}{\partial t} \qquad \text{(V-3b)}$$

this expansion takes the explicit form

$$[G_y \cdot y]^{(1)} = a(t)\{y(t) + \int_{-\infty}^{t} \bar{g}_y(t,\tau)y(\tau)d\tau\} \qquad \text{(V-5b)}$$

The inverse of $A[\ ]$, $A^{-1}[\ ]$, is found by inverting this last equation and then integrating n times. The result is

$$A^{-1}[\psi] = a^{-1}(t)\{\psi\}^{(n)} + \int_{-\infty}^{t} h_y(t,\tau)\{\psi(\tau)\}^{(n)}d\tau \qquad \text{(V-10)}$$

where, by a series of Volterra compositions,

$$h_y(t,\tau) = -\bar{g}_y(t,\tau) + \int_{\tau}^{t} \bar{g}_y(t,\xi)\bar{g}_y(\xi,\tau)d\xi - \int_{\tau}^{t}\int_{\tau}^{\xi} \bar{g}_y(t,\xi)\bar{g}_y(\xi,\sigma)\bar{g}_y(\sigma,\tau)d\sigma d\xi$$

$$+ \ldots \qquad \text{(V-7b)}$$

Applying this inverse to (VI-1) gives the output in terms of the input

$$y = A^{-1}[B[x]]$$

The philosophy is to convert to an integral operator with a functional kernel, (V-2). This kernel is inverted using the integral equation theory of Volterra, which justifies (V-7b). However, one has to use care since impulses are heavily relied upon. Consequently one must employ the theory of distributions and proceed with equations and quantities whose properties can be precisely determined. Calculationally the theory is difficult because the Volterra series of compositions of equations (V-7b) must be used. But it appears that some uses could occur in computer analysis and in synthesis, the latter particularly if factorizations of operators could be given.

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Expression for $[\underset{\sim}{G}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)]^{(k)}$

We first show that, where $\ell = k-1$,

$$[\underset{\sim}{G}(t,\tau)\cdot\underset{\sim}{\psi}(\tau)]^{(k)} = [\frac{\partial^k g(t,\tau)}{\partial t^k}u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)] +$$

$$\sum_{j=0}^{\ell}\sum_{m=0}^{\ell-j}(-1)^{\ell-j-m}\binom{\ell+1}{j}\binom{\ell-j}{m}\frac{\partial^{\ell-m} g(t,\tau)}{\partial t^{\ell-m}}\Big|_{\tau=t}\underset{\sim}{\psi}^{(m)}(t)$$

Demonstration: We give the details for scalar matrix quantities.

$$< [G(t,\tau)\cdot\psi(\tau)]^{(k)},\ \varphi_1(t) > \ = \ < [G(t,\tau)\cdot\psi(\tau)],\ (-1)^k\varphi_1^{(k)}(t) >$$

$$= \ < G(t,\tau),\ (-1)^k\varphi_1^{(k)}(t)\psi(\tau) > \ = \ < \frac{\partial^k G(t,\tau)}{\partial t^k},\ \varphi_1(t)\psi(\tau) >$$

$$= \ < \frac{\partial^k g(t,\tau)u(t-\tau)}{\partial t^k},\ \varphi_1(t)\psi(\tau) >$$

$$= \ < \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j g(t,\tau)}{\partial t^j}u^{(k-j)}(t-\tau),\ \varphi_1(t)\psi(\tau) >$$

$$= \ < \frac{\partial^k g(t,\tau)}{\partial t^k}u(t-\tau),\ \varphi_1(t)\psi(\tau) >$$

$$+ \sum_{j=0}^{k-1}\binom{k}{j} < \frac{\partial^j g(t,\tau)}{\partial t^j}u^{(k-j)}(t-\tau),\ \varphi_1(t)\psi(\tau) >$$

$$= \ < [\frac{\partial^k g(t,\tau)}{\partial t^k}u(t-\tau)\cdot\psi(\tau)],\ \varphi_1(t) >$$

$$+ \sum_{j=0}^{\ell}\binom{\ell+1}{j} < \frac{\partial^j g(t,\tau)}{\partial t^j}\delta^{(\ell-j)}(t-\tau),\ \varphi_1(t)\psi(\tau) >; \quad \ell = k-1$$

