Homework 3

Question 1

a) The input admittance can be obtained by solving the follow matrix equation where A, B, C, and E are defined in the lecture:

$$y(s) = C(sE - A)^{-1}B$$

The actual computation is done using software (MATLAB) where the $(sE - A)^{-1}$ calculation is done using row reduction. Please see following script.

```
%create symbolic variables
syms s L C gm G

%set E,A matrices
E = [0 0 0 0 0 0;0 0 0 -L -L 0; 0 0 C 0 0 0; 0 0 0 0 0 0;0 0 0 0 0 0; 0 0 0 0 0 0];
A = [0 0 0 -1 -1 -1;0 -1 0 0 0 0; 0 0 0 0 1 1; 0 0 0 1 0 0; -gm gm 0 0 1 0; -G 0 G 0 0 1];
Ci = [1 0 0 0 0 0];
B = [1 0 0 0 0 0]';

%Create augmented matrix by appending 6x6 identity matrix
X = [(s*E-A),eye(6)];

%Perform row reduction to get identity on left of augmented matrix
t = X(1,:);
X(1,:) = X(5,:);
X(5,:) = t;
X(1,:) = X(1,:)./gm;
X(6,:) = X(6,:) + X(1,:).*-G;

X(1,:) = X(1,:) + X(2,:);
X(6,:) = X(6,:) + X(2,:)*-G;

X(3,:) = X(3,:)./(s*C);
X(6,:) = X(6,:) + X(3,:)*G;

X(4,:) = X(4,:).*-1;
X(1,:) = X(1,:) + X(4,:).*L*s;
X(2,:) = X(2,:) + X(4,:).*L*s;
X(5,:) = X(5,:) + X(4,:).*-1;
X(6,:) = X(6,:) + X(4,:).*-s*L*G;

X(1,:) = X(1,:) + X(5,:).*(L*s+1/gm);
X(2,:) = X(2,:) + X(5,:).*(L*s);
X(3,:) = X(3,:) + X(5,:).*1/(C*s);
X(6,:) = X(6,:) + X(5,:).*-(G/gm - G/(C*s) + G*L*s);

X(6,:) = X(6,:)./(- G/gm - G*L*s - 1);
X(1,:) = X(1,:) + X(6,:).*-(L*s+1/gm);
X(2,:) = X(2,:) + X(6,:).*-(L*s);
X(5,:) = X(5,:) + X(6,:).*-1;

%Inverse of (sE-A) is right side of augmented matrix
Kred = X(:,(7:12));

%Find y(s)
yred = Ci*Kred*B;
pretty(simplify(yred));
```

Homework 3

Problem 1

b) Poles / Zeros of input admittance

$$Z(s) = \frac{1}{y(s)} = \frac{(sC+G)(1+sLg_m)}{sC[g_m + G(1+sLg_m)]}$$

Zeros

$$(sC+G)(1+sLg_m) = 0 \Rightarrow \boxed{s = \frac{-G}{C}, \quad s = \frac{-1}{Lg_m}}$$

Poles.

$$sC[g_m + G(1+sLg_m)] = 0 \Rightarrow \boxed{s = 0, \quad s = -\frac{(g+G)}{GLg_m}}$$

c) Scale frequency and admittance level such that:

$$\left.\begin{array}{l} s \rightarrow s' = as \\ y \rightarrow y' = by \end{array}\right\} \quad y'(s') = by(as) = b\,\frac{asC[g_m + G(1+asLg_m)]}{(asC+G)(1+asLg_m)} \qquad (1)$$

