1. (a)

$$\int_{-\infty}^{+\infty}|1(t)|^2\,dt = \int_{0}^{+\infty} 1\,dt = \infty \qquad \Rightarrow 1(t) \text{ is not in } L^2$$

(b)

$$\int_{-\infty}^{+\infty}|1'(t)|^2\,dt = \int_{-\infty}^{+\infty}|\delta(t)|^2\,dt = \cancel{\delta(t)}\ \infty \qquad \delta(t) \text{ is not in } L^2$$

(c)

$$f(t) = 1(t-3)\cdot 1(4-t) = \begin{cases} 1 & 3<t<4 \\ 0 & \text{other} \end{cases}$$

$$\int_{-\infty}^{+\infty}|f(t)|^2\,dt = \int_{3}^{4} 1\,dt = 1 < \infty \qquad \Rightarrow f(t) \text{ is in } L^2$$

(d)

$$\int_{-\infty}^{+\infty}\left|\frac{1}{t}\right|^2 dt = 2\int_{0}^{+\infty}\frac{1}{t^2}\,dt = \left.\frac{-2}{t}\right|_{0}^{+\infty} = \infty \qquad \frac{1}{t} \text{ is not in } L^2$$

(e)

$$\int_{-\infty}^{+\infty}\left|e^{-t^2}\right|^2 dt = \int_{-\infty}^{+\infty} e^{-2t^2}\,dt = \sqrt{\frac{\pi}{2}} < +\infty \qquad e^{-t^2} \text{ is in } L^2$$

2. $\mathcal{F}(f(t)) = F(\omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega \qquad f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(\omega) e^{-j\omega t} d\omega$$

For Dirac delta function

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t} d\omega$$

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} f(t)\cdot f^*(t)\, dt$$

$$= \int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(\omega) e^{-j\omega t} d\omega\right) dt$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega)\cdot F^*(\omega') \int_{-\infty}^{+\infty} e^{j(\omega-\omega')t} dt\, d\omega\, d\omega'$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega) F^*(\omega')\cdot 2\pi\delta(\omega-\omega')\, d\omega\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega)\cdot\delta(\omega-\omega')\, d\omega\, F^*(\omega')\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega')\cdot F^*(\omega')\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega$$

Since $\int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty$, $\frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega < \infty$

Fourier transformation is in L^2 if $f(t)$ is in L^2

if $f \in L^2$, $g \in L^2$

$f * g$ is not always square-integrable.

a)

$f_a(s) = \frac{a}{b} \cdot \frac{s-1}{s+1}$

When $a \neq 0$, there is a zero in RHP, $f_a(s)$ is not a PR function

When $a = 0$ $f_a(s)$ is a PR function

PR : $a = 0$

$b \in \mathbb{R}$

BR: ✓ $f_a(s)$ has no poles in RHP & jw axis

✓ $|f_a(jw)| \leq 1 \Rightarrow \left|\frac{a}{b}\right| \sqrt{\frac{(w^2-1)+(2w)^2}{(w^2+1)^2}} \leq 1 \quad w \in (-\infty, +\infty)$

$\Rightarrow \left|\frac{a}{b}\right| \leq 1$

$\Rightarrow |a| \leq |b|$

b). PR: to guarantee no poles and zeros in RHP,

$\Rightarrow \begin{cases} a > 0 \\ b > 0 \end{cases}$ or $\begin{cases} a = 0 \\ b = 0 \end{cases}$

$Re[f_b(jw)] = \frac{(a-w^2)(b-w^2)+w^2ab}{(b^2-w^2)^2+b^2w^2} \geq 0 \quad$ for $\begin{cases} w \in (-\infty, +\infty) \\ a > 0 \\ b > 0 \end{cases}$ or $a = b = 0$

$\Leftrightarrow (a-w^2)(b-w^2) + w^2 ab \geq 0$

When $a = b = 0$

$(a-w^2)(b-w^2) + w^2ab = w^4 \geq 0$

When $a > 0$ & $b > 0$

$f(w^2) = (a-w^2)(b-w^2) + w^2ab = w^4 + (ab-a-b)w^2 + ab$

if $-\frac{ab-a-b}{2} \leq 0$, that is $ab - a - b \geq 0$

$f(w^2)|_{w^2=0} \geq 0 \Rightarrow ab \geq 0$

if $-\frac{ab-a-b}{2} > 0$, that is $ab-a-b<0$

$$f(\omega^2)\Big|_{\omega^2=-\frac{ab-a-b}{2}} \geqslant 0$$

$$\Rightarrow \left(-\frac{ab-a-b}{2}\right)^2 + (ab-a-b)\left(-\frac{ab-a-b}{2}\right) + ab \geqslant 0$$

$$\Rightarrow \quad ab \geqslant \frac{(ab-a-b)^2}{4}$$

In sum

PR: $a=b=0$

or $\begin{cases} a>0,\ b>0 \\ ab \geqslant a+b \end{cases}$ or $\begin{cases} a>0,\ b>0 \\ ab < a+b \\ ab \geqslant \frac{(ab-a-b)^2}{4} \end{cases}$

