Solutions

PROBLEM #1

PART A)

From previous hw

$$Y(s) = \frac{1}{s(C_1 g_2^2 + C_2 g_1^2)} \begin{bmatrix} s^2 C_1 C_2 g_1^2 + g_1^2 g_2^2 & -s^2 C_1 C_2 g_1 g_2 + s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 \\ -s^2 C_1 C_2 g_1 g_2 - s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 & s^2 C_1 C_2 g_2^2 + g_1^2 g_2^2 \end{bmatrix}$$

$$-Y(-s)^T = \frac{1}{s(C_1 g_2^2 + C_2 g_1^2)} \begin{bmatrix} s^2 C_1 C_2 g_1^2 + g_1^2 g_2^2 & -s^2 C_1 C_2 g_1 g_2 + s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 \\ -s^2 C_1 C_2 g_1 g_2 - s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 & s^2 C_1 C_2 g_2^2 + g_1^2 g_2^2 \end{bmatrix}$$

Comparing the elements of both matrices

Element (1,1) $\Rightarrow$ $s^2 C_1 C_2 g_1^2 + g_1^2 g_2^2 = s^2 C_1 C_2 g_1^2 + g_1^2 g_2^2$

Element (1,2) $\Rightarrow$ $-s^2 C_1 C_2 g_1 g_2 + s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 = -s^2 C_1 C_2 g_1 g_2 + s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2$

Element (2,1) $\Rightarrow$ $-s^2 C_1 C_2 g_1 g_2 - s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 = -s^2 C_1 C_2 g_1 g_2 - s(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2$

Element (2,2) $\Rightarrow$ $s^2 C_1 C_2 g_2^2 + g_1^2 g_2^2 = s^2 C_1 C_2 g_2^2 + g_1^2 g_2^2$

$\therefore$ All values of C_1, C_2, g_1, and g_2 satisfy the above equations.

PART B)

Must satisfy the following

1) $Y(s)$ must be real valued for $\sigma > 0$ if C_1, C_2, g_1 & g_2 are real valued
2) There exists a pole @ ∞, assuming ∞ is on $j\omega$ axis there is no pole in $\sigma > 0$
3) $Y(s) + Y^{T*}(s)$ is positive semidefinite in $\sigma > 0$

$$Y^{T*}(s) = \frac{1}{s^*(C_1 g_2^2 + C_2 g_1^2)} \begin{bmatrix} (s^*)^2 C_1 C_2 g_1^2 + g_1^2 g_2^2 & -(s^*)^2 C_1 C_2 g_1 g_2 - s^*(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 \\ -(s^*)^2 C_1 C_2 g_1 g_2 + s^*(C_1 g_1 g_2^2 + C_2 g_1^2 g_2) - g_1^2 g_2^2 & (s^*)^2 C_1 C_2 g_2^2 + g_1^2 g_2^2 \end{bmatrix}$$

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$$Y(s) = \frac{sC_1C_2g_1^2}{C_1g_2^2+C_2g_1^2}\begin{bmatrix} 1 & -g_2/g_1 \\ -g_2/g_1 & (g_2/g_1)^2 \end{bmatrix} + \frac{C_1g_1g_2^2 + C_2g_1^2g_2}{C_1g_2^2+C_2g_1^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{g_1^2g_2^2}{s(C_1g_2^2+C_2g_1^2)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Y^{T*}(s) = \frac{s^*C_1C_2g_1^2}{C_1g_2^2+C_2g_1^2}\begin{bmatrix} 1 & -g_2/g_1 \\ -g_2/g_1 & (g_2/g_1)^2 \end{bmatrix} + \frac{C_1g_1g_2^2 + C_2g_1g_2}{C_1g_2^2+C_2g_1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{g_1^2g_2^2}{s^*(C_1g_2^2+C_2g_1^2)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where $s = \sigma + j\omega$

$$Y(s) + Y^{T*}(s) = \frac{2\sigma C_1C_2g_1^2}{C_1g_2^2+C_2g_1^2}\begin{bmatrix} 1 & -g_2/g_1 \\ -g_2/g_1 & (g_2/g_1)^2 \end{bmatrix} + \frac{2\sigma g_1^2g_2^2}{(C_1g_2^2+C_2g_1^2)(\sigma^2+\omega^2)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Let $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_{1R} + jV_{1I} \\ V_{2R} + jV_{2I} \end{bmatrix}$