When $C = G = 1$

$$y(s)\Big|_{C=G=1} = \frac{s[g_m + 1 + sLg_m]}{(s+1)(1+sLg_m)} = k \qquad (2)$$

Setting (1) = (2)

$$asCg_m + asCG + a^2s^2CGLg_m = K[asC + a^2s^2CLg_m + G + asLGg_m]$$

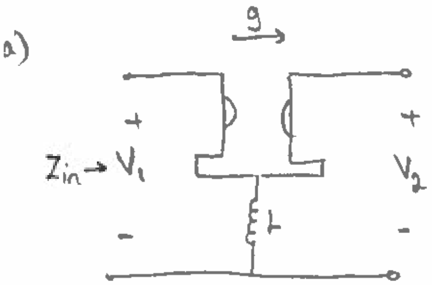
$$\Rightarrow a^2\underbrace{[s^2CGLg_m - ks^2CLg_m]}_{A} + a\underbrace{[sCg_m + sCG - ksC - ksLGg_m]}_{B} + \underbrace{[kG]}_{C} = 0$$

$$\Rightarrow \quad a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad \text{where } A, B, C \text{ defined above}$$

$\therefore$ given the admittance scaling value 'b', we can find frequency scaling 'a' to obtain $C = G = 1$.

Problem 2

a)

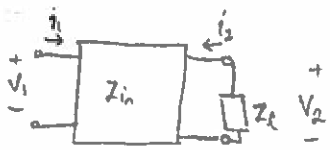


$$Z_L = \begin{bmatrix} sL & sL \\ sL & sL \end{bmatrix}$$ (inductor impedance matrix)

$$Y_g = \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix} \Rightarrow Z_g = \frac{1}{Y_g} = \frac{-1}{g^2}\begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{g} \\ -\frac{1}{g} & 0 \end{bmatrix}$$ (gyrator impedance matrix)

$$\boxed{Z_{in} = Z_g + Z_L = \begin{bmatrix} sL & sL + \frac{1}{g} \\ sL - \frac{1}{g} & sL \end{bmatrix}}$$

b)



$$\Rightarrow \quad V_2 = -i_2 Z_\ell = Z_{21} i_1 + Z_{22} i_2$$

$$\Rightarrow \quad V_2 = Z_{21} i_1 - \frac{Z_{22}}{Z_\ell} V_2$$

$$V_2 = \frac{Z_{21}}{1 + \frac{Z_{22}}{Z_\ell}} i_1$$

$$V_1 = Z_{11} i_1 + Z_{12} i_2$$

$$= Z_{11} i_1 + Z_{12}\left(\frac{-V_2}{Z_\ell}\right)$$

$$= Z_{11} i_1 - \frac{Z_{12}}{Z_\ell} \frac{Z_{21}}{1 + \frac{Z_{22}}{Z_\ell}} i_1$$

$$\frac{V_1}{i_1} = Z_{in} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_\ell + Z_{22}}$$

$$Z_{in} = \frac{Z_{11} Z_{22} - Z_{12} Z_{21} + Z_\ell Z_{11}}{Z_\ell + Z_{22}} = \frac{\Delta Z + Z_\ell Z_{11}}{Z_\ell + Z_{22}}$$

b) continued...

$$\Delta Z = (sL)^2 - \left(sL - \frac{1}{g}\right)\left(sL + \frac{1}{g}\right) = \frac{1}{g^2}$$

$$\boxed{Z_{in} = \frac{\frac{1}{g^2} + Z_\ell sL}{Z_\ell + sL}}$$

c) $$Z_{in}(Z_\ell + sL) = \frac{1}{g^2} + sLZ_\ell$$

$$Z_\ell = \frac{\frac{1}{g^2} - sLZ_{in}}{Z_{in} - sL}$$

For the Richards function:

$$Z(s) = \frac{kZ(k) - sZ(s)}{kZ(s) - sZ(k)} = \frac{1 - \frac{sZ(s)}{kZ(k)}}{\frac{kZ(s)}{Z(k)} - s} \times k$$

Rewrite Z_ℓ:

$$Z_\ell = \frac{1 - sLg^2 Z_{in}}{\frac{Z_{in}}{L} - s} \times \frac{1}{Lg^2}$$

$$\Rightarrow k = \frac{1}{Lg^2} \quad \frac{1}{kZ(k)} = g^2 L \quad \frac{k}{Z(k)} = \frac{1}{L}$$