The scalar product terms of the summation are

$$< \frac{\partial^j g(t,\tau)}{\partial t^j} \delta^{(\ell-j)}(t-\tau),\ \varphi_1(t)\psi(\tau) > =$$

$$< \delta^{(\ell-j)}(t-\tau),\ \frac{\partial^j g(t,\tau)}{\partial t^j}\varphi_1(t)\psi(\tau) > =$$

$$< \delta(t-\tau),\ (-1)^{\ell-j} \frac{\partial^{\ell-j}\{\frac{\partial^j g(t,\tau)}{\partial t^j}\varphi_1(t)\psi(\tau)\}}{\partial t^{\ell-j}} > =$$

$$= < \delta(t-\tau),\ (-1)^{\ell-j} \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} \varphi_1^{(m)}(t) \frac{\partial^{\ell-m} g(t,\tau)}{\partial t^{\ell-m}}\psi(\tau) >$$

$$= < 1,\ (-1)^{\ell-j} \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} \varphi_1^{(m)}(t) \frac{\partial^{\ell-m} g(t,\tau)}{\partial t^{\ell-m}}\Big|_{\tau=t} \psi(t) >$$

$$= \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} < \frac{\partial^{\ell-m} g(t,\tau)}{\partial t^{\ell-m}}\Big|_{\tau=t} \psi(t),\ (-1)^{\ell-j}\varphi_1^{(m)}(t) >$$

$$= \sum_{m=0}^{\ell-j} (-1)^{\ell-j-m} \binom{\ell-j}{m} < \left(\frac{\partial^{\ell-m} g(t,\tau)}{\partial t^{\ell-m}}\Big|_{\tau=t} \psi(t)\right)^{(m)},\ \varphi_1(t) >$$

which proves the assertion, when matrices are investigated component by component.

We can now show (III-10) by induction. From the expression just derived, for $k = 1$, we get

$$[\underset{\sim}{G}(t,\tau)\underset{\sim}{\psi}(t)]^{(1)} = [\frac{\partial \underset{\sim}{g}(t,\tau)}{\partial t} u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)] + (\underset{\sim}{g}(t,\tau)\Big|_{\tau=t}\underset{\sim}{\psi}(t))$$

Assume then for any k, equation (III-10)

$$[\underset{\sim}{G}(t,\tau)\underset{\sim}{\psi}(t)]^{(k)} = \sum_{i=0}^{k-1} [(\frac{\partial^i g(t,\tau)}{\partial t^i})|_{\tau=t} \underset{\sim}{\psi}(t)]^{(k-i-1)} + [\frac{\partial^k g(t,\tau)}{\partial t^k} u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)]$$

Differentiating this

$$[\underset{\sim}{G}(t,\tau)\underset{\sim}{\psi}(t)]^{(k+1)} = \sum_{i=0}^{k-1} [\frac{\partial^i g(t,\tau)}{\partial t^i}|_{\tau=t} \underset{\sim}{\psi}(t)]^{(k-i)} + [\frac{\partial^k g(t,\tau)}{\partial t^k} u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)]^{(1)}$$

Applying the above result for the first derivative to the last term on the right gives

$$[\underset{\sim}{G}(t,\tau)\underset{\sim}{\psi}(t)]^{(k+1)} = \sum_{i=0}^{k-1} [\frac{\partial^i g(t,\tau)}{\partial t^i}|_{\tau=t} \underset{\sim}{\psi}(t)]^{(k-i)} +$$

$$[\frac{\partial^{k+1} g(t,\tau)}{\partial t^{k+1}} u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)] + (\frac{\partial^k g(t,\tau)}{\partial t^k}|_{\tau=t} \underset{\sim}{\psi}(t))$$

$$= \sum_{i=0}^{k} [\frac{\partial^i g(t,\tau)}{\partial t^i}|_{\tau=t} \underset{\sim}{\psi}(t)]^{(k-i)} + [\frac{\partial^{k+1} g(t,\tau)}{\partial t^{k+1}} u(t-\tau)\cdot\underset{\sim}{\psi}(\tau)]$$

which is (III-10) with k replaced by $k+1$. This is then true by induction for all k.

