

$$y(s) \approx \frac{s^2 + a s + b}{s^2 + c s + d}$$

$$\dot{x} = Ax + Bu \quad u = v, y = \dot{L}$$

$$y = Cx + Du$$

11/22/06  
EE610  
(additions of 11/23)

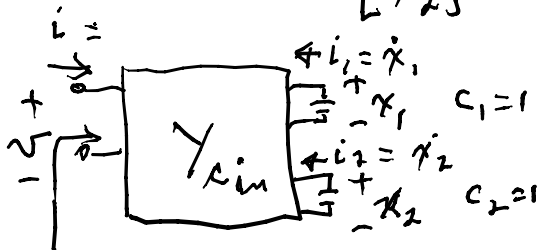
$$x = (sI - A)^{-1} B u$$

$$y = [D + C(sI - A)^{-1} B] u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -d & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a-c \\ b-d \end{bmatrix} [v]$$

$$y(s) = D + C(sI - A)^{-1} B$$

$$[i] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} [v]$$



$$Y_{sim} \begin{bmatrix} v \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{L} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{L} \\ -\dot{x}_1 \\ -\dot{x}_2 \end{bmatrix}$$

$y(s)$

$$\begin{bmatrix} 1 & C \\ -B & -A \end{bmatrix} \begin{bmatrix} v \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \dot{L} \\ \begin{bmatrix} -\dot{x}_1 \\ -\dot{x}_2 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} y \\ -\dot{x} \end{bmatrix}$$

$$\begin{bmatrix} y \\ \dot{x} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ -Y_{21} & -Y_{22} \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix}; \quad A = -Y_{22}$$

$$C = Y_{12}$$

$$B = -Y_{21}$$

$$D = Y_{11}$$

$$Y_{sim} = \begin{bmatrix} D & C \\ -B & -A \end{bmatrix} = \begin{bmatrix} 1 & [a-c] \\ -[b-d] & [d \ 0] \end{bmatrix}$$

If  $y(s)$  is PR then we desire a passive circuit  
i.e. want  $Y_{in}$  uses only passive R's, gyrators &  
transformers.  $P_{in} = V^T I > 0$

$$= \frac{V^T I + I^T V}{2} = \frac{V^T [Y_c V] + V^T Y_c^T V}{2}$$

$$= V^T \left[ \frac{Y_c + Y_c^T}{2} \right] V \geq 0^2 \text{ for all } V's$$

means  $Y_{c, sym} = \frac{Y_c + Y_c^T}{2}$  should be positive

definite. The PR lemma is that there is a  
transformation so that  $Y_{c, sym}$  is positive:

Let  $\hat{x} = T x$

$$\dot{\hat{x}} = T \dot{x} = T A T^{-1} \hat{x} + T B u$$

$$y = C T^{-1} \hat{x} + D u$$

$$y(s) = D + C T^{-1} (s I_{\hat{x}} - T A T^{-1})^{-1} T B$$

$$Y_{c, in} = \begin{bmatrix} D & C \\ -T B & -T A T^{-1} \end{bmatrix} \Rightarrow \begin{bmatrix} D & C T^{-1} \\ -T B & -T A T^{-1} \end{bmatrix} = Y_c$$

$$Y_c = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} Y_{c, in} \begin{bmatrix} 1 & 0 \\ 0 & T^{-1} \end{bmatrix}$$

$$(Y_c + Y_c^T) = \begin{bmatrix} D + D^T & C T^{-1} - B^T T^T \\ (T^{-1})^T C^T - T B & -T A T^{-1} - T^{-1} A^T T^T \end{bmatrix}$$

Math result:  $P A + A^T P = -P^T \Gamma$

given  $\Gamma$  &  $A$   
with eigenvalues

Eigenvalues of  $A$  are poles of  $(sI_k - A)^{-1}$  <sup>of  $A$  in LHP</sup> which are in LHP by PR nature of  $Z(s)$

$P$  is positive definite,  $P = P^{1/2} P^{1/2}$ ;  $P^{1/2}$  is pos. def.

multiply  $PA + A^T P = -\Gamma^T \Gamma$  by  $P^{-1/2}$  on right & left

$$\text{gives } P^{1/2} A P^{-1/2} + P^{-1/2} A^T P^{1/2} = -P^{-1/2} \Gamma^T \Gamma P^{-1/2} = -\hat{\Gamma}^T \hat{\Gamma}$$

$$\hat{\Gamma} = \Gamma P^{-1/2}$$

↓

$$T A T^{-1} + T^{-1} A^T T = -\hat{\Gamma}^T \hat{\Gamma} \Rightarrow \text{choose } T = P^{1/2}$$

$$D + D^T = D_0 D_0^T = P_1$$

$$Y_A + Y_A^T = \begin{bmatrix} D_0^T D_0 & C P^{-1/2} - B^T P^{1/2} \\ P^{-1/2} C^T - P^{1/2} B & \hat{\Gamma}^T \Gamma \end{bmatrix} = \begin{bmatrix} D_0^T \\ \hat{\Gamma}^T \end{bmatrix} \begin{bmatrix} D_0 & \hat{\Gamma} \end{bmatrix}$$

$$\therefore \text{set } C P^{-1/2} - B^T P^{1/2} = D_0^T \hat{\Gamma} \quad ; \quad \Gamma = \hat{\Gamma} P^{1/2}$$

but

$$PA + A^T P = -\Gamma^T \Gamma = -(\hat{\Gamma} P^{1/2})^T (\hat{\Gamma} P^{1/2})$$

