Solutions

1) a) $y(s) = \frac{2s(s^2+6)(s^2+8)}{(s^2+3)(s^2+7)} \Rightarrow Z(s) = \frac{(s^2+3)(s^2+7)}{2s(s^2+6)(s^2+8)}$ this function has all the properties and conditions of theorem 8.2-3

$$Re\{Z(s)\}\big|_{s=j\omega} = Re\left\{\frac{(3-\omega^2)(7-\omega^2)}{2j\omega(6-\omega^2)(8-\omega^2)}\right\} = 0$$

Since $Re\{Z(j\omega)\}=0$ So $y(s)$ is positive real.

$y(-s) = -y(s)$ So $y(s)$ is lossless

b) $Z(s) = \frac{2s(s^2+6)(s^2+8)}{(s^2+3)(s^2+7)}$ this function has all the conditions of theorem 8.2-3

$Re\{Z(j\omega)\}=0 \Rightarrow Z(s)$ is positive real.

$Z(s)$ has alternating poles and zero So $Z(s)$ is lossless and positive real.

c) $Z(s) = \frac{s(s^2+6)(s^2+8)}{(s^2+3)^2(s+7)}$

C is not PR because of multiple poles at $s = \pm j\sqrt{3}$

so condition C in 8.2-3 is not held so $Z(s)$ is not PR

$Z(-s) \neq -Z(s)$ So $Z(s)$ is not lossless.

d) $Y(s) = \tanh(s) \Rightarrow Z(s) = \frac{\cosh(s)}{\sinh(s)}$ $\cosh(s) = \frac{e^s+e^{-s}}{2}$

$\sinh(s) = \frac{e^s-e^{-s}}{2}$

$$\Rightarrow Z(s) = \frac{e^s+e^{-s}}{e^s-e^{-s}}$$

we know that $e^s = 1+s+s^2/2+s^3/6+\cdots$

$e^{-s} = 1-s+\frac{s^2}{2}-s^3/6+\cdots$

$$Z(s) = \frac{2\left(1+\frac{s^2}{2}+\frac{s^4}{4!}+\cdots\right)}{2\left(s+\frac{s^3}{6}+\frac{s^5}{5!}+\cdots\right)} = \frac{1+\frac{s^2}{2}+\frac{s^4}{4!}+\cdots}{s+\frac{s^3}{6}+\frac{s^5}{5!}+\cdots}$$

$Z(s)$ holds all conditions of theorem 8.2-3

$$Re\{Z(j\omega)\} = Re\left\{\frac{1-\frac{\omega^2}{2}+\frac{\omega^4}{4!}+\cdots}{j\left(\omega-\frac{\omega^3}{6}+\frac{\omega^5}{5!}+\cdots\right)}\right\} \Longrightarrow Re\{Z(j\omega)\}=0$$

$Y(s) = \tanh(s)$ is positive real

Now $\tanh(-s) = -\tanh(s)$ so $Y(s)$ is lossless and positive real.

1) b)

$$Y(s) = \frac{5s(s^2+3)(s^2+7)}{21(s^2+1)(s^2+5)}$$

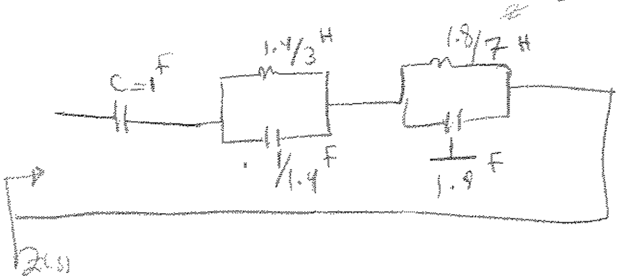
1st foster:

$$Z(s) = \frac{21(s^2+1)(s^2+5)}{5s(s^2+3)(s^2+7)} = \frac{k_1}{s} + \frac{k_2 s}{s^2+3} + \frac{k_3 s}{s^2+7}$$

$$k_1 = s\,Z(s)\Big|_{s=0} = 1$$

$$k_2 = \frac{s^2+3}{s} \times Z(s)\Big|_{s^2=-3} = \frac{21 \times -2 \times 2}{5 \times -3 \times (4)} = \frac{7}{5}$$

$$k_3 = \frac{s^2+7}{s} Z(s)\Big|_{s^2=-7} = \frac{21 \times -6 \times -2}{5 \times -4 \times 7} = \frac{9}{5}$$