BR: $b>0$

$\checkmark$ $f_b(s)$ has no poles in RHP & $j\omega$ axis

$\checkmark$ $|f_b(j\omega)| = \frac{1}{(b-\omega^2)^2+b^2\omega^2}\sqrt{[(a-\omega^2)(b-\omega^2)+\omega^2 ab]^2+(a\omega^3+b\omega^3)^2} \leq 1$

$$\Rightarrow \frac{|a|}{b} \leq 1.$$

$$\Rightarrow |a| \leq b$$

Therefor

$$\begin{cases} b>0 \\ |a| \leq b \end{cases}$$

(c)

P.R To guarantee no zeros and poles in RHP,

$$\Rightarrow ① \begin{cases} a=0 \\ b=0 \end{cases} ② \begin{cases} a>0 \\ b>0 \end{cases} ③ \begin{cases} a=0 \\ b>0 \end{cases} ④ \begin{cases} a>0 \\ b=0 \end{cases}$$

for ①

$$f_c(s) = \frac{s}{s^2+1} \quad \text{P.R}$$

for ③

$$f_c(s) = \frac{s}{s^2+bs+1} \quad \text{P.R when } Re\left[\frac{j\omega}{(j\omega)^2+bj\omega+1}\right] = \frac{b\omega^2}{(1-\omega^2)^2+b^2\omega^2} \geq 0 \Rightarrow b \geq 0$$

$f_c(s)$ is always a P.R function in this case

for ④

$$f_c(s) = \frac{s(s^2+as+1)}{(s^2+1)^2}$$

there will be double-pole in $j\omega$ axis,

$f_c(s)$ is not a P.R function in this case.

for ②

✓ $f_c(s)$ has no poles in RHP

✓ $f_c(s)$ has two poles on $j\omega$ axis

$s = j$

$$Res[f_c(s)] = \frac{j(-1+aj+1)}{(-1+bj+1)(2j)} = \frac{a}{2b} > 0$$

$s = -j$

$$Res[f_c(s)] = \frac{(-j)(a(-j))}{b(-j)\cdot(-2j)} = \frac{a}{2b} > 0$$

✓ $$Re[f_c(j\omega)] = \frac{(b-a)\omega^2}{\omega^4+(b^2-2)\omega^2+1} \geq 0 \text{ for } \begin{cases} a>0 \\ b>0 \\ \omega \in (-\infty, +\infty) \end{cases}$$

$$= \frac{(b-a)\omega^2}{(1-\omega^2)^2+b^2\omega^2} \geq 0$$

$\Rightarrow b-a \geq 0$

~~for~~ In sum

$$\text{P.R} \Leftrightarrow \begin{cases} a=0 \\ b=0 \end{cases} \quad \begin{cases} a>0 \\ b>0 \\ b \geq a \end{cases} \quad \begin{cases} \cancel{a>0}\ a=0 \\ b>0 \end{cases}$$

B.R. ~~is~~

$\because$ There are poles on $j\omega$ axis

$f_c(s)$ is not a B.R function

d). To guarantee no poles in RHP, $a \leq 2$.

When $a=2$, consider relative degree

① $b=1$

$f_d(s)$ is a p.R function

② $b=0$

$f_d(s)$ is a p.R function

When $a=1$,

① $b=2$

P.R

② $b=1$.

P.R

③ $b=0$

P.R

When $a=0$, $b \in \mathbb{R}$

P.R.

Therefore.

$$\begin{cases} a=2 \\ b=1,0 \end{cases} \quad \begin{cases} a=1 \\ b=2 \\ b=1, \\ b=0 \end{cases} \quad \begin{cases} a=0 \\ b \in \mathbb{R} \end{cases}$$

B.R.

$$a=2 \begin{cases} b=1 & \cancel{\text{not}}\ \text{B.R} \\ b=0 & \text{not B.R} \end{cases}$$

$$a=1 \begin{cases} b=2 & \text{B.R} \\ b=1 & \text{B.R.} \\ b=0 & \text{not B.R} \end{cases}$$

$a=0$, $b \in \mathbb{R}$ not B.R.