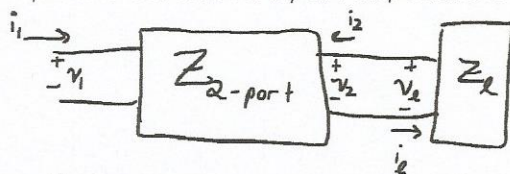


1. For a 2-port with a 2×2 admittance matrix Y when loaded in a 1-port load of admittance y_l we know that the input admittance is $y_{in} = (y_{11}y_l + \Delta_y) / (y_{22} + y_l)$:

a. Give the corresponding formula if the 2-port is described by an impedance matrix, Z , and the load by an impedance z_l .



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$v_l = v_2 = z_l i_l, \quad i_l = -i_2$$

$$v_2 = -z_l i_2$$

$$v_1 = z_{11} i_1 + z_{12} i_2$$

$$v_2 = z_{21} i_1 + z_{22} i_2$$

$$z_{21} i_1 + z_{22} i_2 = -z_l i_2$$

$$z_{21} i_1 = -(z_l + z_{22}) i_2$$

$$\Rightarrow i_2 = \frac{-z_{21}}{z_l + z_{22}} i_1$$

$$v_1 = z_{11} i_1 + z_{12} i_2 = \left(z_{11} + \frac{-z_{12} z_{21}}{z_l + z_{22}} \right) i_1 = \left(\frac{z_{11}(z_l + z_{22}) - z_{12} z_{21}}{z_l + z_{22}} \right) i_1$$

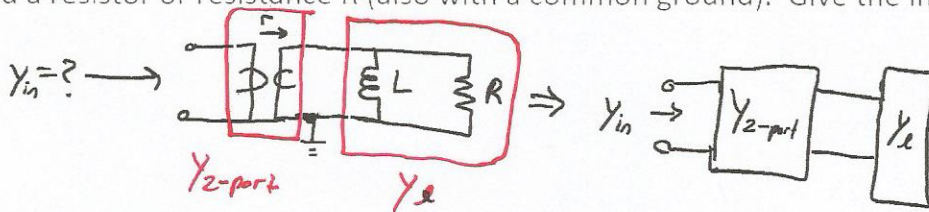
$$= \left(\frac{z_{11} z_l + \cancel{z_{11} z_{22}} - \cancel{z_{12} z_{21}}}{z_l + z_{22}} \right) i_1$$

$$\det(Z) = \Delta_z = z_{11} z_{22} - z_{12} z_{21}$$

$$= \left(\frac{z_{11} z_l + \Delta_z}{z_l + z_{22}} \right) i_1$$

$$\Rightarrow \boxed{Z_{in} = \frac{z_{11} z_l + \Delta_z}{z_l + z_{22}}}$$

b. Draw the circuit of a gyrator, of gyration resistance r , loaded by the parallel combination of an inductor of inductance L and a resistor or resistance R (also with a common ground). Give the input impedance of this circuit.



$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = Y_{2-port} = \begin{bmatrix} 0 & -\frac{1}{r} \\ \frac{1}{r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix}, \quad g = \frac{1}{r}$$

$$\Rightarrow \Delta_y = 0 - (-g^2) = g^2$$

$$Y_{in} = \frac{y_{11} y_l + \Delta_y}{y_l + y_{22}}$$

$$y_l = Y_L \parallel Y_R = \frac{1}{\frac{1}{g_L} + \frac{1}{g_R}} = \frac{g_L g_R}{g_L + g_R}$$

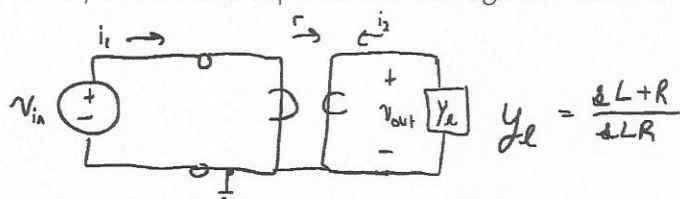
$$g_L = \frac{1}{L}, \quad g_R = \frac{1}{R}$$

