

10/23/06

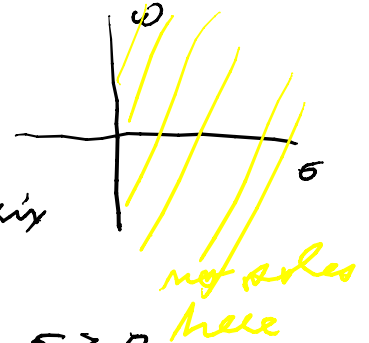
EE 610

If $y(s)$ is PR & scalar then

$S(s)$ is BR $\Rightarrow$ analytic in $\text{Re } s > 0$

and $\frac{1}{n} S^T(s) S(s)$ is

a positive semidefinite matrix
for $\text{Re } s > 0$



if 1×1 matrix then $1 - |S(s)|^2 > 0$ in $\sigma > 0$

$\therefore 1 > |S(s)|^2$ in $\sigma \geq 0$

By the maximum modulus theorem an analytic function in a region assumes the maximum of its magnitude on the boundary of the region.

For a BR function the boundary can be the $j\omega$ axis. Thus $1 \geq |S(j\omega)|^2$ for all real ω .

Ex: $y(s) = AC$; $S(s) = \frac{1-y(s)}{1+y(s)} = \frac{1-AC}{1+AC}$, no pole in $\sigma > 0$ if $C > 0$

$$S(j\omega) = \frac{1-j\omega C}{1+j\omega C}$$

(pole is at $s = -1/C$)

$$|S(j\omega)| = \frac{|1-j\omega C|}{|1+j\omega C|} = \frac{\sqrt{1^2 + (-\omega C)^2}}{\sqrt{1^2 + (\omega C)^2}} = 1$$

Ex: $y(s) = \frac{s^2 + as + a}{s^2 + as + b}$

for what a & b is this PR
here $a \geq 0, b \geq 0$

a) $a=0$ & $b=0$ then $y(s) = 1$ & is PR

b) $b=0, a>0$ then $y(s) = \frac{s^2 + s + a}{a(s+1)} = \frac{s}{a} + 1 + \frac{(a/-1)}{s+1}$

a result is that if extract $j\omega$ axis poles the remainder is PR. Here $\frac{-a}{s+1}$ is not PR

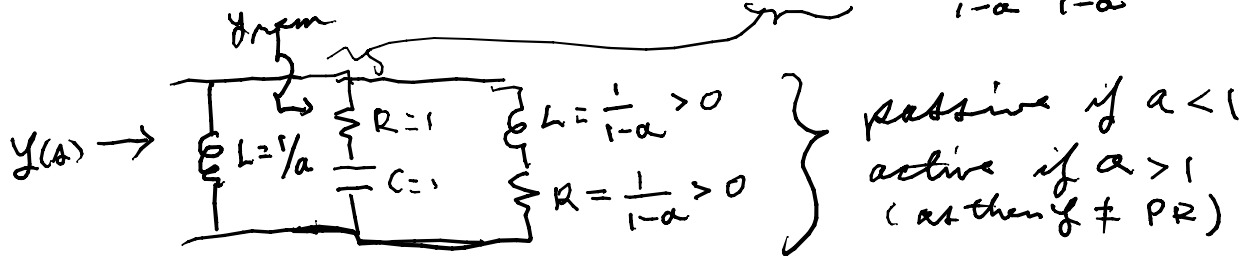
$$y_{rem}(s) = 1 + \frac{-a}{s+1} = \frac{s+1-a}{s+1}$$

$$g(s) = \frac{-a}{s+1} \Rightarrow \begin{cases} L = -1/a \\ R = -1/a \end{cases}$$

is PR if $1-a > 0 \Rightarrow a < 1$

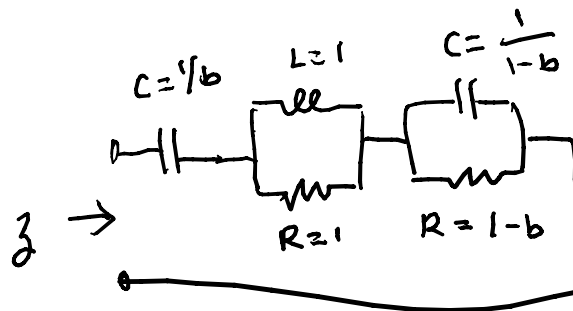
(can not be if $a > 1$ as a zero would be in $\sigma > 0$ at $s = -(1-a)$).

