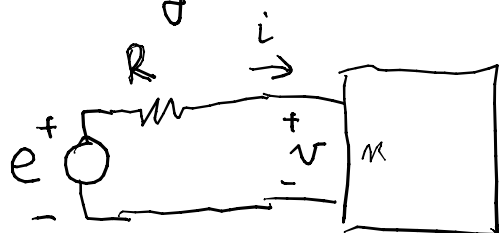


10/18/06
EE610

$$\text{let } y_R(s) = y(s) R(s) = y(s) \left[\frac{2y(s) - 2y(s)}{Ry(s) - 2y(s)} \right]$$

P. L. Richards, Duke Mathematics Journal, 1947,
Vol. 14, #3(?), pp. 777-786

scattering matrix: (use it to prove y_R is PR)



$$e = v + Ri$$

$$\left. \begin{array}{l} 2v^i \triangleq v + Ri = 2e \\ 2v^n \triangleq v - Ri \end{array} \right\} \begin{array}{l} 2v = 2v^i + 2v^n \Rightarrow v = v^i + v^n \\ 2Ri = 2v^i - 2v^n \Rightarrow i = R^{-1}(v^i - v^n) \end{array}$$

$$v^n = S v^i \text{ defines } S$$

$$\frac{v - Ri}{2} = S \left(\frac{v + Ri}{2} \right); \quad i = Yv$$

$$v - RYv = S(v + RYv) \Rightarrow (1_n - RY)v = S(1_n + RY)v$$

$$S = (1_n + RY)^{-1}(1_n - RY); \text{ if } Z = Y^{-1} \text{ exists}$$

$$= [(Y^{-1} + R)Y]^{-1}[Y^{-1} - R]Y = Z(Z + R)^{-1}(Z - R)Y$$

$$\text{note that } (1_n + X)^{-1}(1_n - X) = (1_n - X)(1_n + X)^{-1}$$

as $(1_n - X)(1_n + X) \stackrel{!}{=} (1_n + X)(1_n - X)$
 $1_n - X + X - X^2 \stackrel{!}{=} 1_n + X - X - X^2$

$\therefore$ also $S = (1_n - RY)(1_n + RY)^{-1} = (1_n + RY)^{-1}(1_n - RY)$
 $= (1_n - RZ^{-1})(1_n + RZ^{-1})^{-1}$
 $= [(Z - R)Z^{-1}][(Z + R)Z^{-1}]^{-1}$
 $= (Z - R)Z^{-1}Z(Z + R)^{-1}$

$\Rightarrow S = (Z - R)(Z + R)^{-1}$

no poles of S in $\sigma > 0$
 if $Z(RY)$ is PR

now take $R = Y(k)$

$Z = Y_R$

$S = \frac{y(k) \left[\frac{k y(k) - a y(a)}{k y(a) - a y(k)} \right] - y(k)}{y(k) \left[\frac{k y(k) - a y(a)}{k y(a) - a y(k)} \right] + y(k)}$

let $y(k) = y_k$
 $y(a) = y$

$= \frac{k y_k - a y - k y + a y_k}{k y_k - a y + k y - a y_k} = \frac{(k+a)(y_k - y)}{(k-a)(y_k + y)}$

$= \left(\frac{k+a}{k-a} \right) (-1) \left(\frac{y - y_k}{y + y_k} \right)$

here $k-a$ cancels in
 num & den &
 $k+a$ cancels also.

here S is rational with real coefficients if y is

also S has no poles in $\sigma > 0$ as $\operatorname{Re} y(a) \neq y_k > 0$ when $k > 0$

also $|S(\sigma)| \leq 1$ in $\sigma \geq 0$

$$S = \frac{Z-R}{Z+R}; \quad |S| = \frac{|Z-R|}{|Z+R|} = \frac{\left([(Re Z) - R]^2 + [Im Z]^2 \right)^{1/2}}{\left([Re Z + R]^2 + [Im Z]^2 \right)^{1/2}}$$

as $Re Z > 0$ in $\sigma > 0$ ($\dot{y} \equiv 0$)

