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Oscillations, SE(2)-snakes and motion control: a study of the Roller Racer

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Abstract. This paper is concerned with the problem of motion generation via cyclic variations in selected degrees of freedom (usually referred to as shape variables) in mechanical systems subject to non-holonomic constraints (here the classical system of a disc rolling without sliding on a flat surface). In earlier work, we identified an interesting class of such problems arising in the setting of Lie groups, and investigated these under a hypothesis on constraints, that naturally led to a purely kinematic approach. In the present work, the hypothesis on constraints does not hold, and as a consequence, it is necessary to take into account certain dynamical phenomena. Specifically we concern ourselves with the group $SE(2)$ of rigid motions in the plane and a concrete mechanical realization dubbed the 2-node, 1-module $SE(2)$-snake. In a restricted version, it is also known as the Roller Racer (a patented ride/toy). Based on the work of Bloch, Krishnaprasad, Marsden and Murray, one recognizes in the example of this paper a balance law called the momentum equation, which is a direct consequence of the interaction of the $SE(2)$-symmetry of the problem with the constraints. The systematic use of this type of balance law results in certain structures in the example of this paper. We exploit these structures to demonstrate that the single shape freedom in this problem can be cyclically varied to produce a rich variety of motions of the $SE(2)$-snake. In their study of the snakeboard, a patented modification of the skateboard that also admits the group $SE(2)$ as a symmetry group, Lewis, Ostrowski, Burdick and Murray exploited the same type of balance law as that discussed here to generate motions. A key difference, however, is that, in the present paper, we have only one control variable and thus controllability considerations become somewhat more delicate. In the present paper, we give a self-contained treatment of the geometry, mechanics and motion control of the Roller Racer.

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1. Introduction

The idea of using periodic driving signals to produce rectified movement appears in a number of settings in engineering. Some of the more inventive examples are associated with the design and operation of novel actuators exploiting vibratory transduction (Ueha and Tomikawa 1993, Venkataraman *et al.* 1995). Brockett (1989) develops a mathematical basis for understanding such devices. Elsewhere, in the context of robotic machines with many degrees of freedom designed to mimic snake-like movements (Hirose 1993), periodic variations in the shape parameters are used in an essential way to generate global movements. In Krishnaprasad and Tsakiris (1994a, b), we have developed a general mathematical formulation to study systems of this type. The study of periodic signal generators (also called central pattern generators), as sources of timing signals to compose movements has a long history in the neurophysiology of movement dating back to the early work of Sherrington, Brown and Bernstein.

Recent studies by neurophysiologists (Carling *et al.* 1994) have attempted to bring together principles of motion control based on pattern generation in the spinal cord of the 'lamprey', its compliant body dynamics, and the fluid dynamics of its environment to achieve a comprehensive understanding of the swimming behaviour of such anguilliform animals. These efforts have in part relied on continuum mechanical models of the body, and computational fluid dynamical calculations. There appear to be some unifying themes that underlie this type of neural-mechanical approach to biological locomotion, and the work of the authors and others involving the study of land-based robotic machines subject to the constraint of 'no sliding'. As pointed out in Krishnaprasad (1995), the connecting links between these two streams of research appear to be related to the manner in which systems of coupled oscillators are used to generate finite-dimensional shape variations of the bodies of specialized robot designs, and the associated geometric–mechanical descriptions of the constraints to produce effective motion control strategies (see also the work of Collins and Stewart (1993) for another dynamical systems perspective).

In the present paper, we report on a complete study of an interesting example, the (single module) $SE(2)$-snake, with a view towards deeper appreciation of the above-mentioned connections. In section 2, we present the basic geometry of the configuration space, and the applicable constraints. We also discuss a simplification that reduces the shape freedom to one variable, leading to the Roller Racer. The constraints of 'no sliding' are 'insufficient' to determine the movement of the Roller Racer from shape variations alone. In sections 3 and 4, a model Lagrangian and the action of $SE(2)$, the rigid motion group in the plane as a symmetry group (of the Lagrangian and the constraints) are discussed. A balance law associated to the $SE(2)$-symmetry, the momentum equation, is derived, which is a consequence of the Lagrange–d'Alembert principle (the basic results behind momentum equations are to be found in Bloch *et al.* (1996)). This momentum equation is the key additional data that, together with the constraints, allows us to generate motion control laws. In section 5, we consider controllability and motion control issues.

G-snakes are kinematic chains with configurations taking values in products of several copies of a Lie group G, and subject to non-holonomic constraints (Krishnaprasad and Tsakiris 1994a, Tsakiris 1995). The group G acts on the chain by diagonal action as a symmetry group. The shape space is the quotient by this action. Figure 1 illustrates an $SE(2)$-snake composed of two modules and three nodes, where the configuration space Q is $SE(2)\times SE(2)\times SE(2)$.

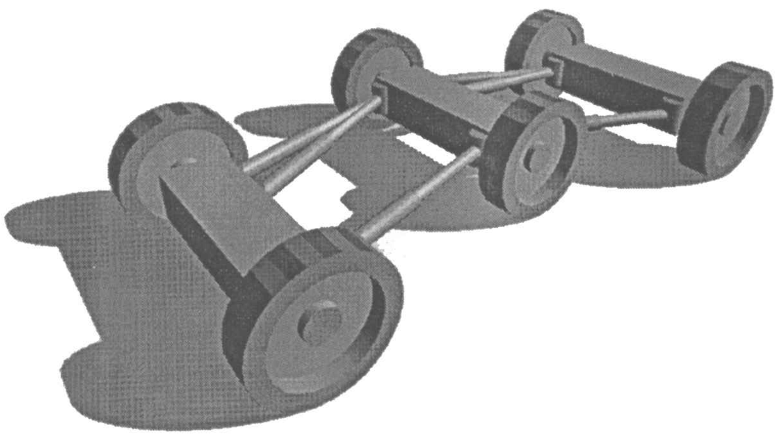


Figure 1. The 2-module $SE(2)$-snake.

The machine in figure 1 is composed of three axles and linearly actuated linkages connecting each adjacent pair of axles, resulting in an assembly of two identical modules. Altering the lengths of the connecting linkages leads to changes in the shapes of component modules. The wheels mounted on each axle are independent and are 'not' actuated, but subject to the constraint of 'no sliding'. In this case there are three constraints, the shape space $\mathcal{S}$ is $SE(2)\times SE(2)$, the constraints define a principal connection on the bundle $(Q, SE(2), \mathcal{S})$, away from a set of non-holonomic singularities, and it is possible to generate global movement of the assembly by periodic variations in the module shapes. The entire situation can be understood at a kinematic level as long as the shapes are control variables (Krishnaprasad and Tsakiris 1994a, b, Tsakiris 1995).

When one of the modules is removed from the machine in figure 1, leaving us with two axles connected by linkages and two non-holonomic constraints, the resulting problem is kinematically under-constrained. It is no longer possible to define a connection without using additional information. It is this type of 1-module $SE(2)$-snake that is of interest here. Matters can be simplified by limiting the extent of shape freedom. In 1972, W. E. Hendricks was awarded US patent No. $3,663,038$ for a toy illustrated in figure 2 and dubbed the Roller Racer, that serves as one such simplification. The rider, on the seat shown, has to merely oscillate the handlebars from side to side to generate forward propulsion, a behaviour for which Hendricks did not claim to have an explanation.

The model of figure 3 will be used in our analysis. Two planar platforms with centres of mass (c.o.m.'s) located at points O_1 and O_2 are connected with a rotary joint at $O_{1,2}$. A pair of idler wheels is attached on each of the platforms, with the axis of the wheels perpendicular to the line connecting the c.o.m. with the joint. A coordinate frame centred at the c.o.m. and with its x-axis along the line $O_iO_{1,2}$ for $i = 1, 2$ connecting the c.o.m. with the joint, will be used to describe the configuration of each platform with respect to a global coordinate system at some reference point O. For simplicity, it will be assumed that the axis of the wheels passes through the c.o.m. of each platform.

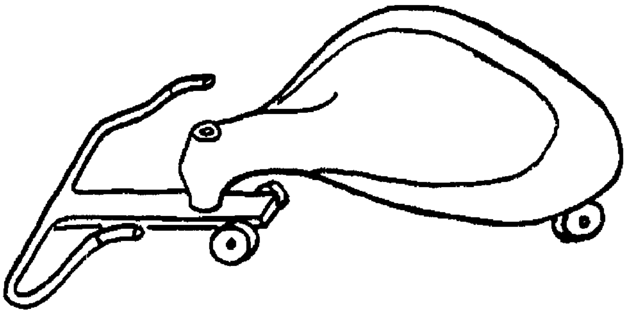


Figure 2. The Roller Racer.

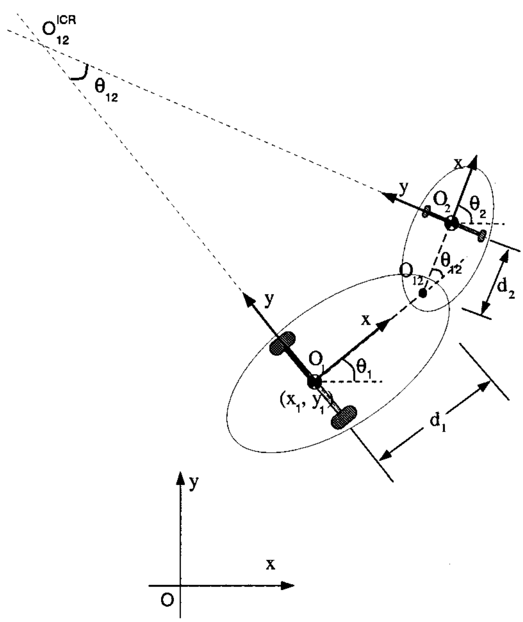


Figure 3. The Roller Racer model.

The effects of the rider's body motion will be ignored at first approximation. Experiments with the Roller Racer and our analysis below, show that, even though these body motions may amplify the resulting motion of the system, the fundamental means of its propulsion is the pivoting of the steering arm around the joint axis and the non-holonomic constraints coming from the wheels' rolling-without-slipping on the plane supporting the vehicle. In this respect, this system is very different from the snakeboard, a variation of the skateboard, where the motion of the rider is essential for the propulsion of the system (Lewis *et al.* 1994). Riderless prototypes of the Roller Racer built at the Intelligent Servosystems Laboratory (ISL) verified this. The propulsion and steering mechanism in these vehicles comes from a rotary motor at the joint $O_{1,2}$, whose torque can be considered as the control of our system. As discussed in Krishnaprasad and Tsakiris (1994b), the purely kinematic analysis of such a system does not allow us to determine the global motion of the system by just the shape variations (the joint velocity in this case), since (unlike the 2-module case) it does not possess a sufficient number of non-holonomic constraints for this to happen. Our goal here is to complement this kinematic analysis with the dynamics of the system, which will provide the necessary information. Thus, certain fundamental behaviours of the system ('straight-line' motion, 'turning' motion) can be achieved by proper oscillatory relative motions of the two platforms. In both numerical simulations and experiments with prototypes, we observed such behaviours, as described in section 6.

