

1. Given $f(s)$ defined as a function of the complex variable s and analytic for all s with $\text{Re}(s) > 0$, a **Richards' function** $r(s, f)$ is defined by [text, p. 361]

$$r(s, f) = f(k) \frac{kf(s) - sf(k)}{kf(k) - sf(s)}$$

This is useful since it is a positive real function of a positive real function, $f(\cdot)$, if k is real and positive, in which case the multiplier $f(k)$ is a positive scale factor. Useful $f(s)$ are admittances $y(s)$ and impedances $z(s)$.

(a) Show that for $f(s)$ rational with real coefficients and k real and positive, $s-k$ is a factor of the numerator and denominator of $r(s, f)$. Further $s+k$ is such a factor if k is a zero of the even part of $f(s)$, that is k satisfies $f(k) + f(-k) = 0$, so that $r(s)$ has one less degree than $f(s)$.

(b) Besides $r(s, f)$ there are a number of other forms of the Richards' function:

$$1/r(s, f), r(s, 1/f), 1/r(s, 1/f)$$

Show that these all have the same degree reduction properties as $r(s, f)$. Show also that they are all positive real with $f(s)$ for k real and positive.

(c) Show that if $f(s)$ is an odd function, that is, $f(s) = -f(-s)$, then any real k [that is not a pole of $f(s)$] can be chosen to yield a degree reduction. Use this to give a cascade synthesis of reactance functions [text, p. 343] (reactance functions are odd positive real rational functions) using a cascade of the gyrator-capacitor 2-ports covered in class. Carry this out on examples 8.3-1 and 8.3-2 [text, pp. 344, 347] using $y(s) = 1/z(s)$.

(d) [hard] Show that $r(s, f)$ is defined and analytic for all s with $\text{Re}(s) > 0$ if the constant k satisfies $\text{Re}(k) > 0$ and that $s-k$ cancels [here k may be complex]. Show that if k is a zero of the even part of $f(s)$ then when $f(s)$ is rational $s+k$ cancels, but it may not if $f(s)$ is not rational [consider for example the positive real function $f(s) = \sqrt{s}$]. Use this to give a synthesis of rational $y(s)$ by cascading two sections found by the use of the Richards' function at k and then at the conjugate of k on the load; the gyrators and capacitors will be complex valued for each of the two cascaded 2-port coupling sections but these can be combined into one real 2-port [see the book by Hazony for details].

(e) [for microwave engineers] The Richards' function is also useful for design on a

scattering basis. There $f(s) = \pm \frac{S(s)-1}{S(s)+1}$ and $r(s, f) = \pm \frac{S_r(s)-1}{S_r(s)+1}$ would be substituted with the +

sign for f an impedance and the - sign for f an admittance and with $S(s)$ the (input) reflection coefficient and $S_r(s)$ the (load) "Richard's" reflection coefficient. Investigate the properties of $S_r(s)$; it should be bounded real if $f(s)$ is positive real and rational if $f(s)$ is rational. Investigate a cascade synthesis using the gyrator-C 2-port evaluated on a scattering basis. Here the scattering variable are: incident voltage $2v^i = v + i$, reflected voltage $2v^r = v - i$, and the Richard's function should be something like $S_r(s) = \frac{(s+k)}{(s-k)} S(s)$.