$$= -(C - B^T P)^T D_0^T D_0^T (C - B^T P)$$

$$= -(C^T - P^T B) D_0^T (C - B^T P)$$

$$= -C^T D_0^T C + C^T D_0^T B^T P + P^T B D_0^T C - P^T B D_0^T B^T P$$

or

$$P[A - B D_0^T C] + [A^T - C^T D_0^T B^T] P + C^T D_0^T C - P B D_0^T B^T P = 0_n \quad (\text{note } P^T = P, D_0^T = D_0)$$

$$PA_1 + A_1^T P + Q - P B (-D_0^T) B^T P = 0 \Rightarrow \text{Riccati Eq.}$$

$A_1 = A - BD_1^{-1}C$ ,  $Q = C^T D_1^{-1}C$ ,  $B_1 = BD_1^{-1}$ ,  $R = -I_k$   
 use `care(A1, B1, Q, R) = [X=P, ...]` in Matlab (=-identity)

which solves for  $X$  in

$$A_1^T X + X A_1 - X B_1 R^{-1} B_1^T X + Q = 0; \quad P = X$$

`systm(P) = P^{1/2}` (`systm(.)` = matlab matrix square root)

guarantees  $2Y_{c, \text{sym}} = \begin{bmatrix} D_0 & \hat{A} \\ \hat{A}^T & \end{bmatrix} \begin{bmatrix} D_0 & \hat{A} \end{bmatrix}$  (which is positive semidefinite)

rank of  $Y_{c, \text{sym}} = \#$  of resistors

note  $V^T Y_{c, \text{sym}} V = \begin{bmatrix} D_0 & \hat{A} \end{bmatrix} V \begin{bmatrix} D_0 & \hat{A} \end{bmatrix}^T V = \text{sum of squares} \geq 0$  for all vectors  $V$

$Y_{c, \text{sym}}$  is realized by ideal transformers and resistors

$$Y_c = Y_{c, \text{sym}} + \underbrace{Y_{c, \text{gyr}}}_{\text{gyrators}}$$

parallel connection

The matlab m file `PRtrans.m` is set up to find

$$Y_c \text{ for } y(s) = \frac{s^2 + as + b}{s^2 + cs + d}$$

Example:  $y(s) = [s^2 + s + 1] / [s^2 + s + 3]$  was previously shown to be PR. PRtrans yields:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 \end{bmatrix} v, \quad \dot{z} = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} v$$

initial

$$Y_{c_i} = \begin{bmatrix} 1 & C \\ -B & -A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}, \quad Y_{c_i, \text{sym}} = \frac{Y_{c_i} + Y_{c_i}^T}{2} = \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

as  $\det(Y_{c_i, \text{sym}}) = -1$ , this is not a positive semidefinite matrix so  $Y_{c_i}$  is not a PR matrix and needs active elements to synthesize it.

$$D_0 = \sqrt{2}, \quad A_1 = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}, \quad Q = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = -1$$

Matlab care gives  $X = \begin{bmatrix} 0.3014 & -0.0254 \\ -0.0254 & 0.1594 \end{bmatrix}$  and matlab

$$\text{sqrtm}(X) = \begin{bmatrix} 0.5484 & -0.268 \\ -0.268 & 0.3983 \end{bmatrix} = X^{1/2} \quad \left( \text{note } X = X^T \text{ \& } X^{1/2} = X^{1/2 T} \text{ and both are positive definite} \right)$$

using  $T = X^{1/2}$  gives

$$Y_c = \begin{bmatrix} 1.0000 & 1.8296 & 0.1232 \\ -0.0537 & 0.7885 & -1.3237 \\ 0.7966 & 2.1405 & 0.2115 \end{bmatrix}$$

Ycsym =

$$\begin{bmatrix} 1.0000 & 0.8880 & 0.4599 \\ 0.8880 & 0.7885 & 0.4084 \\ 0.4599 & 0.4084 & 0.2115 \end{bmatrix}$$

Ycsw =

$$\begin{bmatrix} 0 & 0.9416 & -0.3367 \\ -0.9416 & 0 & -1.7321 \\ 0.3367 & 1.7321 & 0 \end{bmatrix}$$

Using

$$[U, S, V] = \text{svd}(Y_{\text{csym}})$$

gives

$$Y_{c, \text{sym}} = U S V^T$$

U =

$$\begin{bmatrix} -0.7071 & -0.0884 & -0.7016 \\ -0.6279 & -0.3779 & 0.6804 \\ -0.3252 & 0.9216 & 0.2117 \end{bmatrix}$$

S =

$$\begin{bmatrix} 2.0000 & 0 & 0 \\ 0 & 0.0000 & 0 \\ 0 & 0 & 0.0000 \end{bmatrix}$$

V =

$$\begin{bmatrix} -0.7071 & 0.0640 & 0.7042 \\ -0.6279 & 0.4012 & -0.6669 \\ -0.3252 & -0.9138 & -0.2435 \end{bmatrix}$$

which shows that  $Y_{c, sym}$  can be written as

$$Y_{c, sym} = \begin{bmatrix} -0.7071 \\ -0.6279 \\ -0.3252 \end{bmatrix} [2] \begin{bmatrix} -0.7071 & -0.6279 & -0.3252 \end{bmatrix}$$

which is PR and realized by a 3-input, 1-output transformer loaded in a  $\frac{1}{2}$  ohm resistor.  $Y_{c, skew}$  is realized by 3 gyrators (or a 3-input, 2 output transformer loaded in a gyrator). The last 2 ports of this 3-port parallel connection are loaded in unit capacitors (whose voltages are  $\dot{x}$ ), and  $Y(s)$  is seen at the 1st port (realized by only passive elements).