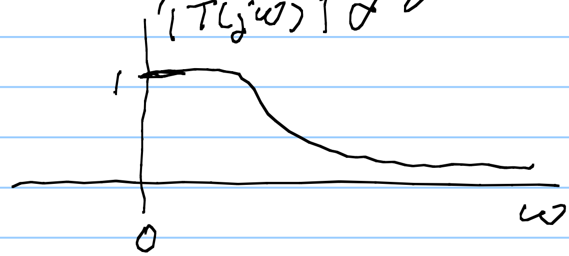
Note Title 9/28/2014

EE 610
10/02/14

Filter functions

LP filter $T(s) = \dfrac{1}{s^n + a_{n-1}s^{n-1} + \dots a_1 s + 1}$, $s = \sigma + j\omega$, a_i real

Maximally flat magnitude = Butterworth



as many derivatives of $|T(j\omega)|\Big|_{\omega=0}$ are 0

$$|T(j\omega)|^2 = T(j\omega)\,T^*(j\omega) = T(j\omega)\,T(-j\omega) \text{ if } a_i \text{ real}$$

$$\frac{d}{d\omega}\left(\frac{1}{|T(j\omega)|^2}\right) = -\frac{1}{|T(j\omega)|^4} \times 2\,\frac{d|T(j\omega)|}{d\omega} \times |T(j\omega)| = \frac{-2}{|T(j\omega)|^3} \cdot \frac{d|T(j\omega)|}{d\omega}$$

$\therefore$ use $\dfrac{d}{d\omega}\left(\dfrac{1}{|T(j\omega)|^2}\right)$ & set as many of similar derivaties to zero as possible

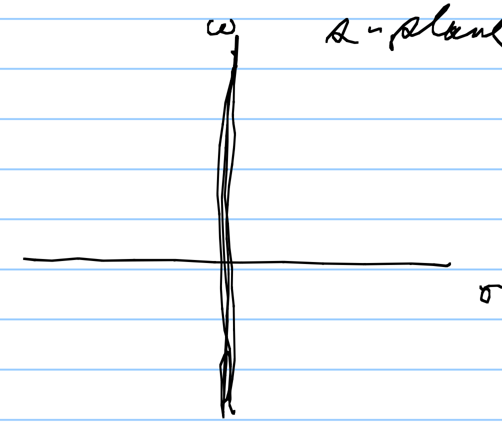
$\therefore$ desire a Taylor series expansion of $1/|T(j\omega)|^2$

$$\frac{1}{|T(j\omega)|^2} = 1 + \underset{\Uparrow\,0}{c_1}\omega + c_2\omega^2 + \dots + (j\omega)^n(-j\omega)^n = 1 + \dots + (+1)^n\omega^{2n}$$

$=$ as $|T(j\omega)|^{-2}$ is even in ω

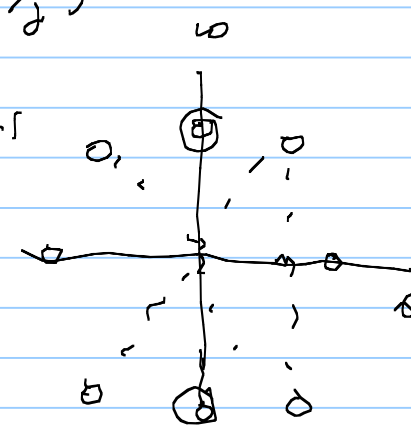
$\Rightarrow$ all coefficients $c_i = 0$, $i = 1, \dots, 4n-1$

$$\frac{1}{|T(j\omega)|^2} = 1 + \omega^{2n} = 1 + (-1)^n\omega^n(-\omega)^n \Rightarrow$$



By $\omega = s/j$ get an analytic continuation to whole s-plane

$$\left.\frac{1}{T(j\omega)T(-j\omega)}\right|_{\omega = s/j} = \frac{1}{T(s)T(-s)} = 1 + (-1)^n \left(\frac{s}{j}\right)^n \left(-\frac{s}{j}\right)^n = 1 + (s^n)(-s)^n$$



zeroes at $0 = 1 + (-1)^n s^{2n} \Rightarrow s^{2n} = (-1)^{n+1}$

$$T(s) \cdot T(-s) = \frac{1}{s^n(-s)^n + 1} = \frac{1}{(-1)^n\left(s^{2n} + (-1)^n\right)}$$

desire $T(s)$; ∴ factor $s^{2n} + (-1)^n \Rightarrow 0 = s^{2n} + (-1)^n$

find $s^{2n} = (-1)^{n+1} = \left(e^{j(\pi + 2k\pi)}\right)^{(n+1)} \Rightarrow s_k = \left(e^{j\frac{(2k+1)\pi}{2n}(n+1)}\right)$

$$s_k = e^{j\left[\frac{(2k+1)\pi}{2} + \frac{(2k+1)\pi}{2n}\right]}$$

stable $T(s)$ has all poles in LHP $\Rightarrow$ choose the LHP poles

Ex: $T(s)$ of degree 3 $\Rightarrow n = 3 = \dfrac{1}{s^3 + a_2 s^2 + a_1 s + 1}$

$$s_k = e^{j\left[\frac{2k+1}{2}\pi + \frac{2k+1}{6}\pi\right]} \qquad k = 0, 1, 2, 3, \ldots \infty, \; 0, -1, -2, \ldots$$

$$s_0 = e^{j\left(\frac{\pi}{2} + \frac{\pi}{6}\right)} = e^{j\frac{4\pi}{6}} = e^{j\frac{2}{3}\pi}$$

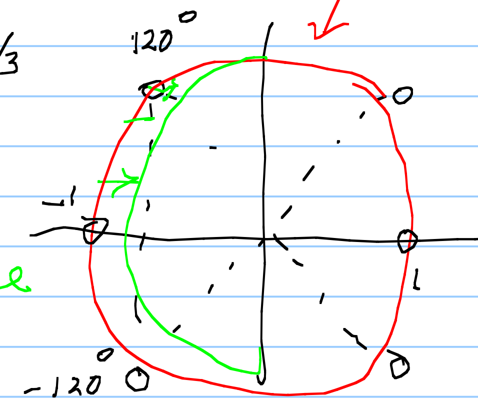
$$s_1 = e^{j\left(\frac{3\pi}{2} + \frac{3\pi}{6}\right)} = e^{j\frac{12}{6}\pi} = e^{j2\pi} = 1$$

$$s_2 = e^{j\left(\frac{5\pi}{2} + \frac{5\pi}{6}\right)} = e^{j\frac{20\pi}{6}} = e^{j\frac{10\pi}{3}} = e^{j\frac{4}{3}\pi}$$

$$s_3 = e^{j\left(\frac{7\pi}{2} + \frac{7\pi}{6}\right)} = e^{j7\pi\left(\frac{4}{6}\right)} = e^{j\frac{14\pi}{3}} = e^{-j2\pi/3}$$

$s_4 =$

$s_5 =$



$$\therefore \quad T(s) = \frac{1}{(s-(-1))\left(s-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right)\left(s-\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\right)}$$

$$= \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{1}{s^3+2s^2+2s+1}$$

max flat $|T(j\omega)|$ low pass

Equal ripple

$$|T(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)}$$

Chebyshev $\Rightarrow$ Tchebishev

$$C_n(\omega) = \cos(n(\cos^{-1}(\omega))) \quad \text{if } 0 \le \omega \le 1$$

$$= \cos(n(\cosh^{-1}(\omega))) \quad \text{if } 1 \le \omega$$