In order to study the dynamics of this system, an alternative to the usual approach of solving the full Lagrange–d'Alembert equations of motion of the system is considered here. In Bloch *et al*. (1996), the notion of the momentum map is examined for systems with non-holonomic constraints and symmetries and its evolution law, the momentum equation, is derived from the Lagrange–d'Alembert equations. By applying this method to the problem at hand, a useful decomposition of the equations of motion is obtained: given a shape-space trajectory (which corresponds to the controls of our system), first we compute the non-holonomic momentum from the momentum equation. This involves only the solution of a linear ordinary differential equation (ODE). Subsequently, we use the momentum to reconstruct the group trajectory, which corresponds to the global motion of the system. The corresponding velocities depend linearly on the momentum. This process is very useful for the derivation of motion control laws for this system and can be extended to 1-module $SE(2)$-snakes with more general shape-changing mechanisms.

2. Kinematics of the Roller Racer

Consider a left-invariant dynamical system on a matrix Lie group G with an n-dimensional Lie algebra $\mathcal{G}$ and a curve $g(\cdot) \subset G$. Then, there exists a curve $\xi(\cdot) \subset \mathcal{G}$ such that

$$\dot{g} = g\xi. \tag{1}$$

Let $\{\mathcal{A}_i, i = 1, \dots, n\}$ be a basis of $\mathcal{G}$ and let $[\ ,\ ]$ be the usual Lie bracket on $\mathcal{G}$ defined by $[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$. Then, there exist constants $\Gamma^k_{i,j}$, called 'structure constants', such that

$$[\mathcal{A}_i, \mathcal{A}_j] = \sum_{k=1}^{n} \Gamma^k_{i,j}\mathcal{A}_k, \quad i,j = 1, \dots, n. \tag{2}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, that is the space of linear functions from $\mathcal{G}$ to $\mathbb{R}$. Let $\{\mathcal{A}^\flat_i, i = 1, \dots, n\}$ be the basis of $\mathcal{G}^*$ such that

$$\mathcal{A}^\flat_i(\mathcal{A}_j) = \delta^j_i, \quad i,j = 1, \dots, n, \tag{3}$$

where δ^j_i is the Kronecker symbol. Then the curve $\xi(\cdot) \subset \mathcal{G}$ can be represented as

$$\xi = \sum_{i=1}^{n} \xi_i\mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}^\flat_i(\xi)\mathcal{A}_i, \tag{4}$$

for $\xi_i \stackrel{\text{def}}{=} \mathcal{A}^\flat_i(\xi) \in \mathbb{R}, i = 1, \dots, n$.

Let now the Lie group G be $SE(2)$, the Special Euclidean group of rigid motions on the plane, and $\mathcal{G}$ be $se(2)$, the corresponding Lie algebra, with the basis

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Observe that

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3, \quad [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2, \quad [\mathcal{A}_2, \mathcal{A}_3] = 0. \tag{6}$$

From equation (4), an element $\xi \in \mathcal{G}$ is represented as

$$\xi = \sum_{i=1}^{3} \xi_i \mathcal{A}_i = \begin{pmatrix} 0 & -\xi_1 & \xi_2 \\ \xi_1 & 0 & \xi_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The homogeneous matrix representation of an element $g \in SE(2)$ with coordinates (x, y, θ) is

$$g = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

From equations (1), (7) and (8), we get

$$\xi_1 = \dot{\phi}, \quad \xi_2 = \dot{x}\cos\phi + \dot{y}\sin\phi, \quad \xi_3 = -\dot{x}\sin\phi + \dot{y}\cos\phi. \tag{9}$$

Let

$$g_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & x_i \\ \sin\theta_i & \cos\theta_i & y_i \\ 0 & 0 & 1 \end{pmatrix} \in SE(2), \text{ for } i = 1, 2,$$

be the configuration of platform i with respect to the global coordinate frame at O, where x_i, y_i and θ_i are indicated in figure 3. Let

$$g_{1,2} = \begin{pmatrix} \cos\theta_{1,2} & -\sin\theta_{1,2} & x_{1,2} \\ \sin\theta_{1,2} & \cos\theta_{1,2} & y_{1,2} \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$$

be the configuration of platform 2 with respect to the coordinate frame of platform 1 at O_1. Because of the special structure of the joint, we have

$$x_{1,2} = d_1 + d_2\cos\theta_{1,2}, \quad y_{1,2} = d_2\sin\theta_{1,2}, \tag{10}$$

where $\theta_{1,2}$ is the relative angle of the two platforms and d_i is the distance of O_i from the joint $O_{1,2}$, as indicated in figure 3. We consider non-negative d_1 and d_2. In fact, we assume $d_1 > 0$. However, we allow for the case $d_2 = 0$ and we examine it in detail.

Since the platforms form a kinetmatic chain, we have

$$g_2 = g_1 g_{1,2}, \tag{11}$$

thus

$$\theta_2 = \theta_1 + \theta_{1,2},$$

$$x_2 = x_1 + x_{1,2}\cos\theta_1 - y_{1,2}\sin\theta_1 = x_1 + d_1\cos\theta_1 + d_2\cos\theta_2,$$

$$y_2 = y_1 + x_{1,2}\sin\theta_1 + y_{1,2}\cos\theta_1 = y_1 + d_1\sin\theta_1 + d_2\sin\theta_2. \tag{12}$$

The system kinematics are a special case of the n-module $SE(2)$-snake (n-VGT) assembly (Krishnaprasad and Tsakiris 1994b, Tsakiris 1995), that is for

$$\xi_i = \begin{pmatrix} 0 & -\xi_1^i & \xi_2^i \\ \xi_1^i & 0 & \xi_3^i \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{G} = se(2)$$

we have

$$\dot{g}_i = g_i \xi_i, \quad i = 1, 2 \tag{13}$$

and

$$\dot{g}_{1,2} = g_{1,2}\xi_{1,2}, \tag{14}$$

where

$$\xi_{1,2} = \begin{pmatrix} 0 & -\xi_1^{1,2} & \xi_2^{1,2} \\ \xi_1^{1,2} & 0 & \xi_3^{1,2} \\ 0 & 0 & 0 \end{pmatrix} = \dot{\theta}_{1,2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

By differentiating (12), we get

$$\dot{\theta}_2 = \dot{\theta}_1 + \dot{\theta}_{1,2},$$

$$\dot{x}_2 = \dot{x}_1 = \dot{\theta}_1[d_1\sin\theta_1 + d_2\sin(\theta_1 + \theta_{1,2})] - \dot{\theta}_{1,2}d_2\sin(\theta_1 + \theta_{1,2}),$$

$$\dot{y}_2 = \dot{y}_1 + \dot{\theta}_1[d_1\cos\theta_1 + d_2\cos(\theta_1 + \theta_{1,2})] + \dot{\theta}_{1,2}d_2\cos(\theta_1 + \theta_{1,2}). \tag{15}$$

From (12) we see that the configuration space for the Roller Racer system is $Q = SE(2) \times S^1$. Its shape space is $\mathcal{S} = S^1$. then, $Q = G \times \mathcal{S}$.

Consider the bases $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ for $\mathcal{G} = se(2)$ (given by (5)) and $\{\mathcal{A}_1^\flat, \mathcal{A}_2^\flat, \mathcal{A}_3^\flat\}$ for its dual space $\mathcal{G}^*$. The 'non-holonomic constraints' on the wheels of the two platforms can be expressed as

$$\xi^1 = \mathcal{A}_3^\flat(\xi_1) = -\dot{x}_1\sin\theta_1 + \dot{y}_1\cos\theta_1 = 0, \tag{16}$$

$$\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -\dot{x}_2\sin\theta_2 + \dot{y}_2\cos\theta_2 = 0. \tag{17}$$

From (15) and (17), we get

$$\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -\dot{x}_1\sin(\theta_1 + \theta_{1,2}) + \dot{y}_1\cos(\theta_1 + \theta_{1,2}) + \dot{\theta}_1(d_1\cos\theta_{1,2} + d_2) + \dot{\theta}_{1,2}d_2 = 0. \tag{18}$$

Observe that for $d_2 = 0$, neither one of the constraints (16) and (18) involves $\dot{\theta}_{1,2}$. From (16) and (18) we get

$$\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\sin\theta_{1,2} + (d_1\cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2\dot{\theta}_{1,2} = 0. \tag{19}$$

It can be easily seen that the non-holonomic constraints (16) and (18) are linearly independent for all $q \in Q$. The 'constraint one-forms' can be defined as

$$\omega_q^1 = -\sin\theta_1 \mathrm{d}x_1 + \cos\theta_1 \mathrm{d}y_1,$$

$$\omega_q^2 = -\sin(\theta_1 + \theta_{1,2})\,\mathrm{d}x_1 + \cos(\theta_1 + \theta_{1,2})\,\mathrm{d}y_1 + (d_1\cos\theta_{1,2} + d_2)\,\mathrm{d}\theta_1 + d_2\,\mathrm{d}\theta_{1,2}. \tag{20}$$

The 'constraint distribution' $\mathcal{D}_q$ is the subspace of T_qQ which is the intersection of the kernels of the constraint one-forms, i.e.

$$\mathcal{D}_q = \mathrm{Ker}\,\omega_q^1 \cap \mathrm{Ker}\,\omega_q^2. \tag{21}$$

Since the constraints are linearly independent, we know that $\mathcal{D}_q$ is always two-dimensional. A basis for $\mathcal{D}_q$ is given by

$$\mathcal{D}_q = \mathrm{sp}\{\xi_Q^1, \xi_Q^2\}, \tag{22}$$

where in the case $d_2 \neq 0$:

$$\begin{aligned} \xi_Q^1 &= d_2\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}}, \\ \xi_Q^2 &= d_2\frac{\partial}{\partial\theta_1} - (d_1\cos\theta_{1,2} + d_2)\frac{\partial}{\partial\theta_{1,2}}, \end{aligned} \tag{23}$$

while in the case $d_2 = 0$:

$$\begin{aligned} \xi_Q^1 &= d_1\cos\theta_{1,2}\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}, \\ \xi_Q^2 &= \frac{\partial}{\partial\theta_{1,2}}. \end{aligned} \tag{24}$$

(Reader: do not confuse ξ_Q^i here with infinitesimal generators of group actions.)