If $a < 1$ then $y_{rem}(s) = \frac{s}{s+1} + \frac{1-a}{s+1} = \frac{1}{1+1/a} + \frac{1}{\frac{s}{1-a} + \frac{1}{1-a}}$



c) $a=0, b>0$

$$z(s) = \frac{1}{y(s)} = \frac{s^2 + s + b}{a(s+1)}$$



passive iff $b \leq 1$

$\therefore y(s)$ is PR if $b \leq 1$ when $a=0$

d) $a>0, b>0$ $y(s) = \frac{s^2 + s + a}{s^2 + s + b}$

for PR, check $\text{Re } y(s)$ in $\sigma > 0$

$$y(s) = \frac{(s+j\omega)^2 + \cancel{6} + j\omega + a}{(s+j\omega)^2 + \cancel{6} + j\omega + b} = \frac{(s^2 - \omega^2 + a + \cancel{6}) + j\omega + 2j\cancel{6}\omega}{(s^2 - \omega^2 + b + \cancel{6}) + j(\omega + 2\cancel{6}\omega)}$$

$$2 \operatorname{Re} y(s) \times |\operatorname{den}|^2 = 2 \operatorname{Re} \left\{ [(s+j\omega)^2 + \cancel{6} + j\omega + a] [(s-j\omega)^2 + \cancel{6} - j\omega + b] \right\}$$

can try to evaluate this $\Rightarrow$ very big mess

Better is to look & see for what a & $b > 0$ is $|S(j\omega)|^2 \leq 1$

$$\text{Here } S(s) = \frac{1-y(s)}{1+y(s)} = \frac{1 - \left(\frac{s^2 + a + a}{s^2 + s + b} \right)}{1 + \left(\frac{s^2 + s + a}{s^2 + s + b} \right)} = \frac{b-a}{2s^2 + 2s + b+a}$$

$$\text{for } s = j\omega: \quad S(j\omega) = \frac{b-a}{-2\omega^2 + b+a + j2\omega} \Rightarrow |S(j\omega)|^2 = \frac{|b-a|^2}{(-2\omega^2 + b+a)^2 + (2\omega)^2}$$

$$|S(j\omega)|^2 = \frac{|b-a|^2}{4\omega^4 - 4\omega^2(b+a-1) + (b+a)^2} = \frac{|b-a|^2}{4\omega^4 + 4\omega^2(1-[b+a]) + (b+a)^2}$$

for BR desire a & b such that this is ≤ 1
 look for minimum of the denominator w.r. ω as this
 maximizes $|S(j\omega)|^2$

$$\begin{aligned} \frac{d \operatorname{den}}{d\omega} &= 16\omega^3 + 8\omega(1-[b+a]) \\ &= 8\omega [2\omega^2 + (1-[b+a])] \Rightarrow 0 \end{aligned}$$

$$\text{at } \omega = 0 \quad |S(0)|^2 = \frac{|b-a|^2}{(b+a)^2} < 1$$

$$\text{zeros @ } \omega = 0; \quad \omega_{\phi, 2} = \pm \frac{1}{\sqrt{2}} \sqrt{(b+a-1)} \Rightarrow \omega_{\phi, 2}^2 = \frac{-1}{2}(1-(b+a))$$