2nd order Foster form:

expand $Y(s)$

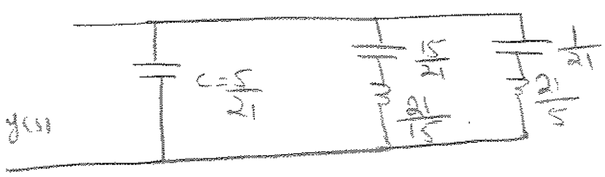
$$Y(s) = \frac{\frac{5}{21}s^5 + \frac{50}{21}s^3 + 5s}{(s^2+1)(s^2+5)} = k_\infty s + \frac{k_1 s}{s^2+1} + \frac{k_2 s}{s^2+5}$$

$$k_1 = \frac{s^2+1}{s} Y(s)\Big|_{s^2=-1} = \frac{5}{21}\frac{(-1+3)(-1+7)}{(-1+5)} = \frac{12}{4} \times \frac{5}{21} = \frac{15}{21}$$

$$k_2 = \frac{s^2+5}{s} Y(s)\Big|_{s^2=-5} = \frac{5}{21}\frac{(-5+3)(-5+7)}{(-5+1)} = \frac{5}{21}$$

$$k_\infty = \frac{5}{21}$$

$$Y(s) = \frac{5}{21}s + \frac{\frac{15}{21}s}{s^2+1} + \frac{\frac{5}{21}s}{s^2+5} = \frac{5}{21}s + \frac{1}{\frac{21}{15}s + \frac{1}{\frac{15}{21}s}} + \frac{1}{\frac{21}{5}s + \frac{1}{\frac{1}{21}s}}$$



1st Cauer form:

$$Z(s) = \frac{21}{5}\frac{(s^2+1)(s^2+5)}{s(s^2+3)(s^2+7)} \Rightarrow y(s) = \frac{5}{21}\frac{s^5+10s^3+21s}{s^4+6s^2+5} = \frac{5s^5+50s^3+105s}{21s^4+126s^2+105}$$

$21s^4+126s^2+105 \,\overline{)\, 5s^5+50s^3+105s}$ $\quad \frac{5}{21}s$

$5s^5+30s^3+25s$

$20s^3+80s \,\overline{)\, 21s^4+126s^2+105}$ $\quad \frac{21}{20}s$

$21s^4+84s^2$

$42s^2+105 \,\overline{)\, 20s^3+80s}$ $\quad \frac{20}{42}s$

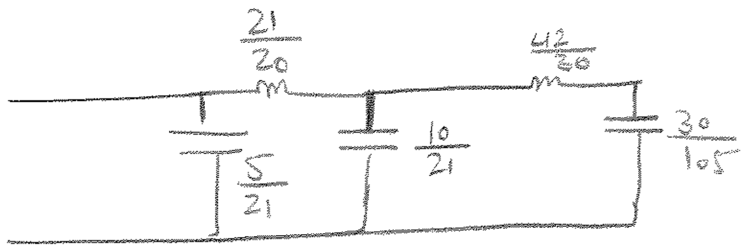
$20s^3+50s$

$30s \,\overline{)\, 42s^2+105}$ $\quad \frac{42}{30}s$

$105 \,\overline{)\, 30s}$ $\quad \frac{30s}{105}$

$30s$

$$\cdot\ y(s) = \frac{5}{21}s + \cfrac{1}{\frac{21}{20}s + \cfrac{1}{\frac{20}{42}s + \cfrac{1}{\frac{42s}{30} + \cfrac{1}{\frac{30}{105}s}}}}$$



2nd Cauer form:

It has a pole at zero

$$K_0 = \frac{21}{5}\,\frac{(0+1)(0+5)}{(0+3)(0+7)} = 1 \qquad \text{so} \quad Z(s) = \frac{1}{s} + \text{whatever is left}$$

$$105s + 50s^3 + 5s^5 \;\big)\; 105 + 126s^2 + 21s^4 \quad \frac{1}{s}$$

$$105 + 50s^2 + 5s^4$$

$$76s^2 + 16s^4 \;\big)\; 105s + 50s^3 + 5s^5 \quad \frac{105}{76s}$$

$$105s + \frac{420}{19}s^3$$

$$\frac{530}{19}s^3 + 5s^5 \;\big)\; 76s^2 + 16s^4 \quad \frac{722}{265}$$

$$76s^2 + \frac{1444}{106}s^4$$

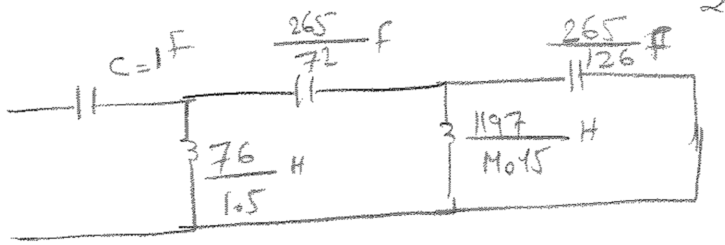
$$\frac{126}{53}s^4 \;\big)\; \frac{530}{19}s^3 + 5s^5 \quad \frac{14045}{1197s}$$

