

10/30/06
ΣΕ 610

State variables:

p. 212 (LTI: constant A, B, C, D)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

u = input; n -vector

x = state; k -vector

y = output; m -vector

Take Laplace transforms, $x(0) = 0$,

$$s\mathcal{L}[x] = A\mathcal{L}[x] + B\mathcal{L}[u]$$

$$\mathcal{L}[y] = C\mathcal{L}[x] + D\mathcal{L}[u]$$

Transfer function: $T(s): \mathcal{L}[y] = T(s)\mathcal{L}[u]$

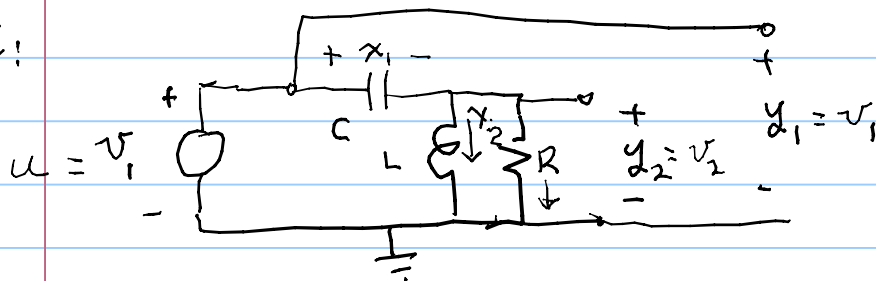
$(sI_k - A)\mathcal{L}[x] = B\mathcal{L}[u]$; if $sI_k - A$ is nonsingular

then $\mathcal{L}[x] = (sI_k - A)^{-1}B\mathcal{L}[u]$

$$\mathcal{L}[y] = \{C(sI_k - A)^{-1}B + D\}\mathcal{L}[u]$$

$$T(s) = D + C(sI_k - A)^{-1}B$$

Ex:



$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$u = [v_1]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ i_L \end{bmatrix}$$

$$C \frac{dx_1}{dt} = C \frac{dv_c}{dt} = \dot{v}_c = \dot{v}_L + \dot{v}_R = x_2 + \frac{1}{R} v_2$$

but $v_1 = x_1 + v_2 \Rightarrow v_2 = -x_1 + u$

or

$$C \frac{dx_1}{dt} = x_2 + \frac{1}{R} (-x_1 + u)$$

$$L \frac{dx_2}{dt} = L \frac{di_L}{dt} = \text{voltage on } L = v_2 = -x_1 + u$$

$$\left. \begin{aligned} \frac{dx_1}{dt} &= \frac{-1}{RC} x_1 + \frac{1}{C} x_2 + \frac{1}{RC} u \\ \frac{dx_2}{dt} &= -\frac{1}{L} x_1 + \frac{1}{L} u \end{aligned} \right\} \Rightarrow \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} u$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} -1/RC & 1/C \\ -1/L & 0 \end{bmatrix} x + \begin{bmatrix} 1/RC \\ 1/L \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \end{aligned}}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u \\ -x_1 + u \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{L}[y_1] \\ \mathcal{L}[y_2] \end{bmatrix} = T(s) [\mathcal{L}[u]] \Rightarrow \begin{bmatrix} T_{11}(s) \\ T_{21}(s) \end{bmatrix} = T(s)$$

$$T(s) = D + C (sI_2 - A)^{-1} B$$

$$(sI_2 - A)^{-1} = \frac{1}{\det(sI_2 - A)} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix};$$

$$\det(sI_2 - A) = \det \begin{bmatrix} s + \frac{1}{RC} & -1/L \\ 1/L & s + 0 \end{bmatrix} = s(s + \frac{1}{RC}) - (-\frac{1}{LC})$$

$$= s^2 + \frac{1}{RC}s + \frac{1}{LC}$$

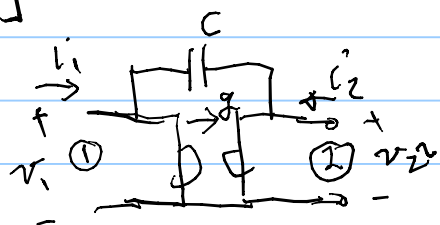
$$sI_2 - A = \begin{bmatrix} s + \frac{1}{RC} & -1/L \\ 1/L & s \end{bmatrix} \Rightarrow (sI_2 - A)^{-1} = \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \begin{bmatrix} s & +1/C \\ -1/L & s + \frac{1}{RC} \end{bmatrix}$$

$$T(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \begin{bmatrix} s & 1/C \\ -1/L & s + \frac{1}{RC} \end{bmatrix} \begin{bmatrix} 1/RC \\ 1/L \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 + \frac{1 \times (-1) \left[\begin{bmatrix} s & 1/C \end{bmatrix} \begin{bmatrix} 1/RC \\ 1/L \end{bmatrix} \right]}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - \frac{(\frac{s}{RC} + \frac{1}{LC})}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \end{bmatrix}$$