$V^{T*} = [V_{1R} - jV_{1I}, V_{2R} - jV_{2I}]$

PROVE $V^{T*}HV \geq 0$ For $Y(s) + Y^{T*}(s)$ To be positive semidefinite

$$[V_{1R} - jV_{1I}, V_{2R} - jV_{2I}] \times H = \frac{2\sigma C_1C_2g_1^2}{C_1g_2^2+C_2g_1^2}\left[V_1^* - \frac{g_2}{g_1}V_2^*, -\frac{g_2}{g_1}V_1^* + \left(\frac{g_2}{g_1}\right)^2 V_2^*\right] +$$

$$\frac{2\sigma g_1^2g_2^2}{(C_1g_2^2+C_2g_1^2)(\sigma^2+\omega^2)}[V_1^* - V_2^*, -V_1^* + V_2^*]$$

$$V^{T*}HV = \overbrace{\frac{2\sigma C_1C_2g_1^2}{C_1g_2^2+C_2g_1^2}}^{\substack{\text{const} \\ N}} \overbrace{\left[V_1^*V_1 - \frac{g_2}{g_1}V_2^*V_1 - \frac{g_2}{g_1}V_1^*V_2 + \left(\frac{g_2}{g_1}\right)^2 V_2^*V_2\right]}^{A} +$$

$$+ \underbrace{\frac{2\sigma g_1^2g_2^2}{(C_1g_2^2+C_2g_1^2)(\sigma^2+\omega^2)}}_{\substack{\text{const} \\ M}} \underbrace{[V_1^*V_1 - V_2^*V_1 - V_1^*V_2 + V_2^*V_2]}_{B}$$

$V_1^*V_1 = V_{1R}^2 + V_{1I}^2$

$V_2^*V_2 = V_{2R}^2 + V_{2I}^2$

$$V_2^* V_1 + V_1^* V_2 = (V_{1R} - jV_{1I})(V_{2R} + jV_{2I}) + (V_{1R} + jV_{1I})(V_{2R} - jV_{2I})$$
$$= 2Re(V_1^* V_2)$$
$$= 2(V_{1R}V_{2R} + V_{1I}V_{2I})$$

$$NA = N\left[V_{1R}^2 + V_{1I}^2 - \frac{g_2}{g_1} 2(V_{1R}V_{2R} + V_{1I}V_{2I}) + \left(\frac{g_2}{g_1}\right)^2 (V_{2R}^2 + V_{2I}^2)\right]$$
$$MB = M\left[V_{1R}^2 + V_{1I}^2 - 2Re(V_1^* V_2) + V_{2R}^2 + V_{2I}^2\right]$$
$$= M\left[(V_{1R} - V_{2R})^2 + (V_{1I} - V_{2I})^2\right] \geq 0$$

g_1 & g_2 = Anything but c_1 & $c_2 > 0$

Because $m > 0$ & $B > 0$ for $\sigma > 0$

$$NA = n\left[\left(V_{1R} - \frac{g_2}{g_1} V_{2R}\right)^2 + V_{1I} - \frac{g_2}{g_1} V_{2I})^2\right] \geq 0$$

Because $n > 0$ & $A > 0$ for $\sigma > 0$

Thus, $Y(s)$ is positive real for any g_1, g_2 values as long as C_1 and C_2 are both non-negative values > 0.

PART C

Lossless $\rightarrow$ $Y(s) + Y^T(-s) = 0$

Using part (a)

$$Y^T(-s) = \frac{1}{s(c_1 g_2^2 + c_2 g_1^2)} \begin{bmatrix} -s^2 c_1 c_2 g_1^2 - g_1^2 g_2^2 & s^2 c_1 c_2 g_1 g_2 - s(c_1 g_1 g_2^2 + c_2 g_1^2 g_2) + g_1^2 g_2^2 \\ s^2 c_1 c_2 g_1 g_2 - s(c_1 g_1 g_2^2 + c_2 g_1^2 g_2) - g_1^2 g_2^2 & s^2 c_1 c_2 g_2^2 + g_1^2 g_2^2 \end{bmatrix}$$