$$\begin{aligned}
 |S(j\omega_{\phi})|^2_{1,2} &= \frac{|b-a|^2}{4 \left[\frac{(-1)^2}{4} (1-[b+a])^2 \right] - \frac{4}{2} (1-[b+a])(1-[b+a]) + (b+a)^2} \\
 &= \frac{|b-a|^2}{(1-[b+a])^2 - 2(1-[b+a]) + (b+a)^2} \\
 &= \frac{|b-a|^2}{-[1-2(b+a) + (b+a)^2] + (b+a)^2} = \frac{|b-a|^2}{2(b+a)-1}
 \end{aligned}$$

as desire $|S(j\omega_{\phi})|^2_{1,2} \leq 1 \Rightarrow |b-a|^2 \leq 2(b+a)-1$

$$b^2 - 2ab + a^2 - 2b - 2a + 1 \leq 0$$

$$f(b,a) = b^2 - 2(1+a)b + [a^2 - 2a + 1] \leq 0$$

here $f(b,a) = 0$ gives boundary:

$$\begin{aligned}
 b &= (1+a) \pm \frac{1}{2} \sqrt{4(1+a)^2 - 4(1-a)^2} = (1+a) \pm \sqrt{4a} \\
 &= (1+a) \pm 2\sqrt{a}, \quad a \geq 0
 \end{aligned}$$

as $b \geq 0$ need choose $b \leq (1+a) + 2\sqrt{a}$ then $S(a) \in \text{BR}$

for any positive a (actually for b between the two values $(1+a) \pm 2\sqrt{a}$ see below)

Next is if $y(a)$ is PR how to construct a circuit.

10/24/06 More detailed result

$$|S(j\omega)|^2 = \frac{|b-a|^2}{4\omega^4 - 4\omega^2[(b+a)-1] + (b+a)^2}$$

If $(b+a) \leq 1$ then the denominator has all terms ≥ 0 and their sum is $\geq (b+a)^2$, as $|b-a|^2/(b+a)^2 < 1$ for $a, b > 0$ then $|S(s)|^2 < 1$ in $\text{Re } s \geq 0$ and $g(s)$ is PR for $b+a < 1$.

For $(a+b) > 1$ we look as above for the minimum of the denominator over ω . This has

$$\frac{d \text{denominator}}{d\omega} = 8\omega(2\omega^2 + [1-(b+a)]) = 0 @ \omega_0 = 0, \omega_{1,2} = \pm \sqrt{\frac{(b+a)-1}{2}}$$

to check if a max or a min

$$\frac{d^2 \text{denominator}}{d\omega^2} = 8(2\omega^2 + [1-(b+a)]) + 32\omega^2 @ \omega_{1,2} \Rightarrow = 32\omega_{1,2}^2 = 32\left(\frac{(b+a)-1}{2}\right) = 16[(b+a)-1] > 0$$

$\therefore \omega_{1,2}$ gives a minimum denominator or $\max_{\omega} |S(j\omega)|^2$,

The denominator value at $\omega_{1,2}$ is

$$\begin{aligned} D(\omega) &= 4\omega_{1,2}^4 - 4\omega_{1,2}^2[(b+a)-1] + (b+a)^2 = \\ &= \frac{4}{4} [(b+a)-1]^2 - \frac{4}{2} [(b+a)-1][(b+a)-1] + (b+a)^2 \\ &= -[(b+a)-1]^2 + (b+a)^2 = 2(b+a)-1 \end{aligned}$$

We derive the range of b & a , $b+a \geq 1$, for which $|S(j\omega_{1,2})|^2 \leq 1$

$$\text{i.e. } |S(j\omega_{1,2})|^2 = \frac{|b-a|^2}{2(b+a)-1} \leq 1 \Rightarrow (b-a)^2 \leq 2(b+a)-1$$

$$\text{or } b^2 - 2ab + a^2 - 2b - 2a + 1 \leq 0 \quad (\text{which is symmetric in } a \text{ \& } b)$$

$$\text{Thus } b^2 - 2(1+a)b + (a-1)^2 \leq 0$$

For this to equal zero $b_{\pm} = (1+a) \pm 2\sqrt{a}$ (as found before)

The function $f(b) = b^2 - 2(1+a)b + (a-1)^2$ is positive for $b \rightarrow \infty$, $b=0$ & has two zeros, one at $b_+ = (1+a) + 2\sqrt{a}$ & one at b_- .