①(b) cont.

$$\Rightarrow Y_{in} = \frac{Y_1 Y_2 + \Delta Y}{Y_2 + Y_{22}} = \frac{\Delta Y}{Y_2} = \frac{g^2}{\frac{\Delta L + R}{\Delta L R}} = \frac{\Delta L R g^2}{\Delta L + R} = Y_{in}$$

$$Z_{in} = \frac{1}{Y_{in}} = \frac{\Delta L + R}{\Delta L R g^2}$$

c. For the above circuit if fed by an input voltage, V_{in} , at the input port, find the Voltage gain V_{out}/V_{in} if the output is the voltage across the load (measured positive with respect to ground).



$$Y_L = \frac{\Delta L + R}{\Delta L R}$$

$$i_1 = -\frac{1}{r} V_{out}$$

$$-i_2 = Y_L V_{out}$$

$$i_2 = \frac{1}{r} V_{in}$$

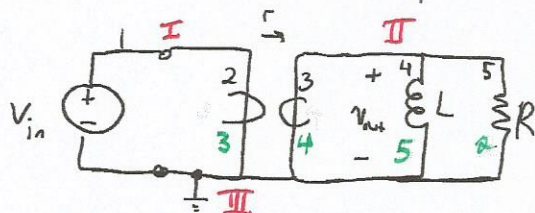
$$\frac{1}{r} V_{in} = -Y_L V_{out}$$

$$(\frac{1}{r} = g)$$

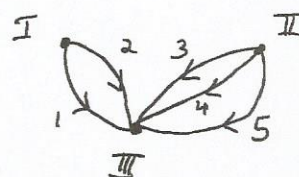
$$\Rightarrow \frac{V_{out}}{V_{in}} = \frac{-g}{Y_L} = \frac{-g}{\frac{\Delta L + R}{\Delta L R}}$$

$$\Rightarrow \frac{V_{out}}{V_{in}} = \frac{-\Delta L R g}{\Delta L + R}$$

d. Use the state equations to check the result of part b.



green = renumbered branches



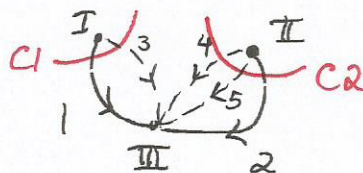
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 \\ 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\Delta L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

Choose tree (1,5) which means these branches will be renumbered as (1,2)

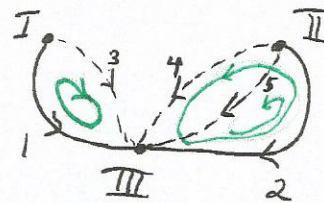
$$C1: 0 = i_1 + 0 \cdot i_2 + i_3 + 0 \cdot i_4 + 0 \cdot i_5$$

$$C2: 0 = 0 \cdot i_1 + i_2 + 0 \cdot i_3 + i_4 + i_5$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$



①② cont.



$$\Rightarrow J = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$AC^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 \\ 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{sL} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \\ 0 & -g \\ 0 & g \\ 0 & \frac{1}{sL} \end{bmatrix}$$

$$BJ^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[AC^T \mid -BJ^T] \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = Av_s - Bv_{\cancel{3}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & \frac{1}{sL} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{in} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & \frac{1}{sL} & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & \frac{1}{sL} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{in} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Semi-State Eq'ns} \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & \frac{1}{sL} & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R} & 0 & 0 & 0 \\ 0 & g & 1 & 0 & 0 \\ 0 & -g & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & \frac{1}{sL} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{in} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right.$$