$$Y(s) = \frac{1}{s(c_1 g_2^2 + c_2 g_1^2)} \begin{bmatrix} s^2 c_1 c_2 g_1^2 + g_1^2 g_2^2 & -s^2 c_1 c_2 g_1 g_2 - s(c_1 g_1 g_2^2 + c_2 g_1^2 g_2) + g_1^2 g_2^2 \\ -s^2 c_1 c_2 g_1 g_2 - s(c_1 g_1 g_2^2 + c_2 g_1^2 g_2) - g_1^2 g_2^2 & s^2 c_1 c_2 g_2^2 + g_1^2 g_2^2 \end{bmatrix}$$

so $y(s) + y^T(-s) = 0$

$Y(s)$ is lossless for any value of c_1, c_2, g_1 & g_2
$\therefore$ this implies the pos. real constraint when also considering part (b)

Therefore combining the efforts of part (b) and (c), $Y(s)$ is positive real & lossless for any g_1 and g_2 values as long as c_1 and c_2 are positive.

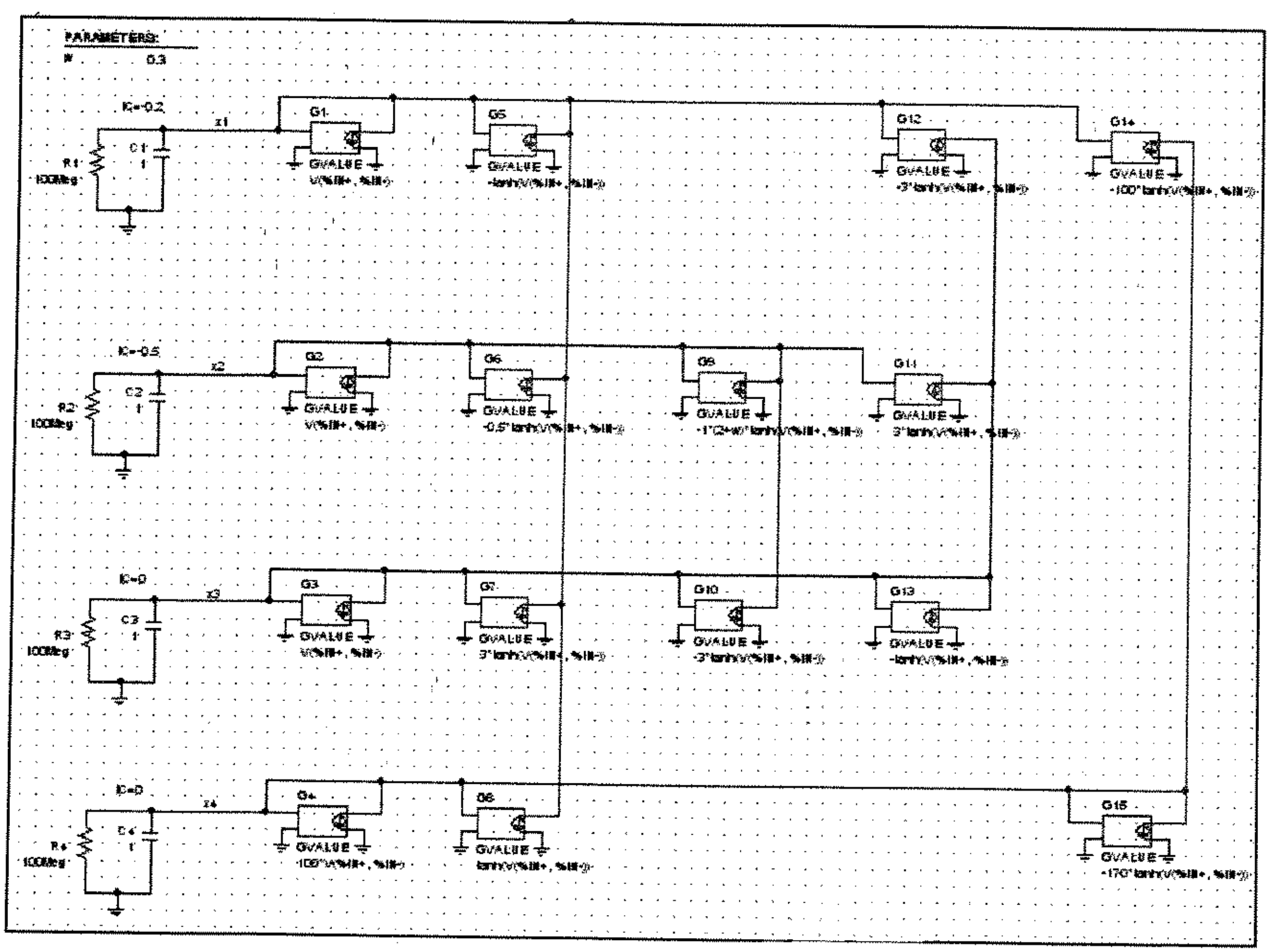
Problem #2

$$\frac{dx}{dt} = Ax + Wv \quad , \quad x(0) = [-0.2, -0.5, 0, 0]^T$$
