

HW 7 Tahmid Sami Rahman

1. unit step response: $g(t) = te^{-3t} 1(t)$
input = $1(t)$.

Taking Laplace transform: $G(s) = \frac{1}{s} H(s)$

$$\Rightarrow \frac{1}{(s+3)^2} = \frac{1}{s} H(s)$$

$$\Rightarrow H(s) = \frac{s}{(s+3)^2} = \frac{s+3-3}{(s+3)^2} = \frac{1}{s+3} - \frac{3}{(s+3)^2}$$

$$\text{So, } \boxed{h(t) = e^{-3t} 1(t) - 3te^{-3t} 1(t)}$$

a) a1: Impulse Response using Laplace transform:

$$\text{Impulse Response, } i(t) = H(s) \cdot 1$$

$$= \frac{s}{(s+3)^2}$$

$$\text{so, } i(t) = e^{-3t} 1(t) - 3te^{-3t} 1(t)$$

a2: Using Convolution: $i(t) = s(t) * h(t)$

$$= \int_{-\infty}^{\infty} s(t-\tau) h(\tau) d\tau$$

$$= h(t)$$

$$= e^{-3t} 1(t) - 3te^{-3t} 1(t)$$

b) b1: causal input $u(t) = 2 \cos 5t 1(t)$

$$U(s) = \frac{2s}{s^2+25} \quad [\text{ROC: } s > 0]$$

$$H(s) = \frac{s}{(s+3)^2}$$

$$\text{So, } Y(s) = U(s)H(s) = \frac{2s^2}{(s^2+25)(s+3)^2}$$

$$\frac{2s^2}{(s^2+25)(s+3)^2} \equiv \frac{As+B}{s^2+25} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$\Rightarrow 2s^2 \equiv (As+B)(s^2+6s+9) + C(s^2+25)(s+3) + D(s^2+25)$$

$$\equiv (A+C)s^3 + (6A+B+3C+D)s^2 + (9A+6B+25C)s + 9B+25C+25D$$

$$C = -A, \quad B = \frac{16}{6}A, \quad 3A + \frac{16}{6}A + D = 2 \Rightarrow 34A + 6D = 12$$

$$24A - 75A + 25D = 0 \Rightarrow D = \frac{51}{25}A \Rightarrow 21A + 25D = 0$$

$$\Rightarrow A = \frac{150}{578}, \quad B = \frac{400}{578}, \quad C = -\frac{150}{578}, \quad D = \frac{306}{578}$$

Common intersection of Region of convergence: $\text{Re}\{s\} > 0$

$$y(t) = \frac{150}{578} \cos 5t \, 1(t) + \frac{80}{578} \sin 5t \, 1(t) - \frac{150}{578} e^{-3t} \, 1(t) + \frac{306}{578} t e^{-3t} \, 1(t).$$

b2: For non-causal input: $u(t) = -2 \cos 5t \, 1(-t)$.

$$U(s) = \frac{2s}{s^2 + 25}$$

Region of convergence: $\text{Re}\{s\} < 0$.

Intersection of ROC: $-3 < \text{Re}\{s\} < 0$.

Hence, -3 would be a causal pole.

$$\text{So, } y(t) = -\frac{150}{578} \cos 5t \, 1(-t) - \frac{80}{578} \sin 5t \, 1(-t) - \frac{150}{578} e^{-3(t+3)} \, 1(t+3) + \frac{306}{578} t e^{-3(t+3)} \, 1(t+3)$$

(2) For coupled coils, with perfect coupling, $k=1$

$$\underline{Z+I} = \begin{bmatrix} sL_{11}+1 & +sL_{12} \\ +sL_{12} & sL_{22}+1 \end{bmatrix}$$

$$\Rightarrow |L_{12}|^2 = L_{11} L_{22}$$

Scattering Matrix, $S = (\underline{Z+I})^{-1} (\underline{Z-I})$

$$= \frac{1}{s^2 L_{11} L_{22} + sL_{11} + sL_{22} + 1 - s^2 L_{12}^2} \begin{bmatrix} sL_{22}+1 & -sL_{12} \\ -sL_{12} & sL_{11}+1 \end{bmatrix} \begin{bmatrix} sL_{11}-1 & +sL_{12} \\ +sL_{12} & sL_{22}-1 \end{bmatrix}$$

$$= \frac{1}{sL_{11} + sL_{22} + 1} \begin{bmatrix} s^2 L_{11} L_{22} + sL_{11} - sL_{22} - 1 & -s^2 L_{12} L_{22} + sL_{12} \\ -s^2 L_{12} L_{22} + sL_{12} & -s^2 L_{12} L_{12} + s^2 L_{11} L_{22} \\ s^2 L_{11} L_{12} + sL_{12} & -s^2 L_{12} L_{12} + s^2 L_{11} L_{22} \\ -s^2 L_{11} L_{12} + sL_{12} & -sL_{11} + sL_{22} - 1 \end{bmatrix}$$

$$= \frac{1}{sL_{11} + sL_{22} + 1} \begin{bmatrix} sL_{11} - sL_{22} - 1 & +2sL_{12} \\ +2sL_{12} & -sL_{11} + sL_{22} - 1 \end{bmatrix}$$

$$\textcircled{3} \text{ a) } Y(s) = \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}$$