$$V_{out} = [0 \ 1 \ 0 \ 0 \ 0] x$$

$$\hookrightarrow V_{out} = v_2$$

$$v_1 = v_{in}$$

$$\frac{v_2}{R} + i_4 + i_5 = 0$$

$$-g v_2 - i_3 = 0$$

$$g v_1 - i_4 = 0 \rightarrow i_4 = g v_1 = g v_{in}$$

$$\frac{1}{sL} v_2 - i_5 = 0 \rightarrow i_5 = \frac{1}{sL} v_2$$

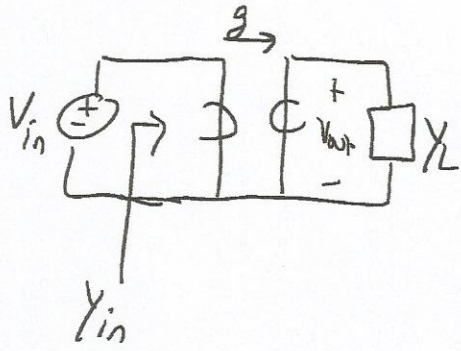
$$\frac{v_2}{R} + g v_{in} + \frac{1}{sL} v_2 = 0 \rightarrow \left(\frac{1}{R} + \frac{1}{sL} \right) v_2 = -g v_{in}$$

$$\rightarrow \left(\frac{sL + R}{sLR} \right) v_{out} = -g v_{in}$$

$$\Rightarrow \boxed{\frac{v_{out}}{v_{in}} = \frac{-sLRg}{sL + R}}$$

$\Rightarrow$ Since part (c) works from state equations, part (b) will also work, seen in next page.

Proof of above statement (can go from (c) $\rightarrow$ (b))



$$\frac{V_{out}}{V_{in}} = \frac{-\Delta L R g}{\Delta L + R}$$

$$V_{in} = \frac{\Delta L + R}{-\Delta L R g} V_{out}$$

$$V_{in} = \dot{V}_{in} Z_{in}$$

$$Y_{in} = \frac{\dot{I}_{in}}{V_{in}}$$

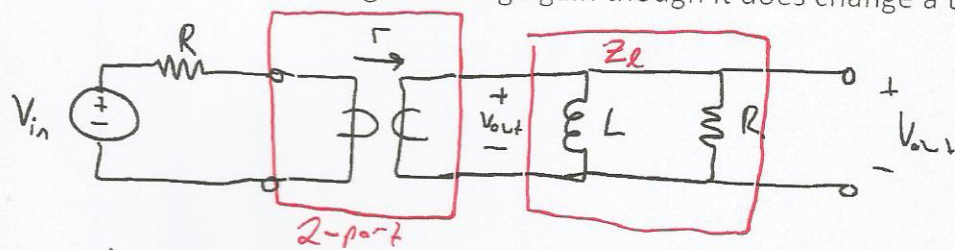
$$\dot{I}_{in} = \dot{I}_1 = -g V_{out}$$

$$\Rightarrow Y_{in} = \frac{+g \cancel{V_{out}}}{\left(\frac{\Delta L + R}{+ \Delta L R g} \right) \cancel{V_{out}}}$$

$Y_{in} = \frac{\Delta L R g^2}{\Delta L + R}$	$\text{and } Z_{in} = \frac{\Delta L + R}{\Delta L R g^2}$
--	--

$\Rightarrow$ Thus results in (b) hold using state equations to derive Voltage gain.

