$1_n - S^{T*}(s)S(s)$ is positive semidefinite in $\sigma \geq 0$, $s = \sigma + j\omega$

$f(t) \Rightarrow F(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt$ = Laplace transform

$S(s)$ has no singularities in $\sigma > 0$ by stability

$= F(j\omega) = F(s)\Big|_{s=j\omega}$

$S(\sigma)$ real, in $\sigma > 0$; real components

these $S(s)$ are bounded real $n \times n$ matrix

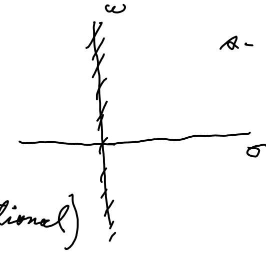
$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} [v^{i*T}(t) v^{i}(t) - v^{r*T}(t) v^{r}(t)]\, dt \geq 0$$

$= 0$ if the circuit is lossless

For lossless $V^{i*T}(j\omega)(1_n - S^{T*}(j\omega)S(j\omega))V^i(j\omega) = 0$, $V^i(j\omega) = F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$ = Fourier transform

for lossless $1_n - S^{T*}(j\omega)S(j\omega) \equiv 0$ for all ω

$\Rightarrow \quad 1_n - S^T(-j\omega)S(j\omega) = 0$



do an analytic continuation into s, $s = \sigma + j\omega$, σ

by $\omega = s/j$ $\quad 1_n - S^T(-s)S(s) = 0_n$ if $s = j\omega$ for (all ω if rational)

$\Rightarrow$ that $1_n - S^T(-s)S(s) \equiv 0$ for all s with $\sigma \geq 0$

$\therefore$ if lossless $S^T(-s)S(s) = 1_n$ or $S^{-1}(s) = S^T(-s)$

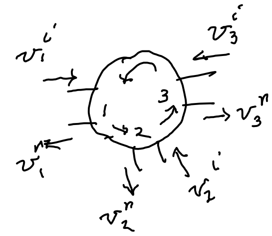
$s \to -s \equiv$ Hermitian conjugate (use a lower notes) $\to$ $s \to s^* = \sigma - j\omega$ = complex conjugate

Ex: $n = 1$

$$S(s) = \frac{1 - Y(s)}{1 + Y(s)} = \frac{1 - \frac{1}{sL}}{1 + 1/sL} = \frac{sL - 1}{sL + 1}$$

$$S^{-1} = \frac{sL+1}{sL-1}; \quad S(-s) = \frac{-sL-1}{-sL+1} = \frac{sL+1}{sL-1} \equiv S^{-1}(s)$$

Ex: $n = 3$ circulator:



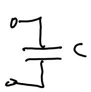
$$S(s) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \begin{bmatrix} v_1^r \\ v_2^r \\ v_3^r \end{bmatrix} = S \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix}$$

$$1_n - S^T(-s)S(s) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{1_3} - \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{S^T} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 1_3 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1_3 - 1_3 = 0_3$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as $1_3 - S^T(-s)S(s) = 0_3$ this 3-port circulator is lossless (as BR)

Ex:



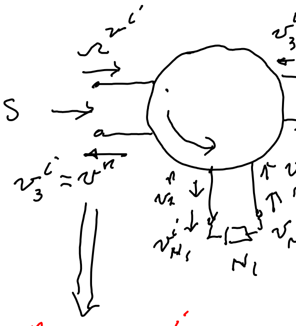
$$S(s) = \frac{1 - Y(s)}{1 + Y(s)} = \frac{1 - sC}{1 + sC}$$

$S(-s) = \frac{1 + sC}{1 - sC} = 1/S(s) \Rightarrow 1 - S^T(-s)S(s) = 0 \Rightarrow$ is lossless if C is real

& the pole @ $s = -1/C$ is not in $\sigma > 0$ (but in $\sigma < 0$)

$\Rightarrow$ $S(s)$ is bounded-real & rational in s

$\Rightarrow$ BR $\Rightarrow$ the capacitor is passive & lossless if C is real & $C \geq 0$