In order to model friction in the bearings of the Roller Racer wheels, the relationship between the wheel angular velocities and the configuration velocities is needed. For each platform i, $i = 1, 2$, let $\phi_{\mathrm{l},i}$ and $\phi_{\mathrm{r},i}$ be the angle of its left and right wheel with respect to some reference position of the wheel. The relationship of the angular velocities $\dot{\phi}_{\mathrm{l},i}$, $\dot{\phi}_{\mathrm{r},i}$, $i = 1, 2$, of the left and right wheel of platform i with the configuration velocities $\dot{q} = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2})^{\mathrm{T}}$ of the system, is as follows:

$$\begin{aligned} \dot{\phi}_{\mathrm{l},1} &= \frac{1}{R_1}\left[-\frac{L_1}{2}\dot{\theta}_1 + \dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1\right], \\ \dot{\phi}_{\mathrm{r},1} &= \frac{1}{R_1}\left[\frac{L_1}{2}\dot{\theta}_1 + \dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1\right], \\ \dot{\phi}_{\mathrm{l},2} &= \frac{1}{R_2}\left[-\frac{L_2}{2}(\dot{\theta}_1 + \dot{\theta}_{1,2}) + (\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\cos\theta_{1,2} + \dot{\theta}_1 d_1\sin\theta_{1,2}\right], \\ \dot{\phi}_{\mathrm{r},2} &= \frac{1}{R^2}\left[\frac{L_2}{2}(\dot{\theta}_1 + \dot{\theta}_{1,2}) + (\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\cos\theta_{1,2} + \dot{\theta}_1 d_1\sin\theta_{1,2}\right], \end{aligned} \tag{25}$$

where R_i and L_i are respectively the wheel radius and the length of the wheel axis of platform i.

3. Symmetry of the Roller Racer

Consider now the effect of symmetries on this system. We first present some material on actions of Lie groups, which is based on Marsden and Ratiu (1994).

Let Q be a smooth manifold. A (left) 'action' of a Lie group G on Q is a smooth mapping $\Phi : G \times Q \to Q$, such that, for all $q \in Q$, $\Phi(e, x) = x$ and, for every $g, h \in G$, $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$. For every $g \in G$, define $\Phi_g : Q \to Q : q \to \Phi(g, q)$. This can be shown to be a diffeomorphism (i.e. one-to-one, onto and both Φ_g and $(\Phi_g)^{-1}$ are smooth).

For $q \in Q$, the 'orbit' (or Φ-orbit) of q is $\mathrm{Orb}(q) \stackrel{\mathrm{def}}{=} \{\Phi_g(q) | g \in G\}$. An action Φ is 'free' if, for each $q \in Q$, $g \mapsto \Phi_g(q)$ is one-to-one, that is the identity e is the only element of G with a fixed point. An action Φ is 'proper' if and only if the map $\tilde{\Phi} : G \times Q \to Q \times Q : (g, q) \mapsto \tilde{\Phi}(g, q) = (q, \Phi(g, q))$ is proper, that is if the set $K \subset Q \times Q$ is compact, then its inverse image $\tilde{\Phi}^{-1}(K)$ is also compact.

Let $\Phi : G \times Q \to Q$ be a smooth action and let $\mathcal{G}$ be the Lie algebra of G. If $\xi \in \mathcal{G}$, then $\Phi^{\xi} : \mathbb{R} \times Q \to Q : (t, q) \mapsto (\exp t\xi, q)$ is an $\mathbb{R}$-action on Q, that is, it is a flow on Q. The corresponding vector field on Q is called the 'infinitesimal generator' of Φ corresponding to ξ, is denoted by $\xi_Q(q)$ and is given by

$$\xi_Q(q) = \frac{\mathrm{d}}{\mathrm{d}t} \Phi(\exp t\xi, q)|_{t=0}. \tag{26}$$

The tangent space to the orbit $\mathrm{Orb}(q)$ of q is then

$$T_q \mathrm{Orb}(q) = \{\xi_Q(q) | \xi \in \mathcal{G}\}. \tag{27}$$

Consider the action Φ of the group $G = SE(2)$ on the configuration space $Q = SE(2) \times S^1$ of the Roller Racer defined by

$$\begin{aligned} &\Phi : \; G \times Q \to Q \\ &(g, (g_1, \theta_{1,2})) \mapsto (gg_1, \theta_{1,2}) \\ &((x, y, \theta), (x_1, y_1, \theta_1, \theta_{1,2})) \mapsto \\ &\qquad (x_1 \cos\theta - y_1 \sin\theta + x, x_1 \sin\theta + y_1 \cos\theta + y, \theta_1 + \theta, \theta_{1,2}), \end{aligned} \tag{28}$$

where $g = g(x, y, \theta) \in G$. The tangent space at $q \in Q$ to the orbit of Φ is given by

$$T_1 \mathrm{Orb}(q) = \mathrm{sp}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta_1}\right\}. \tag{29}$$

Notice that the sum of the subspaces $\mathcal{D}_q$ and $T_q \mathrm{Orb}(q)$ gives the entire $T_q Q$:

$$\mathcal{D}_q + T_q \mathrm{Orb}(q) = T_q Q. \tag{30}$$

In Bloch *et al.* (1996), this is referred to as the 'principal' case. Our goal is to show that the non-holonomic constraints, together with a momentum equation, can specify a connection on the principal fibre bundle $Q \to Q/G$.

An important observation, that we prove below, is that the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ and $T_q \mathrm{Orb}(q)$ is 'non-trivial'. Contrast this with the $(n-1)$-module G-snake with $\dim G = n$, where $T_q Q = \mathcal{D}_q \oplus T_q \mathrm{Orb}(q)$, thus the intersection of $\mathcal{D}_q$ and $T_q \mathrm{Orb}(q)$

is trivial (Krishnaprasad and Tsakiris 1994a, Tsakiris 1995); this is referred to as the 'purely kinematic' case. We specify a basis for $\mathcal{S}_q$ as follows.

Proposition 1. *Consider the intersection*

$$\mathcal{S}_q \overset{\text{def}}{=} \mathcal{D}_q \cap T_q \operatorname{Orb}(q). \tag{31}$$

In the case of $d_1 \neq d_2$, the distribution $\mathcal{S}_q$ is one-dimensional and is given by

$$\mathcal{S}_q = \operatorname{sp}\{\xi_Q^q\}, \tag{32}$$

where

$$\xi_Q^q = (\mathrm{d}_1 \cos\theta_{1,2} + d_2)\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1}. \tag{33}$$

Proof. Consider $X_q \in \mathcal{S}_q$. Since $X_q \in \mathcal{D}_q$, we have $X_q = u_1 \xi_Q^1 + u_2 \xi_Q^2$, for some $u_i \in \mathbb{R}$. Since $X_q \in T_q \operatorname{Orb}(q)$, we have

$$X_q = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial \theta_1},$$

for some $v_i \in R$. In order for X_q to lie in the intersection of the two spaces, we should have

$$u_1 \xi_Q^1 + u_2 \xi_Q^2 = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial \theta_1}. \tag{34}$$

(i) In the case $d_2 \neq 0$, we have from (23):

$$\left[d_2\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_{1,2}}\right] u_1 + \left[d_2 \frac{\partial}{\partial \theta_1} - (d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_{1,2}}\right] u_2 = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial \theta_1}. \tag{35}$$

This corresponds to a system of four equations:

$$\begin{pmatrix} d_2 \cos\theta_1 & 0 & -1 & 0 & 0 \\ d_2 \sin\theta_1 & 0 & 0 & -1 & 0 \\ 0 & d_2 & 0 & 0 & -1 \\ \sin\theta_{1,2} & -(d_1 \cos\theta_{1,2} + d_2) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

When $d_1 \neq d_2$, the 4×5 matrix above is always of maximal rank, thus $\dim \mathcal{S}_q = 5 - 4 = 1$, for all $q \in Q$. Pick $u_1 = (d_1 \cos\theta_{1,2} + d_2) u_5$ and $u_2 = \sin\theta_{1,2} u_5$. Then $v_1 = d_2 \cos\theta_1 (d_1 \cos\theta_{1,2} + d_2) u_5$, $v_2 = d_2 \sin\theta_1 (d_1 \cos\theta_{1,2} + d_2) u_5$ and $v_3 = d_2 \sin\theta_{1,2} u_5$. Thus

$$X_q = \left[(d_1 \cos\theta_{1,2} + d_2)\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1}\right] d_2 u_5,$$

for aribrary u_5. Observe that when $d_1 \neq d_2$, the vector field X_q is non-trivial for all $q \in Q$.

(ii) In the case $d_2 = 0$, we have from (24):

$$\left[d_1 \cos\theta_{1,2}\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}\right]u_1 + u_2\frac{\partial}{\partial\theta_{1,2}} = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1}.$$

From this, we get

$$u_2 = 0, v_1 = d_1 \cos\theta_{1,2}\cos\theta_1 u_1, v_2 = d_1\cos\theta_{1,2}\sin\theta_1 u_1, v_3 = \sin\theta_{1,2}u_1. \tag{36}$$

Therefore

$$X_q = \left[d_1 \cos\theta_{1,2}\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}\right]u_1,$$

for arbitrary u_1. Thus, $\mathcal{S}_q$ is again a one-dimensional distribution.
The two cases can be unified in the expression (33). □

The infinitesimal generators for the action Φ of $SE(2)$ on Q defined in (28), corresponding to the basis elements of $\mathcal{G} = se(2)$ defined in (5), at the point $q \in Q$, are

$$\mathcal{A}^q_{1Q} = -y_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial y_1} + \frac{\partial}{\partial\theta_1}, \quad \mathcal{A}^q_{2Q} = \frac{\partial}{\partial x_1}, \quad \mathcal{A}^q_{3Q} = \frac{\partial}{\partial y_1}. \tag{37}$$

The infinitesimal generator corresponding to $\xi^q = \xi_1\mathcal{A}_1 + \xi_2\mathcal{A}_2 + \xi_3\mathcal{A}_3 \in \mathcal{G}$ is

$$\xi^q_Q = (\xi_2 - y_1\xi_1)\frac{\partial}{\partial x_1} + (\xi_3 + x_1\xi_1)\frac{\partial}{\partial y_1} + \xi_1\frac{\partial}{\partial\theta_1}. \tag{38}$$

A given vector field

$$\xi^q_Q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1}$$

can be considered as the infinitesimal generator of an element $\xi^q \in \mathcal{G} = se(2)$, under the action Φ. This element ξ^q is

$$\xi^q = v_3\mathcal{A}_1 + (v_1 + y_1v_3)\mathcal{A}_2 + (v_2 - x_1v_3)\mathcal{A}_3. \tag{39}$$

The vector field ξ^q_Q in (33) corresponds then to the following element ξ^q of $se(2)$:

$$\begin{aligned}\xi^q = {} & \sin\theta_{1,2}\mathcal{A}_1 + [(d_1\cos\theta_{1,2} + d_2)\cos\theta_1 + y_1\sin\theta_{1,2}]\mathcal{A}_2 \\ & + [(d_1\cos\theta_{1,2} + d_2)\sin\theta_1 - x_1\sin\theta_{1,2}]\mathcal{A}_3.\end{aligned} \tag{40}$$