$$\frac{530}{19}s^3$$

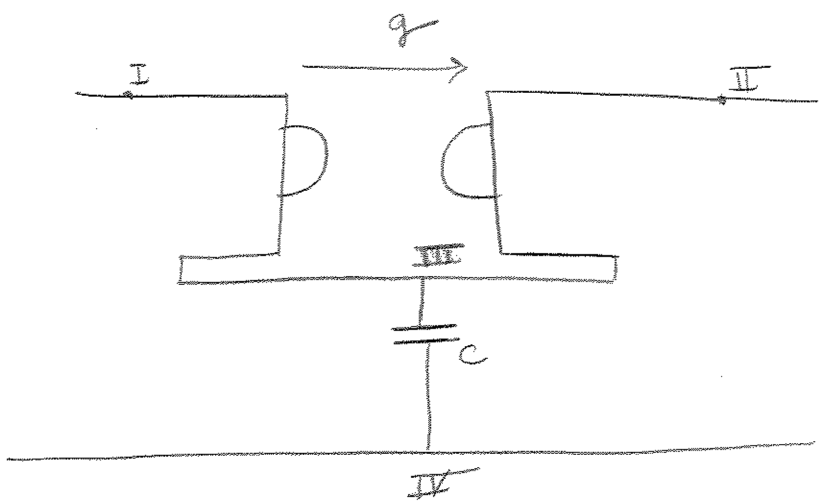
$$5s^5 \;\big)\; \frac{126}{53}s^4 \quad \frac{126}{265s}$$

$$\cancel{\frac{126}{53}}s^4$$

$$Z(s) = \frac{1}{s} + \cfrac{1}{\cfrac{105}{76s} + \cfrac{1}{\cfrac{722}{265s} + \cfrac{1}{\cfrac{14045}{1197s} + \cfrac{1}{\cfrac{126}{265s}}}}}$$



2) Part a)



gyrator: $i_3 = -i_1 - i_2$, $i_2 = -g(V_1 - V_3)$, $i_1 = g(V_2 - V_3)$

$$\Longrightarrow Y_g = \begin{bmatrix} 0 & g & -g & 0 \\ -g & 0 & g & 0 \\ g & -g & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Capacitor $\quad i_3 = sC(V_3 - V_4)$, $i_4 = sC(V_4 - V_3)$

Now we choose node 4 as ground then $\quad Y = \begin{bmatrix} 0 & g & -g \\ -g & 0 & g \\ g & -g & sC \end{bmatrix}$

From y: $i_3 = gV_1 - gV_2 + scV_3$

set $i_3 = 0$ $\Rightarrow scV_3 = gV_2 - gV_1$ $\Rightarrow V_3 = \frac{g}{sc}V_2 - \frac{g}{sc}V_1$

$$\begin{cases} i_1 = gV_2 - gV_3 = gV_2 - g\left(\frac{g}{sc}V_2 - \frac{g}{sc}V_1\right) = \frac{g^2}{sc}V_1 + \left(g - \frac{g^2}{sc}\right)V_2 \\ i_2 = -gV_1 + g\left(\frac{g}{sc}V_2 - \frac{g}{sc}V_1\right) = \left(-g - \frac{g^2}{sc}\right)V_1 + \frac{g^2}{sc}V_2 \end{cases}$$

$$Y = \begin{bmatrix} \frac{g^2}{sc} & g - \frac{g^2}{sc} \\ -g - \frac{g^2}{sc} & \frac{g^2}{sc} \end{bmatrix}$$

$$y_L(s) = \frac{y_{in} \cdot y_{22} - \mathrm{Det}(y)}{y_{11} - y_{in}}$$

$$\mathrm{Det}(y) = \frac{g^2}{sc} \times \frac{g^2}{sc} - \left(-g - \frac{g^2}{sc}\right)\left(g - \frac{g^2}{sc}\right) = \frac{g^4}{s^2c^2} - \left(\frac{g^4}{s^2c^2} - g^2\right) = g^2$$

$$Y_L(s) = \frac{y_{in}\, g^2/sc - g^2}{g^2/sc - y_{in}} = \frac{g^2\left(\frac{1}{sc}y_{in} - 1\right)}{g^2/sc - y_{in}}$$