$v_3^i = v^n_{N_2} = S_2 v^i_{N_2} = S_2 \cdot S_1 v^i_{N_1} = S_2 S_1 v^i$

assume N_1 & N_2 are passive with S_1 & S_2 BR

here $S = S_2 \cdot S_1 \Longrightarrow$ looks like the product of ~~two~~ BR matrices looks to be BR

$v_3^n = v^i_{N_2} = S_1 v^i_{N_1}$

$v^n_{N_1} = S_1 \cdot v^i_{N_1} = S_1 v^i$

$v_3^i = v^n$

$v^n = S_2 \cdot S_1 v^i \Longrightarrow S = S_2 \cdot S_1$

holds also for 3n-port circulators, S_{circ} is $3n \times 3n$

$$S_{circ.} = \begin{bmatrix} 0_n & 0_n & 1_n \\ 1_n & 0_n & 0_n \\ 0_n & 1_n & 0_n \end{bmatrix}$$

then S_1 & S_2 would be $n \times n$

we can synthesize a BR $S(s)$ if we can write it as $S = S_1 \cdot S_2$ with S_1 & S_2 BR $\Longrightarrow$ look at 1×1 lossless BR functions

$$S = (1_n + Y)^{-1}(1_n - Y) \Rightarrow (1_n + Y)S = 1_n - Y \Rightarrow S - 1_n = -YS - Y$$
$$= -Y(S + 1_n)$$
$$Y = (1_n - S)(1_n + S)^{-1}$$
$$= (1_n + S)^{-1}(1_n - S)$$

if S is BR then this $Y(s)$ is PR positive-real

no poles in $\sigma > 0$ (the RHP = right half plane)

if $S(\sigma)$ is real then $Y(\sigma)$ is real, $\sigma > 0$

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} \frac{v^{T*}(t)\, i(t) + i^{T*}(t)\, v(t)}{2}\, dt = \int \frac{1}{2}\left[V^{T*}(j\omega)\, Y(j\omega)\, V(j\omega) + V^{T*}(j\omega)\, Y^{T*}(j\omega)\, V(j\omega)\right] \frac{d\omega}{2\pi}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} V^{T*}(j\omega)\left[Y(j\omega) + Y^{T*}(j\omega)\right] V(j\omega)\, \frac{d\omega}{2\pi} \underset{\text{passive}}{\geq 0} \Rightarrow \frac{Y(j\omega) + Y^{T*}(j\omega)}{2}$$ should be positive semi-definite

Positive-real $n \times n$ matrix has

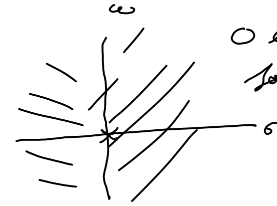
a) $Y(s)$ real for real $s = \sigma > 0$

b) $Y(s)$ is analytic in $\sigma > 0$

c) $Y(j\omega) + Y^{T*}(j\omega)$ is positive semidefinite for (almost all) ω

($= 0_n$ if $\mathcal{E}(\infty) = 0 \Longleftrightarrow$ lossless)

$$\Rightarrow \left[Y(j\omega) + Y^T(-j\omega)\right]\Big|_{\omega = s/j} \Rightarrow Y(s) + Y^T(-s) = 0_n \Rightarrow Y(s) = -Y^T(-s) \text{ in all } s$$

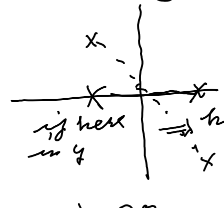


0 everywhere for $Y + Y^T_*$

Check: $y(s) = \frac{1}{Ls}$ (a pole @ $s=0$)

$$\frac{1}{Ls} + \frac{1}{L(-s)} = \frac{1}{Ls} - \frac{1}{Ls} \equiv 0$$

for any lossless 1×1 $y(s)$, $y(s) = -y(-s)$



if here in y $\Rightarrow$ here in $y(-s)$ but this gives a pole in y

shows all poles of $y(s)$, PR, are on the $j\omega$ axis.