By differentiating (40), we get

$$\begin{aligned}\frac{d\xi^q}{dt} = {} & \cos\theta_{1,2}\dot\theta_{1,2}\mathcal{A}_1 + [-d_1\sin\theta_{1,2}\cos\theta_1\dot\theta_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot\theta_1 \\ & + \dot y_1\sin\theta_{1,2} + y_1\cos\theta_{1,2}\dot\theta_{1,2}]\mathcal{A}_2 + [-d_1\sin\theta_{1,2}\sin\theta_1\dot\theta_{1,2} \\ & + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot\theta_1 - \dot x_1\sin\theta_{1,2} - x_1\cos\theta_{1,2}\dot\theta_{1,2}]\mathcal{A}_3.\end{aligned} \tag{41}$$

The corresponding infinitesimal generator is given by (38) as

$$\left[\frac{\mathrm{d}\xi^q}{\mathrm{d}t}\right]_Q = [-d_1 \sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot{\theta}_1 + \dot{y}_1\sin\theta_{1,2}]\frac{\partial}{\partial x_1}$$
$$+ [-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot{\theta}_1 - \dot{x}_1\sin\theta_{1,2}]\frac{\partial}{\partial y_1}$$
$$+ \cos\theta_{1,2}\dot{\theta}_{1,2}\frac{\partial}{\partial\theta_1}. \tag{42}$$

Finally, by differentiating ξ^q_Q in (33), we get

$$\frac{\mathrm{d}\xi^q_Q}{\mathrm{d}t} = [-d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot{\theta}_1]\frac{\partial}{\partial x_1}$$
$$+ [-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot{\theta}_1]\frac{\partial}{\partial y_1}$$
$$+ \cos\theta_{1,2}\dot{\theta}_{1,2}\frac{\partial}{\partial\theta_1}. \tag{43}$$

4. Dynamics of the Roller Racer

4.1. *The Lagrange–d'Alembert equations of motion*

The Lagrangian dynamics of the Roller Racer are set up under the assumption that the mass and linear momentum of platform 2 are much smaller than those of platform 1 and can be ignored. However, the inertia of platform 2 is not ignored. Thus, we consider the following Lagrangian:

$$L(q,\dot{q}) = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}I_{z_1}\dot{\theta}_1^2 + \tfrac{1}{2}I_{z_2}(\dot{\theta}_1 + \dot{\theta}_{1,2})^2, \tag{44}$$

for $q = (x_1, y_1, \theta_1, \theta_{1,2}) \in Q$ and $\dot{q} = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \in T_qQ$, where m_i and I_{z_i} are respectively the mass and moment of inertia of platform i. From (44), we get by differentiation

$$\frac{\partial L}{\partial\dot{q}} = \begin{pmatrix} m_1\dot{x}_1 \\ m_1\dot{y}_1 \\ (I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \\ I_{z_2}\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \end{pmatrix}. \tag{45}$$

The equations of motion of the Roller Racer are derived using the 'Lagrange–d'Alembert principle' for a system with non-holonomic constraints (Vershik and Faddeev 1981, Yang 1992).

Proposition 2 (Lagrange–d'Alembert principle). *In the case of linear constraints on the velocities, the Lagrange–d'Alembert principle for the Roller Racer with the Lagrangian* $L(q,v)$ *given by (44), with* $q = (x_1, y_1, \theta_1, \theta_{1,2}) \in Q$ *and* $v = (v_1, v_2, v_3, v_4) = \dot{q} \in T_qQ$, *takes the form:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial q}\right)\cdot u = \alpha_e \cdot u, \tag{46}$$

where (q,v) *satisfy the non-holonomic constraints*:

$$\omega_q^1(v) = -\sin\theta_1 v_1 + \cos\theta_1 v_2 = 0,$$

$$\omega_q^2(v) = -\sin(\theta_1+\theta_{1,2})v_1 + \cos(\theta_1+\theta_{1,2})v_2 + (d_1\cos\theta_{1,2}+d_2)v_3 + d_2 v_4 = 0, \tag{47}$$

and the test vector $u = (u_1, u_2, u_3, u_4) \in T_qQ$ *satisfies:*

$$\frac{\partial}{\partial v}\omega_q^1(v)\cdot u = -\sin\theta_1 u_1 + \cos\theta_1 u_2 = 0,$$

$$\frac{\partial}{\partial v}\omega_q^2(v)\cdot u = -\sin(\theta_1+\theta_{1,2})u_1 + \cos(\theta_1+\theta_{1,2})u_2 + (d_1\cos\theta_{1,2}+d_2)u_3 + d_2 u_4 = 0, \tag{48}$$

while α_e *is the 1-form describing the external forcing to the system.*

Using Lagrange multipliers, the Lagrange–d'Alembert principle for the case of a system with two linear (in the velocity) non-holonomic constraints, takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial q} = \alpha_e + \lambda_1\frac{\partial\omega_q^1}{\partial v} + \lambda_2\frac{\partial\omega_q^2}{\partial v}, \tag{49}$$

for functions λ_1 *and* λ_2 *on* TQ *and for* (q, v) *such that the non-holonomic constraints (47) are satisfied.*

Consider external forcing to the system described by the 1-form

$$\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}}), \tag{50}$$

where $F_{\theta_{1,2}}$ may be the torque applied by the motor that actuates the rotary joint $O_{1,2}$ and $F_{x_1}, F_{y_1}, F_{\theta_1}$ may be the result of friction in the bearings of the wheels. The equations of motion of the Roller Racer are given below.

Proposition 3 (Lagrange–d'Alembert equations of motion).
(i) In the case $d_2 \neq 0$, *the equations of motion for the Roller Racer are*

$$\begin{aligned}
&(I_{z_2}\sin^2\theta_{1,2} + m_1 d_2^2)\dot{\nu}_1 - I_{z_2}d_1\sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_2 \\
&\quad + I_{z_2}\sin^2\theta_{1,2}\cos\theta_{1,2}\nu_1^2 \\
&\quad - I_{z_2}\sin\theta_{1,2}[d_1(\cos^2\theta_{1,2} - \sin^2\theta_{1,2}) + d_2\cos\theta_{1,2}]\nu_1\nu_2 \\
&\quad - I_{z_2}d_1\sin^2\theta_{1,2}(d_1\cos\theta_{1,2}+d_2)\nu_2^2 \\
&\qquad = d_2(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_{1,2}}, \\
&-I_{z_2}d_1\sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_1 + (I_{z_1}d_2^2 + I_{z_2}d_1^2\cos^2\theta_{1,2})\dot{\nu}_2 \\
&\quad - I_{z_2}d_1\sin\theta_{1,2}\cos^2\theta_{1,2}\nu_1^2 \\
&\quad + I_{z_2}d_1\cos\theta_{1,2}[d_1(\cos^2\theta_{1,2} - \sin^2\theta_{1,2}) + d_2\cos\theta_{1,2}]\nu_1\nu_2 \\
&\quad + I_{z_2}d_1^2(d_1\cos\theta_{1,2}+d_2)\sin\theta_{1,2}\cos\theta_{1,2}\nu_2^2 \\
&\qquad = d_2F_{\theta_1} - r(\theta_{1,2})F_{\theta_{1,2}},
\end{aligned} \tag{51}$$

where $\nu_1 \overset{\text{def}}{=} (1/d_2)(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)$, $\nu_2 \overset{\text{def}}{=} (1/d_2)\dot{\theta}_1$ *and* $r(\theta_{1,2}) \overset{\text{def}}{=} d_1 \cos\theta_{1,2} + d_2$.

(ii) *In the case* $d_2 = 0$, *the equations of motion are*

$$[(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}]\dot{\nu}_1 + I_{z_2} \sin\theta_{1,2}\dot{\nu}_2 + (I_{z_1} + I_{z_2} - m_1 d_1^2) \sin\theta_{1,2} \cos\theta_{1,2}\nu_1\nu_2 = d_1 \cos\theta_{1,2}(F_{x_1} \cos\theta_1 + F_{y_1} \sin\theta_1) + \sin\theta_{1,2}F_{\theta_1},$$

$$I_{z_2} \sin\theta_{1,2}\dot{\nu}_1 + I_{z_2}\dot{\nu}_2 + I_{z_2} \cos\theta_{1,2}\nu_1\nu_2 = F_{1,2}, \tag{52}$$

where

$$\nu_1 \overset{\text{def}}{=} \frac{1}{d_1 \cos\theta_{1,2}}(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1),$$

in the case $\cos\theta_{1,2} \neq 0$, *or*

$$\nu_1 \overset{\text{def}}{=} \frac{1}{\sin\theta_{1,2}}\dot{\theta}_1$$

otherwise and $\nu_2 \overset{\text{def}}{=} \dot{\theta}_{1,2}$.

Proof. The Lagrange–d'Alembert principle (46) for the Lagrangian given by (44), for $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4) \in \mathcal{D}_q$ and for $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$, takes the form:

$$m_1\dot{v}_1 u_1 + m_1\dot{v}_2 u_2 + [(I_{z_1} + I_{z_2})\dot{v}_3 + I_{z_2}\dot{v}_4]u_3 + I_{z_2}(\dot{v}_3 + \dot{v}_4)u_4 = F_{x_1}u_1 + F_{y_1}u_2 + F_{\theta_1}u_3 + F_{\theta_{1,2}}u_4. \tag{53}$$

(i) Let $d_2 \neq 0$

Any $u \in \mathcal{D}_q$ can be represented as $u = \alpha_1\xi_Q^1 + \alpha_2\xi_Q^2$, for ξ_Q^1 and ξ_Q^2 given by (23) and for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Then its components are

$$u_1 = \alpha_1 d_2 \cos\theta_1, \qquad u_2 = \alpha_1 d_2 \sin\theta_1,$$
$$u_3 = \alpha_2 d_2, \qquad u_4 = \alpha_1 \sin\theta_{1,2} - \alpha_2 r(\theta_{1,2}). \tag{54}$$

Similarly, any $v \in \mathcal{D}_q$ can be represented as $v = \nu_1\xi_Q^1 + \nu_2\xi_Q^2$, for some $\nu_1, \nu_2 \in \mathbb{R}$. Its components are

$$v_1 = \nu_1 d_2 \cos\theta_1, \quad v_2 = \nu_1 d_2 \sin\theta_1, \quad v_3 = \nu_2 d_2, \quad v_4 = \nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2}). \tag{55}$$

These relationships can be used to derive ν_1 and ν_2 as follows:

$$\nu_1 = \frac{1}{d_2}(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1), \quad \nu_2 = \frac{1}{d_2}\dot{\theta}_1. \tag{56}$$