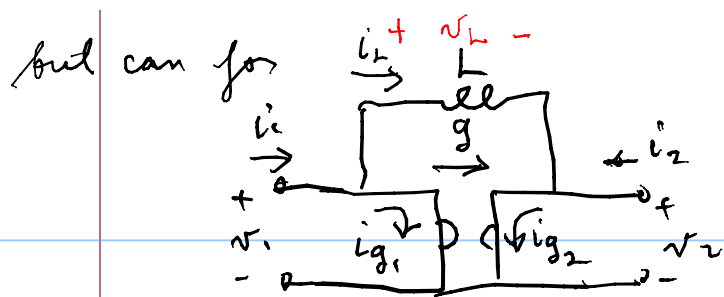
$$= \begin{bmatrix} 1 \\ \frac{s^2}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \end{bmatrix}$$

another example:



$$Y(s) = sC \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$$

for $u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $i = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = y$ then no such state-variable equations for $T(s) \equiv Y(s)$



$$Y(s) = \frac{1}{Ls} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$$

$$\text{let } u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad y = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \quad x = i_L$$

$$v_L = L \frac{di_L}{dt} = L \dot{x} = v_1 - v_2 \Rightarrow \dot{x} = \frac{1}{L} [v_1 - v_2] = \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \Rightarrow \begin{aligned} i_1 &= i_L + i_{g1} = x + g v_2 = x + g u_2 \\ i_2 &= -i_L + i_{g2} = -x - g v_1 \end{aligned}$$

$$y = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} u$$

$$\dot{x} = \begin{bmatrix} 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} u$$

$$Y(s) = D + C(sI_1 - A)^{-1}B = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{sL} \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} + \frac{1}{sL} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{sL} & g - \frac{1}{sL} \\ -g - \frac{1}{sL} & \frac{1}{sL} \end{bmatrix}$$

$$\dot{x} = Ax$$

$$x(0) \neq 0$$

$$x(t) = e^{-At} x(0) ; \quad e^{-At} = \sum_{k=0}^{\infty} \frac{(-At)^k}{k!} \quad ; \quad A \text{ is } k \times k$$

$$e^{-At} = 1_k - \frac{At}{1!} + \frac{(-At)^2}{2!} + \dots$$

$$= 1_k - At + \frac{A^2 t^2}{2} + \dots$$

$$s1_k - A \Rightarrow \quad \mathcal{L}[\dot{x}] = \mathcal{L}[Ax] = A\mathcal{L}[x]$$

$$\parallel$$

$$s\mathcal{L}[x] - x(0) = A\mathcal{L}[x]$$

$$(s1_k - A)\mathcal{L}[x] = x_0$$

$$\mathcal{L}[x] = (s1_k - A)^{-1} x_0$$

$$\Rightarrow x(t) = e^{-At} x_0 \Rightarrow x(t) = \mathcal{L}^{-1}[(s1_k - A)^{-1}] x_0$$

$$e^{-At} = \mathcal{L}^{-1}[(s1_k - A)^{-1}] = \mathcal{L}^{-1}\left[\frac{[\text{cofactors}]}{\det(s1_k - A)}\right]$$

poles _A determine nature of e^{-At} ; poles are zeros of $\det(s1_k - A)$
 of $\frac{[\text{cofactors}]}{\det(s1_k - A)}$

call $\det(s1_k - A)$ the characteristic equation

$P_A(s) = \det(s1_k - A)$; s that are zeros of $P_A(s)$ call natural frequencies as they

are frequencies showing up in e^{-At} .

If we set $\lambda \Rightarrow A \Rightarrow \det(A - \lambda I) = 0$

$\therefore A$ as a matrix satisfies the characteristic equation

$$P_A(\lambda) = \det(\lambda I_k - A) = \text{of degree } k$$

$$\therefore A^k + \dots = 0$$

$$\text{Ex: } A = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}; \quad R = \frac{1}{2}, C = \frac{1}{3}, L = 1$$
$$\Rightarrow A = \begin{bmatrix} -6 & 3 \\ -1 & 0 \end{bmatrix}$$

$$P_A(\lambda) = \det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda + 6 & -3 \\ +1 & \lambda \end{bmatrix} = \lambda^2 + 6\lambda + 3$$

$$\therefore A = \begin{bmatrix} -6 & 3 \\ -1 & 0 \end{bmatrix} \text{ satisfies } A^2 + 6A + 3I_2 = 0$$

$$\text{or } A^2 = -6A - 3I_2 = -6 \begin{bmatrix} -6 & 3 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 33 & -18 \\ 6 & -3 \end{bmatrix}$$

as a check

$$A \cdot A = \begin{bmatrix} -6 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -6 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 36-3 & (-6)(3) \\ -1(-6) & -3 \end{bmatrix} = \begin{bmatrix} 33 & -18 \\ +6 & -3 \end{bmatrix}$$