Van der Pol equation: 09/19/05

$$\ddot{x} + \underbrace{\epsilon(x^2-1)}_{\text{damping}}\dot{x} + \omega_0^2 x = 0 \quad ; \quad \left.\begin{array}{l} x(0) = \\ \dot{x}(0) = \end{array}\right\} \text{given}$$

damping $\left\{ \Rightarrow \begin{array}{l} \text{stable if } \epsilon(x^2-1) > 0 \\ \text{unstable if } \epsilon(x^2-1) < 0 \end{array}\right.$

(linear if $\epsilon = 0$) if $\epsilon \neq 0$ is structurally stable (comes back to a "fixed" oscillation for any $\neq 0$ initial conditions)

$$\epsilon(x^2-1)\dot{x} = \frac{d}{dt}\left[\epsilon g(x)\right]$$

$$= \epsilon \frac{dg(x)}{dx} \cdot \frac{dx}{dt} \; ; \; g(x); \; \frac{dg(x)}{dx} = x^2 - 1 \Rightarrow \frac{x^3}{3} - x = g(x)$$

$$\frac{d}{dt}\frac{dx}{dt} + \frac{d}{dt}\left[\epsilon\left(\frac{x^3}{3} - x\right)\right] + \omega_0^2 x = 0$$

$$= \frac{d}{dt}\left\{\frac{dx}{dt} + \epsilon\left(\frac{x^3}{3} - x\right)\right\} + \omega_0^2 x = 0$$

set $x_1 = \dfrac{dx}{dt} + \epsilon\underbrace{\left(\dfrac{x^3}{3} - x\right)}_{g(x)}$

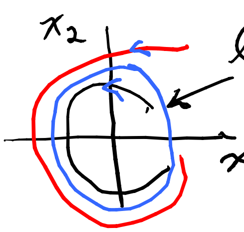
and $\dfrac{dx_1}{dt} + \omega_0^2 x = 0$

Let $x_2 = x$

$$\frac{dx_1}{dt} = -\omega_0^2 x_2$$

$$\frac{dx_2}{dt} = x_1 - \epsilon\left(\frac{x_2^{\cancel{1}\,3}}{3} - x_2\right)$$

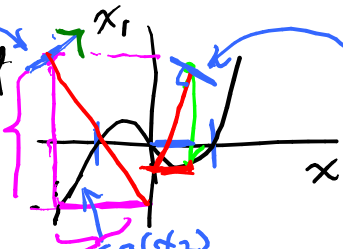
$$\frac{d\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{dt} = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\epsilon g(x) \end{bmatrix}$$



$$\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_1 - \epsilon g(x_2)}{-\omega_0^2 x_{\cancel{1}\,2}}$$

$$\frac{dx_1}{dx_2} = \frac{-\omega_0^2 x_2}{x_1 - \epsilon g(x_2)}$$

look at



normalize $\omega_0 = 1$

To normalize $\omega_0 = 1$:

$$\frac{d^2x}{dt\,dt} + \epsilon f(x)\frac{dx}{dt} + \omega_0^2 x = 0 \; , \quad \text{let } \hat{t} = \Omega t \Rightarrow t = \frac{1}{\Omega}\hat{t}$$

$$\Omega^2 \frac{d^2x}{d\hat{t}^2} + \Omega \epsilon f(x) \frac{dx}{d\hat{t}} + \omega_0^2 x = 0 \qquad \div \text{ by } \Omega^2$$

$$\Rightarrow \frac{d^2x}{d\hat{t}^2} + \frac{\epsilon f(x)}{\Omega} \frac{dx}{d\hat{t}} + \frac{\omega_0^2}{\Omega^2} x = 0 \qquad \Rightarrow \Omega = \omega_0$$

$$\frac{d^2x}{d\hat{t}^2} + \frac{\epsilon}{\Omega} \cdot f(x) \frac{dx}{d\hat{t}} + x = 0$$

Now set up PSpice

$$\frac{dx_1}{dt} = -\omega_0^2 x_2$$

$$\frac{dx_2}{dt} = x_1 - \epsilon g(x_2) \qquad g(x) = \frac{1}{3}x^3 - x$$