By differentiating (55) we get

$$\dot{v}_1 = \dot{\nu}_1 d_2 \cos\theta_1 - \nu_1 d_2 \sin\theta_1 \dot{\theta}_1,$$
$$\dot{v}_2 = \dot{\nu}_1 d_2 \sin\theta_1 + \nu_1 d_2 \cos\theta_1 \dot{\theta}_1,$$
$$\dot{v}_3 = \dot{\nu}_2 d_2,$$
$$\dot{v}_4 = \dot{\nu}_1 \sin\theta_{1,2} - \dot{\nu}_2 r(\theta_{1,2}) + \nu_1 \cos\theta_{1,2}\dot{\theta}_{1,2} + \nu_2 d_1 \sin\theta_{1,2}\dot{\theta}_{1,2}. \tag{57}$$

Introducing (54) and (57) in (53), we get

$$\begin{aligned} m_1\dot{\nu}_1 d_2^2\alpha_1 &+ [(I_{z_1} + I_{z_2})\dot{\nu}_2 d_2 + I_{z_2}(\dot{\nu}_1 \sin\theta_{1,2} - \dot{\nu}_2 r(\theta_{1,2}) + \nu_1\cos\theta_{1,2}\dot{\theta}_{1,2} \\ &+ \nu_2 d_1 \sin\theta_{1,2}\dot{\theta}_{1,2})]d_2\alpha_2 + I_{z_2}[\dot{\nu}_2 d_2 + \dot{\nu}_1 \sin\theta_{1,2} - \dot{\nu}_2 r(\theta_{1,2}) \\ &+ \nu_1 \cos\theta_{1,2}\dot{\theta}_{1,2} + \nu_2 d_1 \sin\theta_{1,2}\dot{\theta}_{1,2}] \cdot [\sin\theta_{1,2}\alpha_1 - r(\theta_{1,2})\alpha_2] \\ &= F_{x_1} d_2 \cos\theta_1 \alpha_1 + F_{y_1} d_2 \sin\theta_1 \alpha_1 + F_{\theta_1} d_2 \alpha_2 \\ &\quad + F_{\theta_{1,2}}[\sin\theta_{1,2}\alpha_1 - r(\theta_{1,2})\alpha_2], \end{aligned} \tag{58}$$

for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$. From (55), we have

$$\dot{\theta}_{1,2} \equiv v_4 = \nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2}). \tag{59}$$

Since α_1, α_2 are arbitrary, equation (58) splits (after using (59)) in the following two equations:

$$\begin{aligned} (I_{z_2}\sin^2\theta_{1,2} + m_1 d_2^2)\dot{\nu}_1 &- I_{z_2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_2 \\ &+ I_{z_2}\sin\theta_{1,2}(\cos\theta_{1,2}\nu_1 + d_1\sin\theta_{1,2}\nu_2)(\nu_1\sin\theta_{1,2} - \nu_2 r(\theta_{1,2})) \\ &= (F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1)d_2 + F_{\theta_{1,2}}\sin\theta_{1,2}, \\ -I_{z_2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_1 &+ (I_{z_1} d_2^2 + I_{z_2} d_1^2 \cos^2\theta_{1,2})\dot{\nu}_2 \\ &- I_{z_2} d_1 \cos\theta_{1,2}(\nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2})) \\ &= F_{\theta_1} d_2 - F_{\theta_{1,2}} r(\theta_{1,2}). \end{aligned} \tag{60}$$

By rearranging terms, we obtain (51). It can be easily seen that the first of the equations (51) is equation (46) with test vector $u = \xi_Q^1$, while the second is (46) with $u = \xi_Q^2$.

(ii) Let $d_2 = 0$
Any $u \in \mathcal{D}_q$ can be represented as $u = \alpha_1\xi_Q^1 + \alpha_2\xi_Q^2$, for ξ_Q^1 and ξ_Q^2 given by (24) and for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Then its components are

$$u_1 = \alpha_1 d_1 \cos\theta_{1,2}\cos\theta_1, \qquad u_2 = \alpha_1 d_1 \cos\theta_{1,2}\sin\theta_1,$$
$$u_3 = \alpha_1 \sin\theta_{1,2}, \qquad u_4 = \alpha_2. \tag{61}$$

Similarly, any $v \in \mathcal{D}_q$ can be represented as $v = \nu_1\xi_Q^1 + \nu_2\xi_Q^2$, for some $\nu_1, \nu_2 \in \mathbb{R}$. Its components are

$$v_1 = \nu_1 d_1 \cos\theta_{1,2} \cos\theta_1, \qquad v_2 = \nu_1 d_1 \cos\theta_{1,2} \sin\theta_1,$$
$$v_3 = \nu_1 \sin\theta_{1,2}, \qquad v_4 = \nu_2. \tag{62}$$

These relationships can be used to derive ν_1 and ν_2 as follows:

$$\begin{aligned}\nu_1 &= \frac{1}{d_1 \cos\theta_{1,2}} (\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1), \quad \text{in the case } \cos\theta_{1,2} \neq 0,\\ &= \frac{1}{\sin\theta_{1,2}} \dot{\theta}_1, \quad \text{otherwise},\\ \nu_2 &= \dot{\theta}_{1,2}.\end{aligned} \tag{63}$$

By differentiating (62) we get

$$\begin{aligned}\dot{v}_1 &= \dot{\nu}_1 d_1 \cos\theta_{1,2} \cos\theta_1 - \nu_1 d_1 (\cos\theta_{1,2} \sin\theta_1 \dot{\theta}_1 + \sin\theta_{1,2} \cos\theta_1 \dot{\theta}_{1,2}),\\ \dot{v}_2 &= \dot{\nu}_1 d_1 \cos\theta_{1,2} \sin\theta_1 + \nu_1 d_1 (\cos\theta_{1,2} \cos\theta_1 \dot{\theta}_1 - \sin\theta_{1,2} \sin\theta_1 \dot{\theta}_{1,2}),\\ \dot{v}_3 &= \dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2},\\ \dot{v}_4 &= \dot{\nu}_2.\end{aligned} \tag{64}$$

Introducing (61) and (64) in (53), we get

$$\begin{aligned}&m_1(\dot{\nu}_1 d_1 \cos\theta_{1,2} \cos\theta_1 - \nu_1 d_1 \sin\theta_{1,2} \cos\theta_1 \dot{\theta}_{1,2}) d_1 \cos\theta_{1,2} \cos\theta_1 \alpha_1\\ &\quad + m_1(\dot{\nu}_1 d_1 \cos\theta_{1,2} \sin\theta_1 - \nu_1 d_1 \sin\theta_{1,2} \sin\theta_1 \dot{\theta}_{1,2}) d_1 \cos\theta_{1,2} \sin\theta_1 \alpha_1\\ &\quad + [(I_{z_1} + I_{z_2})(\dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2}) + I_{z_2} \dot{\nu}_2] \sin\theta_{1,2} \alpha_1\\ &\quad + I_{z_2}(\dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2} + \dot{\nu}_2) \alpha_2\\ &\qquad = F_{x_1} d_1 \cos\theta_{1,2} \cos\theta_1 \alpha_1 + F_{y_1} d_1 \cos\theta_{1,2} \sin\theta_1 \alpha_1\\ &\qquad\quad + F_{\theta_1} \sin\theta_{1,2} \alpha_1 + F_{\theta_{1,2}} \alpha_2,\end{aligned} \tag{65}$$

for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$. Since α_1, α_2 are arbitrary, (65) splits in the following two equations:

$$\begin{aligned}&[(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}] \dot{\nu}_1 + I_{z_2} \sin\theta_{1,2} \dot{\nu}_2\\ &\quad + (I_{z_1} + I_{z_2} - m_1 d_1^2) \sin\theta_{1,2} \cos\theta_{1,2} \nu_1 \dot{\theta}_{1,2}\\ &\qquad = d_1 \cos\theta_{1,2} (F_{x_1} \cos\theta_1 + F_{y_1} \sin\theta_1) + \sin\theta_{1,2} F_{\theta_1},\\ &\qquad\qquad I_{z_2} \sin\theta_{1,2} \dot{\nu}_1 + I_{z_2} \dot{\nu}_2 + I_{z_2} \cos\theta_{1,2} \nu_1 \dot{\theta}_{1,2} = F_{1,2}.\end{aligned} \tag{66}$$

By using $\dot{\theta}_{1,2} \equiv v_4 = \nu_2$ from (62) and by rearranging terms, we obtain (52). Again, the first of the equations (52) is equation (46) with $u = \xi_Q^1$, while the second is (46) with $u = \xi_Q^2$. □

Remark 1. The quantities $\dot{\nu}_1$ and $\dot{\nu}_2$ above can be interpreted as accelerations in the constrained directions.

Suppose 'friction' is present in the joints of the Roller Racer wheels with their axes. We consider a simple viscous friction model, where the frictional forces are introduced in the Lagrange–d'Alembert equations through the following Rayleigh dissipation function that involves the wheel angular velocities $\dot{\phi}_{l,i}, \dot{\phi}_{r,i}$, $i = 1, 2$ defined in (25):

$$
\begin{aligned}
\mathcal{R} &= \frac{1}{2}k_1\dot{\phi}_{l,1}^2 + \frac{1}{2}k_1\dot{\phi}_{r,1}^2 + \frac{1}{2}k_2\dot{\phi}_{l,2}^2 + \frac{1}{2}k_2\dot{\phi}_{r,2}^2 \\
&= \frac{k_1}{R_1^2}\left[\frac{L_1^2}{4}\dot{\theta}_1^2 + (\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)^2\right] \\
&\quad + \frac{k_2}{R_2^2}\left[\frac{L_2^2}{4}(\dot{\theta}_1 + \dot{\theta}_{1,2})^2 + \mathrm{g}((\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\cos\theta_{1,2} + \dot{\theta}_1 d_1 \sin\theta_{1,2})^2\right],
\end{aligned}
\tag{67}
$$

where $k_1 > 0$ and $k_2 > 0$ are friction coefficients, $q = (x_1, y_1, \theta_1, \theta_{1,2})^{\mathrm{T}} \in Q$ and $\dot{q} = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2})^{\mathrm{T}}$. The external force 1-form α_e is $\alpha_e = \mathcal{T}_{1,2} - \partial\mathcal{R}/\partial\dot{q}$, where $\mathcal{T}_{1,2} \overset{25}{=} (0,\ 0,\ 0,\ \tau_{1,2})^{\mathrm{T}}$, with $\tau_{1,2}$ being the torque applied by the motor at the joint $O_{1,2}$. The corresponding components of α_e are

$$
\begin{aligned}
F_{x_1} &= -2\left[\left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\right. \\
&\qquad \left. + \frac{k_2}{R_2^2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\theta}_1\right]\cos\theta_1, \\
F_{y_1} &= -2\left[\left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\right. \\
&\qquad \left. + \frac{k_2}{R_2^2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\theta}_1\right]\sin\theta_1, \\
F_{\theta_1} &= -2\left[\frac{k_2}{R_2^2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\right. \\
&\qquad \left. + \left(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\left(\frac{L_2^2}{4} + d_1^2\sin^2\theta_{1,2}\right)\right)\dot{\theta}_1\right], \\
F_{\theta_{1,2}} &= \tau_{1,2} - 2\frac{k_2}{R_2^2}\frac{L_2^2}{4}\dot{\theta}_{1,2}.
\end{aligned}
\tag{68}
$$

4.2. *Non-holonomic momentum and the momentum equation*

The symmetries of an unconstrained system or of a system with holonomic constraints, described by the invariance of its Lagrangian with respect to a Lie group action, imply the existence of conserved quantities, called momenta (Noether Theorem). For systems with non-holonomic constraints, whose Lagrangian and constraints are invariant with respect to an appropriate Lie group action, it is still possible to define momentum-like quantities. These, however, are not necessarily conserved; instead, they evolve according to a law called the momentum equation (Bloch *et al.* 1996).