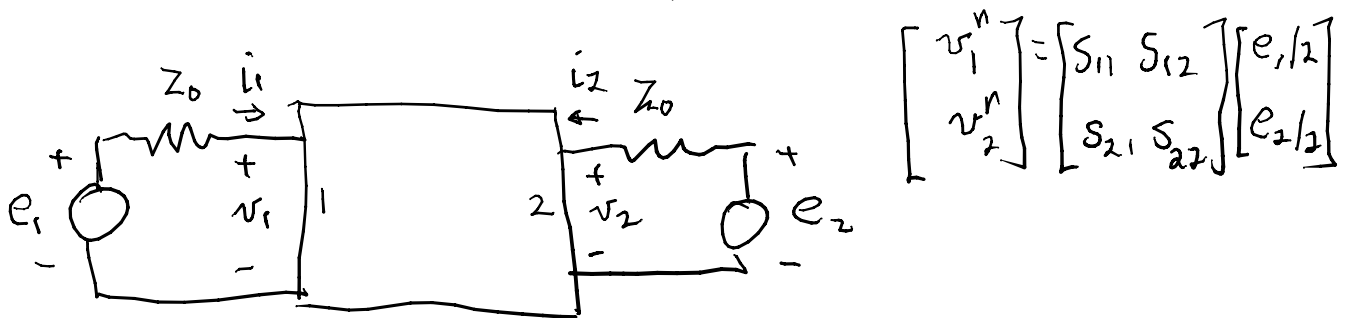
then $|S| \leq 1$

also true for $\sigma = 0$, $|S(j\omega)| \leq 1$

if y is PR then we say S is bounded real, BR
if S is BR then y is PR if rational

so for y PR, S is BR and then so is S for $\mathcal{I}_R$

are interested in $S_{21}, S_{11}, S_{22}, S_{12}$ for RF



$$\begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e_{1/2} \\ e_{2/2} \end{bmatrix}$$

$$2v_1^i = v_1 + Z_0 i_1 = e_1, \quad 2v_2^r = v_2 - Z_0 i_2$$

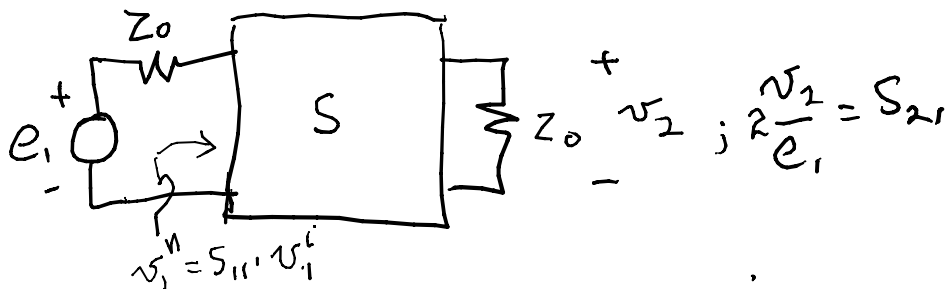
$$\text{choose } e_2 = 0; \quad v_2^r = S_{21} e_{1/2} = \frac{v_2 - Z_0 i_2}{2} \quad e_2 = 0$$

$$\Downarrow$$

$$v_2 = -Z_0 i_2$$

$$v_2^n = S_{21} \frac{e_1}{2} = \frac{v_2 - (-v_2)}{2} = \frac{2v_2}{2} = v_2$$

$$S_{21} = \left. \frac{v_2}{v_1^i} \right|_{v_2^i=0} = \text{load voltage gain of the circuit}$$



$$v^n = S v^i; \text{ if } v_2^i = 0 \text{ then } v_1^n = S_{11} v_1^i \Rightarrow S_{11} = \text{reflection coefficient}$$

$$\begin{aligned} \text{Power}(j\omega) &= \text{Re} \{ V^T I^* \} \\ \left. \begin{aligned} 2V^i &= V + Z_0 I \\ 2V^n &= V - Z_0 I \end{aligned} \right\} \begin{aligned} V &= V^i + V^n \\ I &= Z_0^{-1} (V^i - V^n) \end{aligned} \\ &= \text{Re} \{ (V^{iT*} + V^{nT*}) (Z_0^{-1}) (V^i - V^n) \} \end{aligned}$$

$$\text{if } Z_0 = Z_0 \mathbf{1}_n, \quad Z_0^{-1} = \frac{1}{Z_0} \mathbf{1}_n \Rightarrow \frac{1}{Z_0} \text{Re} \{ (V^{iT*} + V^{nT*}) (V^i - V^n) \}$$

$$\begin{aligned} \text{Power}(j\omega) &= \frac{1}{Z_0} \text{Re} \left[V^{iT*} V^i - V^{nT*} V^n + \underbrace{V^{nT*} V^i}_{\text{as a scalar}} - \underbrace{V^{iT*} V^n}_{\text{purely imaginary}} \right] \\ &= \underbrace{V^{iT*} V^i - V^{nT*} V^n}_{\text{as a scalar}} - \underbrace{(V^{iT*} V^n)^*}_{\text{purely imaginary}} \end{aligned}$$

$$\begin{aligned}
P_{\text{power}}(j\omega) &= \frac{1}{Z_0} \left[V^{iT*} V^i - V^{nT*} V^n \right] \\
&= \frac{1}{Z_0} \left[V^{iT*} V^i - V^{iT*T*} S S V^i \right] \\
&= \frac{1}{Z_0} V^{iT*} \left[I_m - S^{T*} S \right] V^i
\end{aligned}$$

if passive $P(j\omega) \geq 0 \Rightarrow I_m - S^{T*}(j\omega) S(j\omega)$ is a
positive semidefinite Hermitian
form

for BR S , $S^*(j\omega) = S(-j\omega)$

or $I_m - S^T(-j\omega) S(j\omega)$

if lossless then $P(j\omega) = 0 \Rightarrow I_m = S^T(-j\omega) S(j\omega)$

do an analytic continuation from $j\omega$ to $\omega = \sigma/j$

$\therefore I_m = S^T(-A) S(A)$ in $\sigma \geq 0$ & $\sigma < 0$