Richard function $R(s) = \frac{kZ(s) - sZ(k)}{k(Z(k) - sZ(s))}$

$$Y_L(s) = \frac{g^2\left(\frac{1}{sc} - \frac{1}{y_{in}}\right)}{\frac{g^2}{sc}\frac{1}{y_{in}} - 1} = \frac{g^2\left(\frac{1}{sc} - Z_{in}\right)}{\frac{g^2}{sc}Z_{in} - 1} = \frac{g^2 - g^2 sc Z_{in}}{g^2 Z_{in} - sc}$$

$$Z_L(s) = \frac{1}{Y_L(s)} = \frac{g^2 Z_{in}(s) - sc}{g^2 - g^2 sc Z_{in}(s)} = \frac{g^2 c\left(\frac{1}{c} Z_{in}(s) - \frac{s}{g^2}\right)}{g^2 c\left(\frac{1}{c} - s Z_{in}(s)\right)} = \frac{\frac{1}{c} Z_{in}(s) - \frac{s}{g^2}}{\frac{1}{c} - s Z_{in}(s)}$$

Comparing this with the Richards function:

$Z(s) = Z_{in}(s)$, $k = 1/c$, $Z(k) = \frac{1}{g^2}$ and $kZ(k) = \frac{1}{c}$

So $$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)} = \frac{\frac{1}{c} Z_{in}(s) - \frac{s}{g^2}}{\frac{1}{c} - s Z_{in}(s)} = Z_L(s)$$

2) part b) $$y(s) = \frac{2(s^2+2s+4)}{(s^2+s+6.25)}$$

$$y(-s) = \frac{2(s^2-2s+4)}{s^2-s+6.25}$$

$$Ev\{y(s)\} = \frac{y(s)+y(-s)}{2}$$ solve for the zeros of $Ev\{y(s)\}$

$\Rightarrow k = -0.6614 + 2.136j \Rightarrow y(k) = -2.3812 + j0.8544$

$$y(k) = 2\,\frac{k^2+2k+4}{k^2+k+6.25}$$

$g_1(k) = y(k) = -2.3812 + j0.8544$

$$c_1(k) = \frac{y(k)}{k} = 0.68 + 0.9042j$$

$$R(s) = \frac{sy(s) - ky(k)}{sy(k) - ky(s)} = \frac{s \cdot 2\,\frac{s^2+2s+4}{s^2+s+6.25} - ky(k)}{sy(k) - k \cdot 2\,\frac{s^2+2s+4}{s^2+s+6.25}}$$

$$y_L(s) = \frac{y(k)}{R(s)}$$

using Matlab

$$R(s) = \frac{-0.74411\,s - 0.82679 - 2.670001j - 0.2670001\,j\cdot s}{s - 0.62499\times10^{-1} + 1.41283j}$$

$$Ev(y_L(s)) = 0 \Rightarrow \frac{y_L(s) + y_L(-s)}{2} = 0 \Rightarrow s = 0.6614378 - 2.1360009j$$

$s = k^*$ this number is equal to conjugate of k

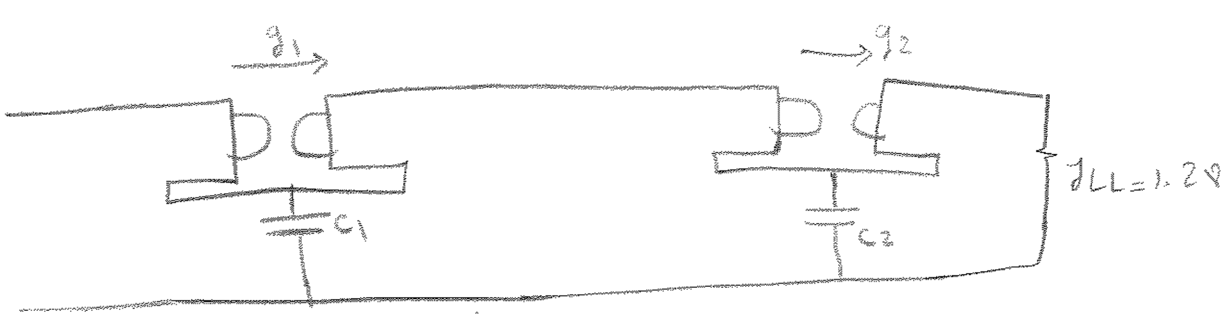
$$y_L(k^*) = -1.90494 + 0.68352j$$

$$g_2(k) = y_L(k^*) = -1.90494 + 0.68352j$$

$$c_2(k) = \frac{y_L(k^*)}{k^*} = -0.04 - 0.90421j$$

$$R(s) = \frac{s\,y_L(s) - k^* y_L(k^*)}{s\,y_L(k^*) - k^* y_L(s)}, \quad y_{LL}(s) = \frac{y_L(k^*)}{R(s)}$$

$$R(1) = -1.48824 + 0.534j \Rightarrow y_{LL}(1) = 1.28$$



$g_1 = -2.38118 + 0.8544j$

$C_1 = 0.68 + 0.90421j$

$g_2 = -1.90494 + 0.68352j$

$C_2 = -0.04 - 0.90421j$