1st Foster: $Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+2} + \frac{Ds+E}{s^2+4}$

$$(s^2+1)(s^2+3) \equiv A(s^2+2)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)s(s^2+2)$$

$$\Rightarrow s^4 + 4s^2 + 3 \equiv (A+B+D)s^4 + (C+E)s^3 + (6A+4B+2D)s^2 + (4C+2E)s + 8A$$

$$C=E=0, \quad A = \frac{3}{8}$$

$$A+B+D = 1$$

$$\Rightarrow B+D = \frac{5}{8} \Rightarrow 2B+2D = \frac{5}{4}$$

$$6A+4B+2D = 4$$

$$\Rightarrow \frac{9}{4}A + 4B + 2D = 4$$

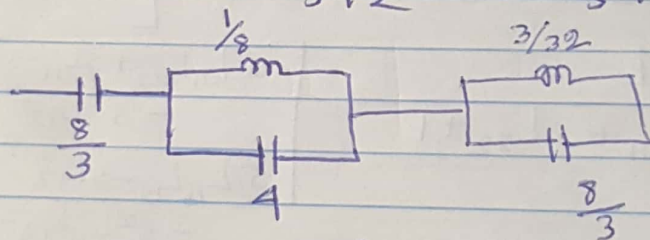
$$\Rightarrow 4B + 2D = \frac{7}{4}$$

$$2B = \frac{2}{4} = \frac{1}{2}$$

$$B = \frac{1}{4}$$

$$D = \frac{5}{8} - \frac{2}{8} = \frac{3}{8}$$

$$Z(s) = \frac{\frac{3}{8}}{s} + \frac{\frac{1}{4}s}{s^2+2} + \frac{\frac{3}{8}s}{s^2+4}$$



2nd Foster: $Y(s) = \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)} \equiv As + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+3}$

$$\Rightarrow s(s^2+2)(s^2+4) \equiv As(s^4+4s^2+3) + (Bs+C)(s^2+3) + (Ds+E)(s^2+1)$$

$$3C+E=0 \Rightarrow E=-3C$$

$$\Rightarrow s^5 + 6s^3 + 8s = As^5 + (4A+B+D)s^3 + (C+E)s^2 + (3A+3B+D)s + (3C+E)$$

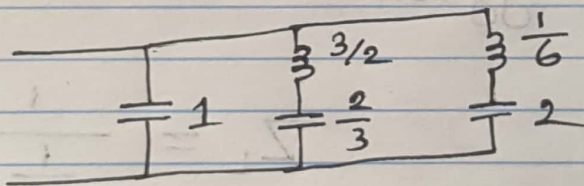
$$A=1, \quad 4A+B+D=6 \Rightarrow B+D=2$$

$$C=E=0$$

$$3B+D=5$$

$$B=\frac{3}{2}, \quad D=\frac{1}{2}$$

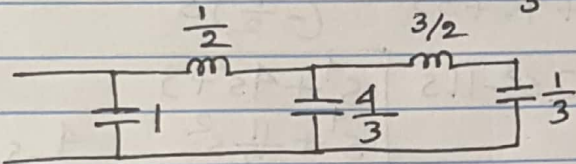
$$Y(s) = s + \frac{\frac{3}{2}s}{s^2+1} + \frac{\frac{1}{2}s}{s^2+3}$$



1st Case: $Y(s) = \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}$

$$\begin{array}{r}
 s \\
 s^4 + 4s^2 + 3 \overline{) s^5 + 6s^3 + 8s} \\
 \underline{s^5 + 4s^3 + 3s} \quad \frac{1}{2}s \\
 2s^3 + 5s \\
 \underline{s^4 + 4s^2 + 3} \quad \frac{4}{3}s \\
 \underline{s^4 + \frac{5}{2}s^2} \\
 \frac{3}{2}s^2 + 3 \\
 \underline{2s^3 + 5s} \quad \frac{3}{2}s \\
 \underline{2s^3 + 4s} \\
 s \\
 \underline{\frac{3}{2}s^2 + 3} \\
 \frac{3}{2}s^2 \\
 \underline{\frac{3}{2}s^2} \\
 3 \quad \frac{5}{3} \\
 \underline{\frac{5}{3}} \\
 0
 \end{array}$$