2. For the circuit of problem 1 assume that the input port is fed by a resistor of resistance R in series with V_{in} . Use the scattering matrix to find the new voltage gain, $V_{out}/V_{in}(s)$. For that normalize the two equal resistors to 1 by dividing all impedances by R before calculating $V_{out}/V_{in}(s)$ [note that this normalization does not change a voltage gain though it does change a transfer admittance]



$$Z_{in} = \frac{Z_{11}Z_L + \Delta Z}{Z_L + Z_{22}}$$

Normalize all impedances:

$$Z_{L_{nR}} = \frac{\Delta L}{R}$$

$$Z_{R_{nR}} = \frac{R}{R} = 1$$

$$Z_{gy_{nR}} = \frac{r}{R}$$

$$Z_{L_{nR}} = Z_{L_{nR}}(s) \parallel Z_{R_{nR}} = \left(\frac{\Delta L}{R} \right) \parallel (1) = \frac{\frac{\Delta L}{R} \cdot 1}{\frac{\Delta L}{R} + 1}$$

$$= \frac{\frac{\Delta L}{R}}{\frac{\Delta L + R}{R}} = \frac{\Delta L}{\Delta L + R} = Z_{L_{nR}}$$

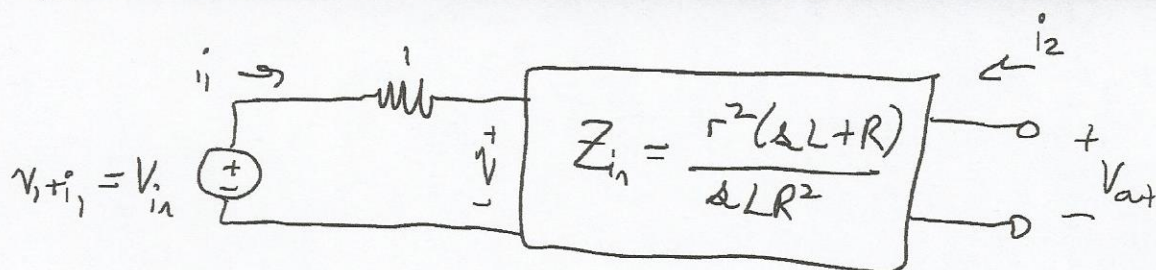
$$Y_{2\text{-port}_{nR}} = \begin{bmatrix} 0 & -\frac{R}{r} \\ +\frac{R}{r} & 0 \end{bmatrix} \rightarrow Z_{2\text{-port}_{nR}} = Y_{2\text{-port}_{nR}}^{-1}$$

$$Z_{2\text{-port}_{nR}} = \frac{1}{0 + (+\frac{R}{r})(\frac{r}{R})} \begin{bmatrix} 0 & +\frac{R}{r} \\ -\frac{R}{r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & +\frac{r}{R} \\ -\frac{r}{R} & 0 \end{bmatrix}$$

$$\Delta_{Z_{2\text{-port}_{nR}}} = 0 - \left(-\frac{r}{R} \right) \left(\frac{r}{R} \right) = \left(\frac{r}{R} \right)^2$$

$$\Rightarrow Z_{in_{nR}} = \frac{0 \cdot Z_L + \left(\frac{r}{R} \right)^2}{\frac{\Delta L}{\Delta L + R} + 0} = \frac{\frac{r^2}{R^2}}{\frac{\Delta L}{\Delta L + R}} = \frac{r^2(\Delta L + R)}{\Delta L R^2}$$

(2) Continued -



$$S = \frac{V^r}{V^i} = \frac{Z_{in} - 1}{Z_{in} + 1}$$

$V = V_{out}$
since all elements
of Z_{in} are
parallel

$$= \frac{\frac{r^2(\Delta L + R) - \Delta L R^2}{\cancel{\Delta L R^2}}}{\frac{r^2(\Delta L + R) + \cancel{\Delta L R^2}}{\cancel{\Delta L R^2}}}$$

$$S = \frac{V^r}{V^i} = \frac{\Delta L(r^2 - R^2) + Rr^2}{\Delta L(r^2 + R^2) + Rr^2} = \frac{(\Delta L + R)r^2 - \Delta L R^2}{(\Delta L + R)r^2 + \Delta L R^2}$$