Appendix 2

Expression for $[\underset{\sim}{G}\cdot\underset{\sim}{\psi}^{(n+1)}]$

Here we show the validity of equation (VI-3). First we can obtain, with $\underset{\sim}{\psi} = \underset{\sim}{\psi}^{(n+1)}$,

$$[\underset{\sim}{G}\cdot\underset{\sim}{\Psi}^{(n+1)}] = (-1)^{n+1}[\frac{\partial^{n+1}\underset{\sim}{g}(t,\tau)}{\partial\tau^{n+1}}u(t-\tau)\cdot\underset{\sim}{\Psi}(\tau)] +$$

$$\sum_{j=0}^{n}\sum_{m=0}^{n-j}(-1)^{n-m}\binom{n+1}{j}\binom{n-j}{m}\left(\frac{\partial^{n}\underset{\sim}{g}(t,\tau)}{\partial t^{n-j}\partial\tau^{j}}\Big|_{\tau=1}\underset{\sim}{\Psi}(t)\right)^{(m)}$$

Demonstration: Again we give the details for the scalar matrix case.

$$< [G\cdot\psi], \varphi_1 > = < G(t,\tau), \varphi_1(t)\psi(\tau) >$$

$$= < G(t,\tau), \varphi_1(t)\Psi^{(n+1)}(\tau) > = < (-1)^{n+1}\frac{\partial^{n+1}G(t,\tau)}{\partial\tau^{n+1}}, \varphi_1(t)\Psi(\tau) >$$

$$= < (-1)^{n+1}\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\frac{\partial^{j}g(t,\tau)}{\partial\tau^{j}}u^{(n+1-j)}(t-\tau), \varphi_1(t)\Psi(\tau) >$$

$$= < (-1)^{n+1}\frac{\partial^{n+1}g(t,\tau)}{\partial\tau^{n+1}}u(t-\tau), \varphi_1(t)\Psi(\tau) >$$

$$+ \sum_{j=0}^{n}(-1)^{j}\binom{n+1}{j} < \frac{\partial^{j}g(t,\tau)}{\partial\tau^{j}}u^{(n+1-j)}(t-\tau), \varphi_1(t)\Psi(\tau) >$$

$$= < (-1)^{n+1}[\frac{\partial^{n+1}g(t,\tau)}{\partial\tau^{n+1}}u(t-\tau)\cdot\Psi(\tau)], \varphi_1(t) >$$

$$+ \sum_{j=0}^{n}(-1)^{j}\binom{n+1}{j} < \frac{\partial^{j}g(t,\tau)}{\partial\tau^{j}}\delta^{(n-j)}(t-\tau), \varphi_1(t)\Psi(\tau) >$$

The scalar product terms of the summation are

$$< \frac{\partial^j g(t,\tau)}{\partial\tau^j}\delta^{(n-j)}(t-\tau),\ \varphi_1(t)\Psi(\tau) > =$$

$$< \delta^{(n-j)}(t-\tau),\ \frac{\partial^j g(t,\tau)}{\partial\tau^j}\varphi_1(t)\Psi(\tau) > =$$

$$< \delta(t-\tau),\ (-1)^{n-j}\ \frac{\partial^{n-j}\{\frac{\partial^j g(t,\tau)}{\partial\tau^j}\varphi_1(t)\Psi(\tau)\}}{\partial t^{n-j}} > =$$

$$< \delta(t-\tau),\ (-1)^{n-j}\sum_{m=0}^{n-j}\binom{n-j}{m}\frac{\partial^n g(t,\tau)}{\partial t^{n-j}\partial\tau^j}\varphi_1^{(m)}(t)\Psi(\tau) > =$$

$$< 1,\ (-1)^{n-j}\sum_{m=0}^{n-j}\binom{n-j}{m}\frac{\partial^n g(t,\tau)}{\partial t^{n-j}\partial\tau^j}\Big|_{\tau=t}\Psi(t)\varphi_1^{(m)}(t) > =$$

$$\sum_{m=0}^{n-j}(-1)^{n-j}\binom{n-j}{m} < \frac{\partial^n g(t,\tau)}{\partial t^{n-j}\partial\tau^j}\Big|_{\tau=t}\Psi(t),\ \varphi_1^{(m)}(t) > =$$

$$\sum_{m=0}^{n-j}(-1)^{n-j-m}\binom{n-j}{m} < \left(\frac{\partial^n g(t,\tau)}{\partial t^{n-j}\partial\tau^j}\Big|_{\tau=t}\Psi(t)\right)^{(m)},\ \varphi_1(t) >$$

which proves the desired formula, when we consider matrices component by component.