It is easy to verify that, for the Roller Racer, the non-holonomic constraints (16) and (18) and the Lagrangian (44) are invariant under the action Φ given by (28) (Krishnaprasad and Tsakiris 1998).

Momentum-like quantities can be defined for a constrained system by

$$p = \frac{\partial L}{\partial v} \cdot u,$$

where $v \in T_q Q$ and $u \in \mathcal{D}_q$, the constraint distribution. In the present case, it is particularly advantageous (see Proposition 5 below) to restrict u to $\mathcal{S}_q$, the intersection of the subspaces $\mathcal{D}_q$ and $T_q\,\mathrm{Orb}(q)$.

We define, then, the 'non-holonomic momentum' as

$$p \stackrel{\text{def}}{=} \sum_i \frac{\partial L}{\partial \dot{q}^i} (\xi_Q^q)^i, \tag{69}$$

where $\xi_Q^q \in \mathcal{S}_q$. From (45) and (33), we get the non-holonomic momentum for the Roller Racer:

$$p = m_1(d_1 \cos\theta_{1,2} + d_2)(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}] \sin\theta_{1,2}. \tag{70}$$

The non-holonomic momentum p given by equation (70) is (up to a scale factor) the angular momentum about the point of intersection O_{12}^{ICR} of the two wheel axles (cf. figure 3). It can be easily seen that

$$O_1 O_{12}^{\mathrm{ICR}} = \frac{d_1 \cos\theta_{1,2} + d_2}{\sin\theta_{1,2}},$$

when $\sin\theta_{1,2} \neq 0$.

Let

$$\Delta(\theta_{1,2}) \stackrel{\text{def}}{=} (I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1(d_1 \cos\theta_{1,2} + d_2)^2. \tag{71}$$

For $d_1 \neq d_2$, we have $\Delta > 0$, for all $q \in Q$. Also, let

$$\delta(\theta_{1,2}) \stackrel{\text{def}}{=} I_{z_2} \sin^2\theta_{1,2} + m_1 d_2(d_1 \cos\theta_{1,2} + d_2). \tag{72}$$

Proposition 4. *The angular velocity $\dot{\theta}_1$ is an affine function of the non-holonomic momentum*

$$\dot{\theta}_1 = \frac{1}{\Delta(\theta_{1,2})} [\sin\theta_{1,2}\ \ p - \delta(\theta_{1,2})\dot{\theta}_{1,2}]. \tag{73}$$

Proof. Multiplying both sides of (70) by $\sin\theta_{1,2}$ and using (18), we get

$$\begin{aligned} \sin\theta_{1,2}\ \ p = {} & m_1(d_1 \cos\theta_{1,2} + d_2)[(d_1 \cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2\dot{\theta}_{1,2}] \\ & + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}] \sin^2\theta_{1,2}. \end{aligned} \tag{74}$$

Solving for $\dot{\theta}_1$, the result follows. □

The momentum equation presented in the next result is derived in Bloch *et al.* (1996) from the Lagrange–d'Alembert principle by considering only variations that satisfy the constraints and that depend on the symmetry, as it is expressed by a free group action. The equation does not depend on internal torques and depends only on the shape variables and not on the group variables. It is given below for the case where external torques are not present.

Proposition 5. *Consider a Lagrangian L which is invariant under the action Φ of a group G on a configuration space Q. Let $\mathcal{D}_q$ be a constraint distribution on T_qQ and consider the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ with the tangent space to the orbit of Φ at q. Let $\xi Q^q \in \mathcal{S}_q$ and let ξ^q be the corresponding element of the Lie algebra $\mathcal{G}$. The evolution of the non-holonomic momentum p, defined as in equation (69), satisfies the equation*

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \sum_i \frac{\partial L}{\partial \dot{q}^i} \left[\frac{\mathrm{d}\xi^q}{\mathrm{d}t} \right]_Q^i. \tag{75}$$

This result generalizes the classical Noether Theorem, which specifies conserved quantities for solutions of the Euler–Lagrange equations (Arnold 1978, Abraham and Marsden 1985, Marsden and Ratiu 1994). In the present paper, the subspace $\mathcal{S}_q$ is one-dimensional and one has a scalar p. In general, one could have a vector non-holonomic momentum.

Proposition 6 (momentum equation without external forces). *The momentum equation for the Roller Racer is*

$$\frac{\mathrm{d}p}{\mathrm{d}t} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2, \tag{76}$$

where

$$A_1^4(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}\beta(\theta_{1,2}) \sin\theta_{1,2} \tag{77}$$

and

$$A_2^4(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{m_1}{\Delta(\theta_{1,2})}\lambda(\theta_{1,2})\gamma(\theta_{1,2}), \tag{78}$$

with

$$\begin{aligned}
\beta(\theta_{1,2}) &\stackrel{\text{def}}{=} (I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2), \\
\gamma(\theta_{1,2}) &\stackrel{\text{def}}{=} -I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2}, \\
r(\theta_{1,2}) &\stackrel{\text{def}}{=} d_1\cos\theta_{1,2} + d_2, \quad \lambda(\theta_{1,2}) \stackrel{\text{def}}{=} d_1 + d_2\cos\theta_{1,2}, \quad I \stackrel{\text{def}}{=} I_{z_1} + I_{z_2}.
\end{aligned} \tag{79}$$

Proof. From (75), (45) and (42) we get

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} = {}& -m_1 d_1(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\sin\theta_{1,2}\dot{\theta}_{1,2} \\
& + m_1(d_1\cos\theta_{1,2} + d_2)(-\dot{x}_1\sin\theta_1 + \dot{y}_1\cos\theta_1)\dot{\theta}_1 \\
& + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}]\cos\theta_{1,2}\dot{\theta}_{1,2}.
\end{aligned} \tag{80}$$

Introducing the non-holonomic constraints (16) and (18), in (80) above, we get

$$\frac{\mathrm{d}p}{\mathrm{d}t} = [(I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2)]\dot{\theta}_1\dot{\theta}_{1,2} + [I_{z_2}\cos\theta_{1,2} - m_1 d_1 d_2]\dot{\theta}_{1,2}^2. \tag{81}$$

By substituting $\dot{\theta}_1$ from (73) in the above expression, the result follows. □

Proposition 7. *The solution of the momentum equation (76) is*

$$p(t) = \Phi(t, t_0) p(t_0) + \int_{t_0}^{t} \Phi(t, \tau) A_2^4(\theta_{1,2}(\tau)) \dot{\theta}_{1,2}^2(\tau)\, \mathrm{d}\tau, \tag{82}$$

where

$$\Phi(t, t_0) \stackrel{\text{def}}{=} \exp\left[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau)) \dot{\theta}_{1,2}(\tau)\, \mathrm{d}\tau\right] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}} \tag{83}$$

is the state transition matrix of the time-varying linear ordinary differential equation (76).

Proof. Equation (76) is a first-order linear time-varying ODE with state transition matrix $\Phi(t, t_0)$. Thus, (82) is obvious. To compute the state transition matrix $\Phi(t, t_0)$, observe that we get from (71)

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\theta_{1,2}} = 2\beta(\theta_{1,2}) \sin \theta_{1,2}. \tag{84}$$

From this and from the definition of A_1^4 in (76) we get

$$A_1^4(\theta_{1,2}) = \frac{\beta(\theta_{1,2})}{\Delta(\theta_{1,2})} \sin \theta_{1,2} = \frac{1}{2\Delta} \frac{\mathrm{d}\Delta}{\mathrm{d}\theta_{1,2}}. \tag{85}$$

Thus

$$\Phi(t, t_0) = \exp\left[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau)) \dot{\theta}_{1,2}(\tau)\, \mathrm{d}\tau\right] = \exp\left[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A_1^4(\theta_{1,2})\, \mathrm{d}\theta_{1,2}\right]$$

$$= \exp\left[\int_{\Delta(\theta_{1,2}(t_0))}^{\Delta(\theta_{1,2}(t))} \frac{1}{2} \frac{\mathrm{d}\Delta}{\Delta}\right] = \exp\left[\ln\left(\sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}}\right)\right] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}}. \tag{86}$$

□

Equation (82) can be used to derive qualitative information about the momentum, which can be useful in motion control.

Proposition 8 (sign of the non-holonomic momentum). *Assume $d_1 > d_2$.*

- *(a) Let $I_{z_1} d_2 > I_{z_2} d_1$.*
  Assume further that the initial momentum of the system is non-positive. Then, the momentum p is negative at all subsequent times.
- *(b1) Let $I_{z_1} d_2 < I_{z_2} d_1$.*
  Suppose, in addition, that the angle $\theta_{1,2}$ remains in an $\tilde{\epsilon}$-neighbourhood of $\theta_{1,2} = 0$, with $\tilde{\epsilon} \le \cos^{-1}(I_{z_1} d_2 / I_{z_2} d_1)$. Assume further that the initial momentum of the system is non-negative. Then, the momentum p is positive at all subsequent times.
- *(b2) Let $I_{z_1} d_2 < I_{z_2} d_1$.*
  Suppose, in addition, that the angle $\theta_{1,2}$ remains outside an $\tilde{\epsilon}$-neighbourhood of $\theta_{1,2} = 0$, with $\tilde{\epsilon} \le \cos^{-1}(I_{z_1} d_2 / I_{z_2} d_1)$. Assume further that the initial momentum of the system is non-positive. Then, the momentum p is negative at all subsequent times.

Proof. Since $d_1 > d_2$, we know that $\Delta > 0$ and $\lambda = d_1 + d_2 \cos\theta_{1,2} > 0$, for all $\theta_{1,2}$.