$$v = \tanh(x)$$

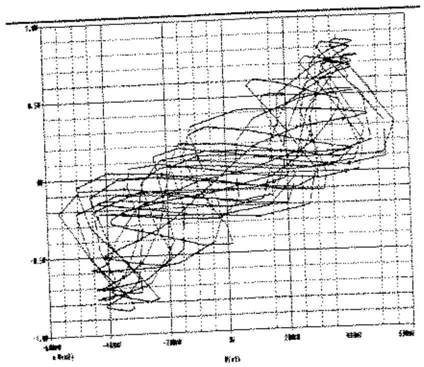
$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -100 \end{bmatrix} , \quad W = \begin{bmatrix} 1 & 0.5 & -3 & -1 \\ 0 & 2+\omega & 3 & 0 \\ 3 & -3 & 1 & 0 \\ 100 & 0 & 0 & 170 \end{bmatrix}$$

(a)
$$\frac{dx_1}{dt} = -x_1 + (\tanh x_1 + \tfrac{1}{2}\tanh x_2 - 3\tanh x_3 - \tanh x_4)$$
$$\frac{dx_2}{dt} = -x_2 + (2+\omega)\tanh x_2 + 3\tanh x_3 \qquad x_1(0) = -0.2$$
$$\frac{dx_3}{dt} = -x_3 + 3(\tanh x_1 - \tanh x_2) + \tanh x_3 \qquad x_2(0) = -0.5$$
$$\frac{dx_4}{dt} = -100x_4 + 100\tanh x_1 + 170\tanh x_4 \qquad x_3(0) = 0$$
$$x_4(0) = 0$$

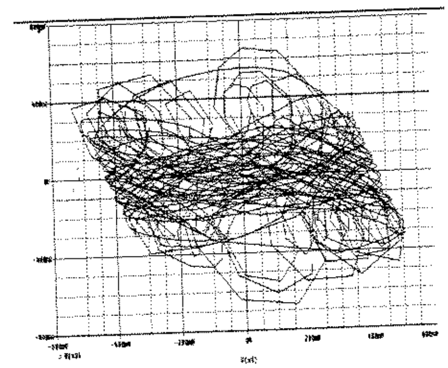
Schematic:



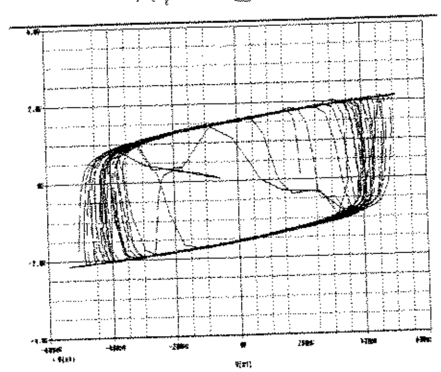
X_2 vs X_1



X_3 vs X_1



X_4 vs X_1

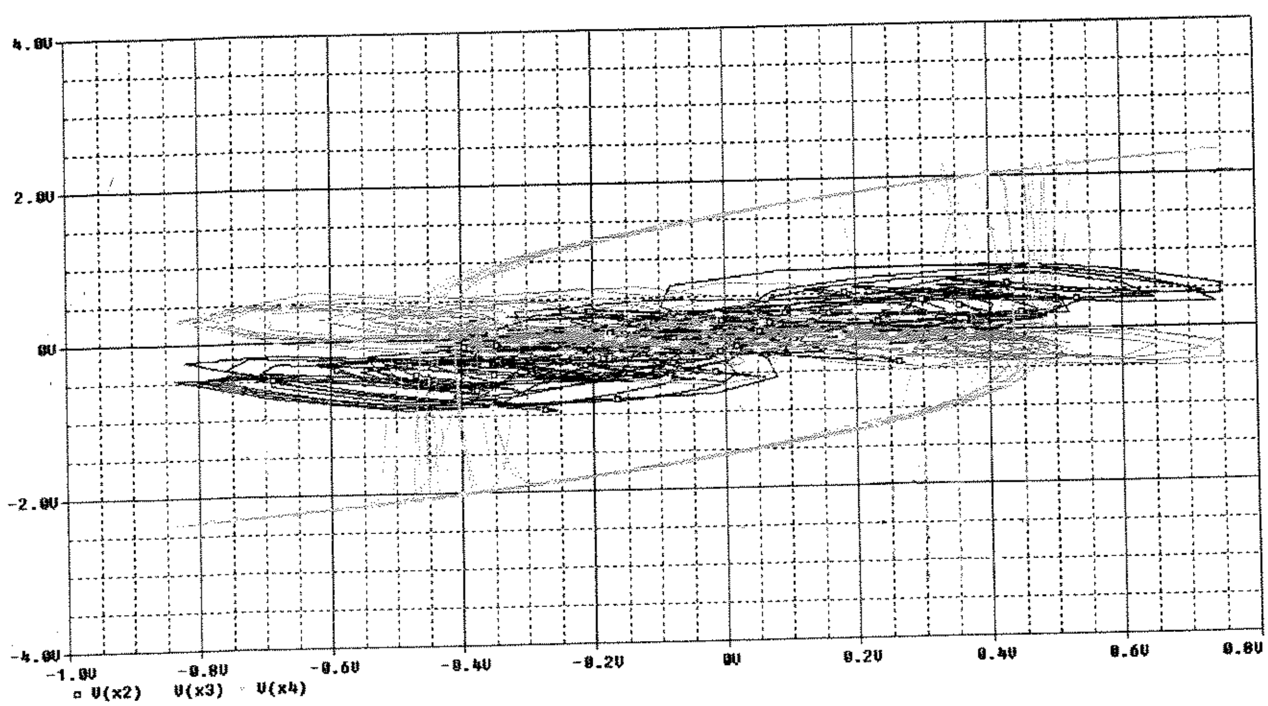


($w = -0.4$)

definite limit cycle

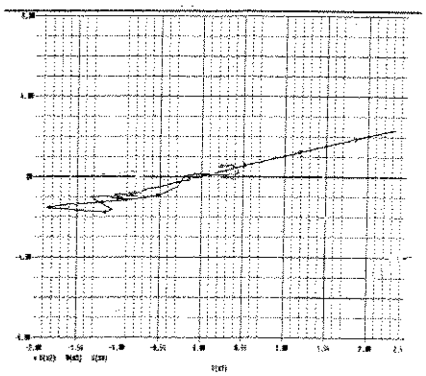
(b) Schematic:

$w = 0.3$ - CHAOS

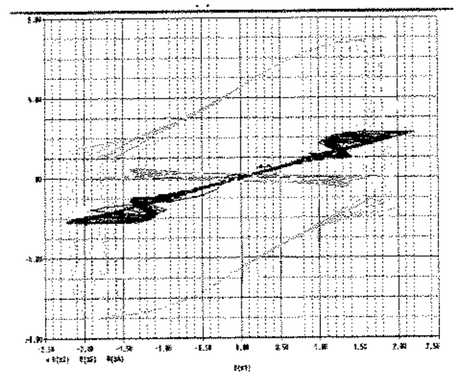


c)

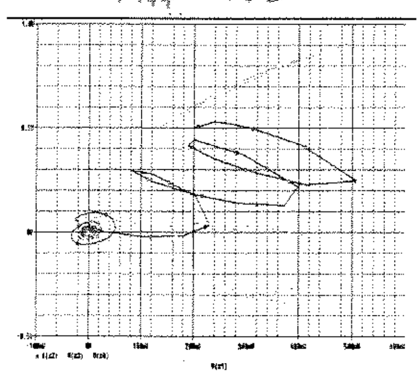
$A_{44} = -37$



$A_{44} = -38$



$A_{44} = -185$



The chaotic behavior is seen to increase from $A_{44} = -38$ to $A_{44} = -185$.