$$V^i = \frac{1}{2} V_s, \quad V_s = V_{in}$$

$$V^r = \frac{1}{2}(V - i) \quad , \quad i = \frac{V}{Z_{in}}$$

$$= \frac{V}{2} \left(1 - \frac{1}{Z_{in}}\right)$$

$$, \quad V = V_{out}$$

$$S = \frac{V^r}{V^i} = \frac{\frac{1}{2} V_{out} \left(1 - \frac{1}{Z_{in}}\right)}{\frac{1}{2} V_{in}} = \frac{(\Delta L + R)r^2 - \Delta L R^2}{(\Delta L + R)r^2 + \Delta L R^2}$$

$$1 - \frac{1}{Z_{in}} = 1 - \frac{\Delta L R^2}{r^2(\Delta L + R)} = \frac{r^2(\Delta L + R) - \Delta L R^2}{r^2(\Delta L + R)}$$

$$\frac{V_{out}}{V_{in}} \left(\frac{r^2(\cancel{\Delta L + R}) - \Delta L R^2}{r^2(\Delta L + R)} \right) = \frac{(\cancel{\Delta L + R})r^2 - \Delta L R^2}{(\Delta L + R)r^2 + \Delta L R^2}$$

$\Rightarrow$

$$\frac{V_{out}}{V_{in}}(\Delta) = \frac{r^2(\Delta L + R)}{(\Delta L + R)r^2 + \Delta L R^2}$$

3. A given circuit has the following semi-state equations

$$(I) \quad 3dx_1/dt = -5x_1 + 7x_2 + 2V_{in1}$$

$$(II) \quad 0 = -7x_1 + 9x_2 - 4V_{in2}$$

$$(III) \quad V_{out} = 2x_1 + 3x_2$$

a) Using $V_{out} = C(sE - A)^{-1}BV_{in}$ calculate the the transfer function (1x2) matrix.

b) By eliminating x_2 give the state variable equations and from it check the transfer function.

@ This is a MISO system so $A \in 2 \times 2$, $B \in 2 \times 2$, $C \in 2 \times 1$
with general form:

$$s \begin{bmatrix} E_0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} V_{in1} \\ V_{in2} \end{bmatrix}$$

$$(1) \quad sE_0 = A_{11}x_1 + A_{12}x_2 + B_{11}V_{in1} + B_{12}V_{in2}$$

$$(2) \quad 0 = A_{21}x_1 + A_{22}x_2 + B_{21}V_{in1} + B_{22}V_{in2}$$

$$y = V_{out} = [C_1 \ C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(3) \quad V_{out} = C_1x_1 + C_2x_2$$

where equation (I) $\rightarrow$ (1), (II) $\rightarrow$ (2), (III) $\rightarrow$ (3)

thus, $E = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -5 & 7 \\ -7 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$

$$C = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$V_{out} = C(sE - A)^{-1}BV_{in}$$

$$= \begin{bmatrix} 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} s - \begin{bmatrix} -5 & 7 \\ -7 & 9 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} V_{in}$$

$$\begin{bmatrix} 3s+5 & -7 \\ 7 & -9 \end{bmatrix}^{-1} = \frac{1}{-27s-45+49} \begin{bmatrix} -9 & 7 \\ -7 & 3s+5 \end{bmatrix}$$

$$V_{out} = \frac{-1}{27s-4} \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} -9 & 7 \\ -7 & 3s+5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} V_{in}$$

$$= \frac{-1}{27s-4} \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} -18 & -28 \\ -14 & -12s-20 \end{bmatrix} V_{in}$$

③ @ cont.

$$V_{out} = \frac{-1}{27s-4} [-36-42 \quad -56-36s-60] V_{in}$$

$$V_{out} = \frac{1}{27s-4} [78 \quad 36s+116] V_{in}$$

⑥ (I) $\begin{cases} 3s x_1 = -5x_1 + 7x_2 + 2V_{in_1} \\ 0 = -7x_1 + 9x_2 - 4V_{in_2} \\ V_{out} = 2x_1 + 3x_2 \end{cases} \rightarrow x_2 = \frac{1}{9}(7x_1 + 4V_{in_2}) \quad (\text{II})$