(a) In the case $I_{z_1} d_2 > I_{z_2} d_1$ we have $\gamma = -I_{z_1} d_2 + I_{z_2} d_1 \cos\theta_{1,2} < 0$, for all $\theta_{1,2}$. Thus $A_2^4 = (m_1/\Delta)\lambda\gamma < 0$, for all $\theta_{1,2}$ and, thus, the second term of (82) is negative. If $p(t_0) \leq 0$, then $p(t) < 0$, $\forall t > t_0$.

(b1) In the case $I_{z_1} d_2 < I_{z_2} d_1$, by our choice of the $\tilde{\epsilon}$-neighbourhood we have $\gamma > 0$, for all $\theta_{1,2}$ in this neighbourhood. Then, $A_2^4 = (m_1/\Delta)\lambda\gamma > 0$ and the second term of (82) is positive. If $p(t_0) \geq 0$, then $p(t) > 0$, $\forall t > t_0$.

(b2) In the case $I_{z_1} d_2 < I_{z_2} d_1$, by our choice of the $\tilde{\epsilon}$-neighbourhood we have $\gamma < 0$, for all $\theta_{1,2}$ outside this neighbourhood. Then, $A_2^4 = m_1/\Delta\lambda\gamma < 0$ and the second term of (82) is negative. If $p(t_0) \leq 0$, then $p(t) < 0$, $\forall t > t_0$. □

As shown in Bloch *et al.* (1996), the momentum equation (76) can be derived directly from the Lagrange–d'Alembert principle (46), by considering a test vector u in the space $\mathcal{S}_q \subset \mathcal{D}_q \subset T_q Q$. This approach is used below to derive the momentum equation for the case when external forces, of the type considered in equation (50), are acting on the Roller Racer.

Proposition 9 (momentum equation with external forces). *Consider external forcing to the system described by the 1-form $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$. The non-holonomic momentum evolves according to the equation below:*

$$\frac{\mathrm{d}p}{\mathrm{d}t} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2} p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2} F_{\theta_1}. \quad (87)$$

Proof. Consider the Lagrange–d'Alembert principle (53) with u restricted to $\mathcal{S}_q \subset \mathcal{D}_q$, instead of belonging to the whole $\mathcal{D}_q$.

(i) For $d_2 \neq 0$, the vectors $v \in \mathcal{D}_q$ and $\dot{v}$ in this equation are given by (55) and (57), while $u \in \mathcal{S}_q \subset \mathcal{D}_q$ is given by (54), where $\alpha_1 = r(\theta_{1,2})/d_2$ and $\alpha_2 = (\sin\theta_{1,2})/d_2$ (cf. equation (33)). Thus, (53) takes the form (58), with α_1 and α_2 as specified above, which gives

$$\begin{aligned}(I_{z_2}\sin^2\theta_{1,2} + m_1 d_2 r(\theta_{1,2}))\dot{\nu}_1 + (I_{z_1} d_2 - I_{z_2} d_1 \cos\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_2 \\ + I_{z_2}(\nu_1\cos\theta_{1,2} + \nu_2 d_1 \sin\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2} \\ = r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2}, \quad (88)\end{aligned}$$

from which we get

$$\begin{aligned}\delta(\theta_{1,2})\dot{\nu}_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\,\dot{\nu}_2 = -I_{z_2}(\nu_1\cos\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2} \\ + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2}.\end{aligned} \quad (89)$$

Consider the non-holonomic momentum defined in (70), that is

$$p = m_1 r(\theta_{1,2})(v_1\cos\theta_1 + v_2\sin\theta_1) + [(I_{z_1} + I_{z_2})v_3 + I_{z_2} v_4]\sin\theta_{1,2}, \quad (90)$$

with $v = (v_1,\ v_2,\ v_3,\ v_4) \in T_q Q$. By restricting v to $\mathcal{D}_q$, we get for p (using, for $d_2 \neq 0$, the expression (55)):

$$
\begin{aligned}
p &= m_1 r(\theta_{1,2}) d_2 \nu_1 + (I_{z_1} + I_{z_2}) d_2 \sin\theta_{1,2}\nu_2 \\
&\quad + I_{z_2} \sin\theta_{1,2}(\sin\theta_{1,2}\nu_1 - r(\theta_{1,2})\nu_2) \\
&= \delta(\theta_{1,2})\nu_1 - \gamma(\theta_{1,2}) \sin\theta_{1,2}\nu_2. \qquad (91)
\end{aligned}
$$

The last of the equations (55) (the one for $v_4 \equiv \dot{\theta}_{1,2}$) and equation (91) are linear in ν_1 and ν_2. By solving them, we get

$$
\nu_1 = \frac{1}{d_2\Delta(\theta_{1,2})}[r(\theta_{1,2})p - \gamma(\theta_{1,2}) \sin\theta_{1,2}\dot{\theta}_{1,2}],
$$

$$
\nu_2 = \frac{1}{d_2\Delta(\theta_{1,2})}[\sin\theta_{1,2}p - \delta(\theta_{1,2})\dot{\theta}_{1,2}]. \qquad (92)
$$

By differentiating (91), we get

$$
\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= \delta(\theta_{1,2})\dot{\nu}_1 - \gamma(\theta_{1,2}) \sin\theta_{1,2}\dot{\nu}_2 \\
&\quad + \left(\frac{\partial\delta}{\partial\theta_{1,2}}\nu_1 - \frac{\partial\gamma}{\partial\theta_{1,2}}\sin\theta_{1,2}\nu_2 - \gamma(\theta_{1,2}) \cos\theta_{1,2}\nu_2\right)\dot{\theta}_{1,2}. \qquad (93)
\end{aligned}
$$

Replacing the first two terms of the right-hand side above with their expression from (89), using (92) and using the definitions (77) and (78), we get (87).

(ii) For $d_2 = 0$, the vectors $v \in \mathcal{D}_q$ and $\dot{v}$ in equation (53) are given by (62) and (64), while $u \in \mathcal{S}_q \subset \mathcal{D}_q$ is given by (61), where $\alpha_1 = 1$ and $\alpha_2 = 0$ (cf. equation (33)). From (53) we get

$$
\begin{aligned}
\Delta(\theta_{1,2})\dot{\nu}_1 + I_{z_2} \sin\theta_{1,2}\dot{\nu}_2 &= -\beta(\theta_{1,2}) \sin\theta_{1,2}\dot{\theta}_{1,2}\ \nu_1 \\
&\quad + r(\theta_{1,2})(F_{x_1} \cos\theta_1 + F_{y_1} \sin\theta_1) + \sin\theta_{1,2}F_{\theta_1}. \qquad (94)
\end{aligned}
$$

From the definition of the non-holonomic momentum (equation (70)) and by restricting the corresponding v to $\mathcal{D}_q$, we get

$$
p = \Delta(\theta_{1,2})\ \nu_1 + I_{z_2} \sin\theta_{1,2}\nu_2. \qquad (95)
$$

From this and (62), we can get ν_1 and ν_2 as functions of p and $\dot{\theta}_{1,2}$

$$
\nu_1 = \frac{1}{\Delta(\theta_{1,2})}(p - I_{z_2} \sin\theta_{1,2}\dot{\theta}_{1,2}), \quad \nu_2 = \dot{\theta}_{1,2}. \qquad (96)
$$

By differentiating p, we get

$$
\frac{\mathrm{d}p}{\mathrm{d}t} = \Delta(\theta_{1,2})\dot{\nu}_1 + I_{z_2} \sin\theta_{1,2}\dot{\nu}_2 + 2\beta(\theta_{1,2}) \sin\theta_{1,2}\dot{\theta}_{1,2}\ \nu_1 + I_{z_2} \cos\theta_{1,2}\dot{\theta}_{1,2}\ \nu_2. \qquad (97)
$$

Replacing the first two terms of the right-hand side above with their expression from (94), using (96) and using the definitions (77) and (78) for $d_2 = 0$, we get (87). □

In the case when viscous friction, of the type considered in section 4.1, is present, the momentum equation takes the form below.

Proposition 10 (momentum equation with friction). *In the presence of friction, the non-holonomic momentum evolves according to the equation*

$$\frac{\mathrm{d}p}{\mathrm{d}t} = [A_1^4(\theta_{1,2})\dot{\theta}_{1,2} - A_1^5(\theta_{1,2})]\, p + \ [A_2^4(\theta_{1,2})\dot{\theta}_{1,2} + A_2^5(\theta_{1,2})]\, \dot{\theta}_{1,2}, \tag{98}$$

where

$$A_1^5(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\sin\theta_{1,2} + \eta_2(\theta_{1,2})r(\theta_{1,2})],$$

$$A_2^5(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\delta(\theta_{1,2}) + \eta_2(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2}],$$

$$\eta_1(\theta_{1,2}) \stackrel{\text{def}}{=} 2\left[\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4} + \frac{k_2}{R_2^2}d_1\lambda(\theta_{1,2})\right]\sin\theta_{1,2},$$

$$\eta_2(\theta_{1,2}) \stackrel{\text{def}}{=} 2\frac{k_1}{R_1^2}r(\theta_{1,2}) + 2\frac{k_2}{R_2^2}\lambda(\theta_{1,2})\cos\theta_{1,2}. \tag{99}$$

If $d_1 > d_2$, *then* $A_1^5(\theta_{1,2}) > 0$, *for all* $\theta_{1,2}$.

Proof. We consider the momentum equation (87) with an external force 1-form α_e which is due to friction and to the torque $\tau_{1,2}$ at the joint $O_{1,2}$. Thus, α_e has the form (68). The force-related terms from the right-hand side of (87) take, then, the form

$$r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} = -\eta_1(\theta_{1,2})\dot{\theta}_1 - \eta_2(\theta_{1,2})(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1), \tag{100}$$

where η_1 and η_2 are defined in (99).

(i) Let $d_2 \neq 0$. For $v = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \in \mathcal{D}_q$, we have from (55)

$$\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 = \nu_1\, d_2 \quad \text{and} \quad \dot{\theta}_1 = \nu_2\, d_2,$$

for $\nu_1, \nu_2 \in \mathbb{R}$. Thus

$$r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} = -\eta_1\nu_2 d_2 - \eta_2\nu_1 d_2.$$

Using (92), we get

$$\begin{aligned} r(\theta_{1,2})&(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \\ &= -\frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\sin\theta_{1,2} + \eta_2(\theta_{1,2})r(\theta_{1,2})]p \\ &\quad + \frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\delta(\theta_{1,2}) + \eta_2(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2}]\dot{\theta}_{1,2}. \end{aligned} \tag{101}$$

From this, the result follows.

(ii) Let $d_2 = 0$. For $v \in \mathcal{D}_q$, we have from (62)

$$\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 = \nu_1\, d_1\cos\theta_{1,2} \quad \text{and} \quad \dot{\theta}_1 = \nu_1\, \sin\theta_{1,2},$$

for $\nu_1, \nu_2 \in \mathbb{R}$. From (96) and (100), the result follows.