$f(b)$ has a minimum at $\frac{df(b)}{db} = 0 = 2b - 2(1+a) \Rightarrow b = 1+a$

and its value is $(1+a)^2 - 2(1+a)^2 + (1-a)^2 = -(1+a)^2 + (1-a)^2 < 0$ if $a > 0$

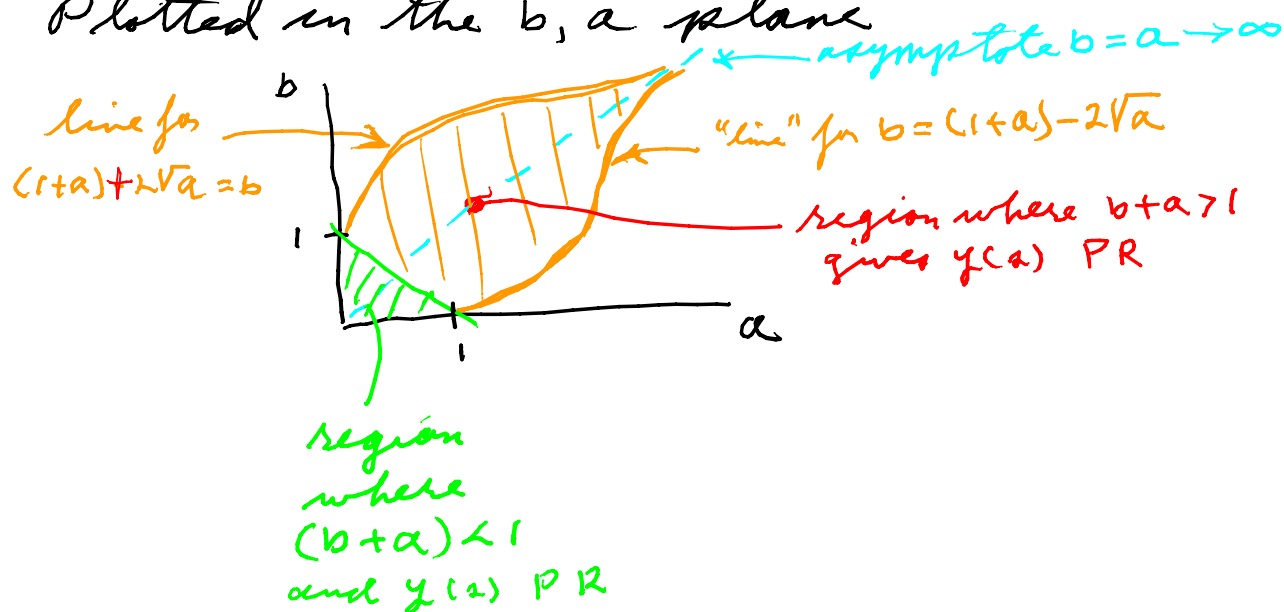
Thus, given $a > 0$ and $b+a \geq 1$ then f is PR if

$$(1+a) - 2\sqrt{a} \leq b \leq (1+a) + 2\sqrt{a}$$

and by symmetry given $b > 0$ and $(b+a) \geq 1$ f is PR if

$$(1+b) - 2\sqrt{b} \leq a \leq (1+b) + 2\sqrt{b}$$

Plotted in the b, a plane



Matlab plot of boundaries, $\Phi(\cdot) = \text{unit step}$

$$b1(a) := (1 + a) + 2 \cdot \sqrt{a}$$

$$b2(a) := [(1 + a) - 2 \cdot \sqrt{a}] \cdot \Phi(a - 1)$$

$$a := 0, 0.1 \dots 10$$

$$b4(a) := (1 - a) \cdot \Phi(1 - a)$$