In this formula let $n = 0$, then

$$[\underset{\sim}{G}\cdot\underset{\sim}{\Psi}^{(1)}] = (-1)[\frac{\partial \underset{\sim}{g}(t,\tau)}{\partial\tau}u(t-\tau)\cdot\underset{\sim}{\Psi}(\tau)] + \underset{\sim}{g}(t,\tau)\underset{\sim}{\Psi}(t)$$

Now assume, for any n that

$$[\underset{\sim}{G}\cdot\underset{\sim}{\Psi}^{(n+1)}] = \sum_{i=0}^{n}(-1)^i\left(\frac{\partial^i \underset{\sim}{g}(t,\tau)}{\partial\tau^i}\Big|_{\tau=t}\right)\underset{\sim}{\Psi}^{(n-i)}(t) + (-1)^{n+1}[\frac{\partial^{n+1}\underset{\sim}{g}(t,\tau)}{\partial\tau^{n+1}}u(t-\tau)\underset{\sim}{\Psi}(\tau)]$$

Then letting $\underset{\sim}{\Psi} = \underset{\sim}{\Phi}^{(1)}$ this gives

$$[\underset{\sim}{G}\cdot\underset{\sim}{\Phi}^{(n+2)}] = \sum_{i=0}^{n} (-1)^i \left(\frac{\partial^i g(t,\tau)}{\partial\tau^i}\Big|_{\tau=t}\right)\underset{\sim}{\Phi}^{(n+1-i)}(t) + (-1)^{n+1}\left[\frac{\partial^{n+1} g(t,\tau)}{\partial\tau^{n+1}} u(t-\tau)\cdot\underset{\sim}{\Phi}^{(1)}(\tau)\right]$$

Applying the above result for $n = 0$ to the very right hand term gives

$$[\underset{\sim}{G}\cdot\underset{\sim}{\Phi}^{(n+2)}] = \sum_{i=0}^{n} (-1)^i \left(\frac{\partial^i g(t,\tau)}{\partial\tau^i}\Big|_{\tau=t}\right)\underset{\sim}{\Phi}^{(n+1-i)}(t)$$

$$+ (-1)^{n+1}\left[\frac{\partial^{n+1} g(t,\tau)}{\partial\tau^{n+1}}\Big|_{\tau=t} \underset{\sim}{\Phi}(t) + (-1)\left[\frac{\partial^{n+2} g(t,\tau)}{\partial\tau^{n+2}} u(t-\tau)\cdot\underset{\sim}{\Phi}(\tau)\right]\right\}$$

$$= \sum_{i=0}^{n+1} (-1)^i \left(\frac{\partial^i g(t,\tau)}{\partial\tau^i}\Big|_{\tau=t}\right)\underset{\sim}{\Phi}^{(n+1-i)} + (-1)^{n+2}\left[\frac{\partial^{n+2} g(t,\tau)}{\partial\tau^{n+2}} u(t-\tau)\cdot\underset{\sim}{\Phi}(\tau)\right]$$

which is the previous formula with n replaced by $n+1$, and, thus by induction the result of equation (IV-3) is proven.

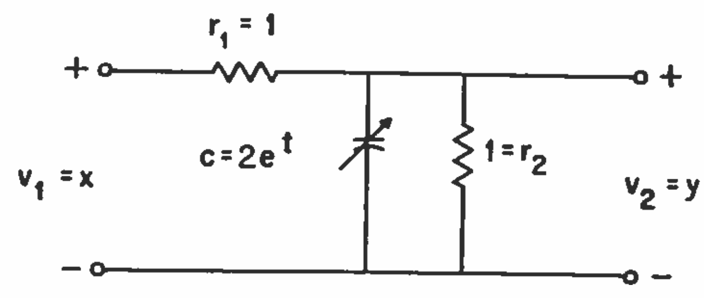

Figure 1. Example Network