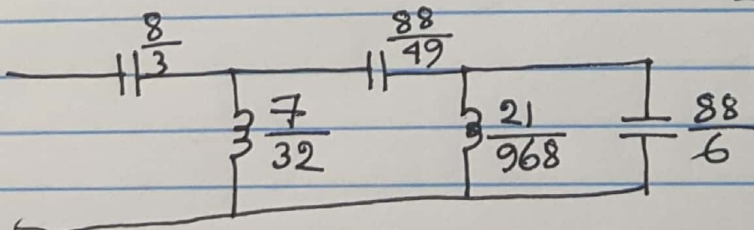
$$Y(s) = s + \frac{1}{2s} + \frac{1}{\frac{4}{3}s} + \frac{1}{\frac{3}{2}s} + \frac{1}{\frac{5}{3}}$$



2nd Case: $Y(s) = \frac{1}{Z(s)} \Rightarrow Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$

$$\begin{array}{r}
 \frac{3}{8s} \\
 8s + 6s^3 + s^5 \overline{) 3 + 4s^2 + s^4} \\
 \underline{3 + \frac{9}{4}s^2 + \frac{3}{8}s^4} \quad \frac{32}{7s} \\
 \frac{7}{4}s^2 + \frac{5}{8}s^4 \\
 \underline{8s + 6s^3 + s^5} \quad \frac{49}{88s} \\
 \underline{8s + \frac{20}{7}s^3} \\
 \frac{22}{7}s^3 + s^5 \\
 \underline{\frac{7}{4}s^2 + \frac{5}{8}s^4} \\
 \frac{7}{4}s^2 + \frac{49}{88}s^4 \quad \frac{968}{21s} \\
 \underline{\frac{6}{88}s^4} \\
 \frac{22}{7}s^3 + s^5 \\
 \underline{\frac{22}{7}s^3} \\
 s^5 \\
 \underline{\frac{6}{88}s^4} \\
 \frac{6}{88}s^4 \\
 \underline{\frac{6}{88}s^4} \\
 0
 \end{array}$$

$$Z(s) = \frac{3}{8s} + \frac{1}{\frac{32}{7s}} + \frac{1}{\frac{49}{88s}} + \frac{1}{\frac{968}{21s}} + \frac{1}{\frac{6}{88s}}$$



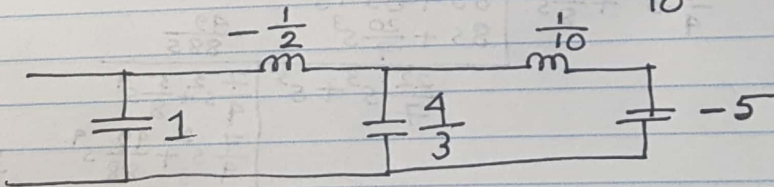
b) Replacing 2 by -2

$$Y(s) = \frac{s(s^2-2)(s^2+4)}{(s^2+1)(s^2+3)}$$

1st Caucere:

$$\begin{array}{r} s \\ s^4 + 4s^2 + 3 \end{array} \begin{array}{r} s + 2s^3 - 8s \\ s^5 + 4s^3 + 3s \\ -2s^3 - 11s \\ s^4 + 4s^2 + 3 \\ s^4 + \frac{11}{2}s^2 \\ -\frac{3}{2}s^2 + 3 \\ -2s^3 - 11s \\ -2s^3 + 4s \\ -15s \end{array} \begin{array}{r} (-\frac{1}{2})s \\ \frac{4}{3}s \\ \frac{1}{10}s \\ -\frac{1}{(-5)s} \end{array}$$

$$Y(s) = s + \frac{1}{(-\frac{1}{2})s} + \frac{1}{\frac{4}{3}s} + \frac{1}{\frac{1}{10}s} + \frac{1}{(-5)s}$$

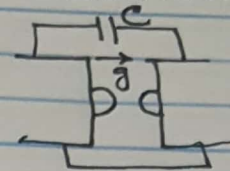


c) clearly, part b has some negative values of inductance/capacitance which is due to right half plane poles for $Z(s)$.

4) y_{21} need not be PR if Y is PR.

Let's consider the Richards' section

$$Y = \begin{bmatrix} sC & -sC - g \\ -sC + g & sC \end{bmatrix}$$



Here, $y_{21}(s) = -sC + g$

$y_{21}(s)$ is PR if $\operatorname{Re}\{y_{21}(s)\} > 0$, for all s .

$$\operatorname{Re}\{y_{21}(s)\} = -\sigma C + g.$$

Clearly, $\operatorname{Re}\{y_{21}(s)\}$ need not be positive always.

So, $y_{21}(s)$ need not be PR.