Plugging (II) into (I):

$$\begin{aligned} 3s x_1 &= -5x_1 + 7\left(\frac{1}{9}\right)(7x_1 + 4V_{in_2}) + 2V_{in_1} \\ &= \left(-5 + \frac{49}{9}\right)x_1 + 2V_{in_1} + \frac{28}{9}V_{in_2} \\ &= \frac{4}{9}x_1 + 2V_{in_1} + \frac{28}{9}V_{in_2} \end{aligned}$$

$$\begin{aligned} \tilde{A}x_1 &= \frac{4}{27}x_1 + \frac{2}{3}V_{in_1} + \frac{28}{27}V_{in_2} \\ E=I, & \quad \tilde{A} = \frac{4}{27} \quad \tilde{B} = \begin{bmatrix} \frac{2}{3} & \frac{28}{27} \end{bmatrix} \end{aligned}$$

Plugging (II) into (III)

$$\begin{aligned} V_{out} &= 2x_1 + 3\left(\frac{1}{9}\right)(7x_1 + 4V_{in_2}) = \left(2 + \frac{7}{3}\right)x_1 + \frac{4}{3}V_{in_2} \\ &= \frac{13}{3}x_1 + \frac{4}{3}V_{in_2} \\ \tilde{C} &= \frac{13}{3} \quad D = \begin{bmatrix} 0 & \frac{4}{3} \end{bmatrix} \end{aligned}$$

→ since general form:
 $y = Cx + Du$

State Variable Eqns:

$$\begin{aligned} \tilde{I}_1 s x_1 &= \tilde{A}x_1 + \tilde{B}u, & \tilde{A} &= \frac{4}{27}, \tilde{B} = \begin{bmatrix} \frac{2}{3} & \frac{28}{27} \end{bmatrix} \\ V_{out} &= \tilde{C}x_1 + Du, & \tilde{C} &= \frac{13}{3}, D = \begin{bmatrix} 0 & \frac{4}{3} \end{bmatrix} \end{aligned}$$

$$\text{Trans}(s) = \tilde{C}(sI_1 - \tilde{A})^{-1}\tilde{B} + D$$

$$= \frac{13}{3} \left(s - \frac{4}{27}\right)^{-1} \begin{bmatrix} \frac{2}{3} & \frac{28}{27} \end{bmatrix} + \begin{bmatrix} 0 & \frac{4}{3} \end{bmatrix}$$

$$\frac{13}{3} \left(\frac{27s-4}{27}\right)^{-1} \begin{bmatrix} \frac{2}{3} & \frac{28}{27} \end{bmatrix} + \begin{bmatrix} 0 & \frac{4}{3} \end{bmatrix}$$

③ ⑤ cont.

$$\text{Trans}(s) = \frac{13}{3} \left(\frac{\overset{9}{27}}{27s-4} \right) \left[\frac{2}{3} \quad \frac{28}{27} \right] + \left[0 \quad \frac{4}{3} \right]$$

$$= \frac{\cancel{3}(39)}{27s-4} \left[\frac{2}{3} \quad \frac{28}{\cancel{27}^9} \right] + \left[0 \quad \frac{4}{3} \right]$$

$$= \left[\frac{78}{27s-4} \quad \frac{13(28) + 4(27s-4)}{3(27s-4)} \right]$$

$$= \frac{1}{27s-4} \left[78 \quad \frac{364 + 4(27)s - 16}{3} \right]$$

$$= \frac{1}{27s-4} \left[78 \quad \frac{\overset{116}{\cancel{3}48} + 4\overset{9}{(27)s}}{\cancel{3}} \right]$$

$$\text{Trans}(s) = \frac{1}{27s-4} \left[78 \quad 36s + 116 \right]$$

This matches the transfer function from part (a) ✓