It is easy to check from the definitions (99) that

$$A_1^5(\theta_{1,2}) \geq \frac{2}{\Delta(\theta_{1,2})}\frac{k_2}{R_2^2}\lambda^2(\theta_{1,2}). \tag{102}$$

If $d_1 > d_2$, then $\Delta(\theta_{1,2}) > 0$ and also $\lambda(\theta_{1,2}) > 0$ for all $\theta_{1,2}$. Thus, $A_1^5(\theta_{1,2}) > 0$. □

Comparing the momentum equations (98) and (76), it is possible to identify the extra terms that are due to friction.

When the shape of the Roller Racer is held constant ($\dot{\theta}_{1,2} = 0$), the momentum equation for the model without friction (76) takes the form $\dot{p} = 0$, i.e. the momentum is conserved. However, in the same case, the momentum equation for the model with friction (98) takes the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -A_1^5(\theta_{1,2}(0))p, \tag{103}$$

thus

$$p(t) = \mathrm{e}^{-A_1^5(\theta_{1,2}(0))t} p(0), \tag{104}$$

for a constant $A_1^5(\theta_{1,2}(0)) > 0$, which is the rate at which the momentum decreases exponentially and at which the system will come to rest. This provides a 'braking mechanism' for the Roller Racer, which has been noticed in experiments with the ISL's Roller Racer prototypes.

4.3. *Reconstruction of group motion*

Assume that a shape-space trajectory $\theta_{1,2}(\cdot) \subset \mathcal{S}$ has been specified. The corresponding non-holonomic momentum can be determined from the solution of the momentum equation (76) in the case of the Roller Racer model without external forces, or from the solution of the momentum equation (98) in the case of the Roller Racer model with friction. From the definition of the non-holonomic momentum (equation (70)) and from the non-holonomic constraints (equations (16) and (18)), we can reconstruct the group trajectory $g_1(\cdot) = g_1(x_1(\cdot), y_1(\cdot), \theta_1(\cdot)) \subset SE(2)$. This can be done by first specifying $(\dot{x}_1, \dot{y}_1, \dot{\theta}_1)$ and then integrating to find (x_1, y_1, θ_1).

Proposition 11 (reconstruction of group trajectory). *For $g_1 = g_1(x_1, y_1, \theta_1) \in SE(2)$, the corresponding curve in the Lie algebra $\xi_1 = g_1^{-1}\dot{g}_1$ is given by*

$$\xi 1 = \xi_1^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_1 + \xi_2^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_2, \tag{105}$$

where for $d_1 \neq d_2$, the components of ξ_1 are

$$\xi_1^1(\theta_{1,2}, \dot{\theta}_{1,2}) = \dot{\theta}_1 = \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot{\theta}_{1,2}], \tag{106}$$

$$\xi_2^1(\theta_{1,2}, \dot{\theta}_{1,2}) = \dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1 = \frac{1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2}) \sin\theta_{1,2}\dot{\theta}_{1,2}]. \tag{107}$$

The group trajectory is given by solving

$$\dot{\theta}_1 = \xi_1^1 = \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot{\theta}_{1,2}],$$

$$\dot{x}_1 = \cos\theta_1\ \xi_2^1 = \frac{\cos\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}],$$

$$\dot{y}_1 = \sin\theta_1\ \xi_2^1 = \frac{\sin\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}]. \tag{108}$$

The solution can be obtained by quadratures.

Proof. Equation (106) is immediate from (73).

When $d_1 \neq d_2$, either $\sin\theta_{1,2} \neq 0$ or $r(\theta_{1,2}) = d_1\cos\theta_{1,2} + d_2 \neq 0$.

First consider $\sin\theta_{1,2} \neq 0$. From equations (19) and (73) we get

$$(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\sin\theta_{1,2} = \frac{r(\theta_{1,2})}{\Delta}\sin\theta_{1,2}\, p - \frac{\gamma(\theta_{1,2})}{\Delta}\sin^2\theta_{1,2}\ \dot{\theta}_{1,2}. \tag{109}$$

Since $\sin\theta_{1,2} \neq 0$, equation (107) follows.

Now let $r(\theta_{1,2}) \neq 0$. From equation (70) we get

$$m_1 r(\theta_{1,2})(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) = p - (I_{z_1} + I_{z_2})\sin\theta_{1,2}\dot{\theta}_1 - I_{z_2}\sin\theta_{1,2}\dot{\theta}_{1,2}. \tag{110}$$

From equation (73) we get

$$m_1 r(\theta_{1,2})(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) = p - \frac{I_{z_1} + I_{z_2}}{\Delta}\sin^2\theta_{1,2}\, p + \frac{(I_{z_1} + I_{z_2})\delta - I_{z_2}\Delta}{\Delta}\sin\theta_{1,2}\dot{\theta}_{1,2}. \tag{111}$$

Observe that $(I_{z_1} + I_{z_2})\delta(\theta_{1,2}) - I_{z_2}\Delta(\theta_{1,2}) = -m_1 r(\theta_{1,2})\gamma(\theta_{1,2})$. Then

$$m_1 r(\theta_{1,2})(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) = \frac{m_1 r(\theta_{1,2})}{\Delta}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\ \dot{\theta}_{1,2}]. \tag{112}$$

Since $r(\theta_{1,2}) \neq 0$, equation (107) follows.

Finally, equations (108) are immediate from (107) and (16).

Now observe that from (106) and (107) we get

$$\begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \sin\theta_{1,2} & -\delta(\theta_{1,2}) \\ r(\theta_{1,2}) & -\gamma(\theta_{1,2})\sin\theta_{1,2} \end{pmatrix}\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} \stackrel{\text{def}}{=} B(\theta_{1,2})\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} \tag{113}$$

and notice that

$$r(\theta_{1,2})\delta(\theta_{1,2}) - \gamma(\theta_{1,2})\sin^2\theta_{1,2} = d_2\Delta(\theta_{1,2}), \tag{114}$$

therefore

$$\det B(\theta_{1,2}) = \frac{d_2}{\Delta(\theta_{1,2})}. \tag{115}$$

Thus, in the case $d_2 = 0$, given a group trajectory $\xi_1 \subset \mathcal{G}$, we cannot always solve (113) for p and $\dot{\theta}_{1,2}$.

When $d_1 \neq d_2$ and $d_2 \neq 0$, from (113) and (115), we get

$$\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} = B^{-1}(\theta_{1,2})\begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} = \frac{1}{d_2}\begin{pmatrix} -\gamma(\theta_{1,2})\sin\theta_{1,2} & \delta(\theta_{1,2}) \\ -r(\theta_{1,2}) & \sin\theta_{1,2} \end{pmatrix}\begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix}. \tag{116}$$

4.4. *Principal fibre bundles and connections*

The following material on principal fibre bundles and connections is based on Bleecker (1981) and Nomizu (1956). These references consider principal fibre bundles where the group action is a right action. Here, we consider left actions and modify appropriately the definition of a principal fibre bundle, as for instance done in Yang (1992).

Let $\mathcal{S}$ be a differentiable manifold and G a Lie group. A differentiable manifold Q is called a (differentiable) 'principal fibre bundle', if the following conditions are satisfied:

(1) G acts on Q to the left, freely and differentiably:

$$\Phi : \; G \times Q \to Q : \; (g, q) \mapsto g \cdot q \stackrel{\text{def}}{=} \Phi_g \cdot q. \tag{117}$$

(2) $\mathcal{S}$ is the quotient space of Q by the equivalence relation induced by G, i.e. $\mathcal{S} = Q/G$ and the canonical projection $\pi : Q \to \mathcal{S}$ is differentiable.

(3) Q is locally trivial, i.e. every point $s \in \mathcal{S}$ has a neighbourhood U such that $\pi^{-1}(U) \subset Q$ is isomorphic with $U \times G$, in the sense that $q \in \pi^{-1}(U) \mapsto (\pi(q), \phi(q)) \in U \times G$ is a diffeomorphism such that $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(g \cdot q) = g\phi(q), \forall g \in G$.

For $s \in \mathcal{S}$, the 'fibre over' s is a closed submanifold of Q which is differentiably isomorphic with G. For any $q \in Q$, the 'fibre through' q is the fibre over $s = \pi(q)$. When $Q = \mathcal{S} \times G$, then Q is said to be a 'trivial' principal fibre bundle (figure 4).

Definition 1. Let $(Q, \mathcal{S}, \pi, G)$ be a principal fibre bundle. The kernel of $T_q\pi$, denoted $V_q \stackrel{\text{def}}{=} \{v \in T_qQ \mid \; T_q\pi(v) = 0\}$, is the subspace of T_qQ tangent to the fibre through q and is called the 'vertical subspace'. A 'connection' on the principal fibre bundle is a choice of a tangent subspace $H_q \subset T_qQ$ at each point $q \in Q$, called the 'horizontal subspace', such that:

(1) $T_qQ = H_q \oplus V_q$.
(2) For every $g \in G$ and $q \in Q$, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.
(3) H_q depends differentiably on q.

4.5. *The non-holonomic connection*

In this section, we explicitly realize a connection on the bundle $(Q, \mathcal{S}, \pi, G)$, the non-holonomic connection of Bloch *et al.* (1996) for the Roller Racer.

Consider the 'kinetic energy inner product' $\ll \, , \, \gg$ specified by the Lagrangian (44):

$$\ll v, \tilde{v} \gg \stackrel{\text{def}}{=} v^{\mathrm{T}} \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & I_{z_1} + I_{z_2} & I_{z_2} \\ 0 & 0 & I_{z_2} & I_{z_2} \end{pmatrix} \tilde{v}, \tag{118}$$

for $v, \tilde{v} \in T_qQ$.

Proposition 12. *The orthogonal complement H_q of the subspace $\mathcal{S}_q$ with respect to the constraint subspace $\mathcal{D}_q$, that is*

$$\mathcal{S}_q \oplus H_q = \mathcal{D}_q, \tag{119}$$

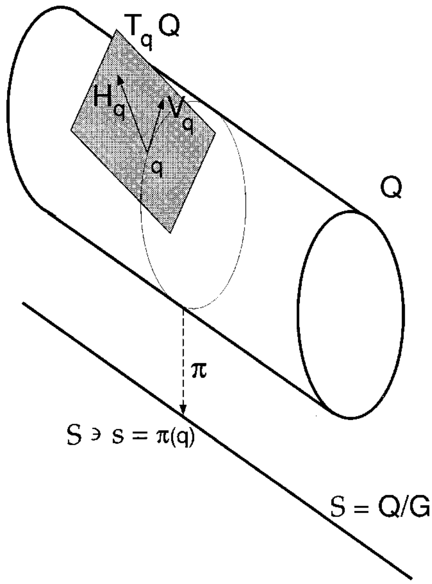


Figure 4. Connection on a principal fibre bundle.

where orthogonality is defined with respect to the kinetic energy inner product $\ll , \gg$, *is given by*

$$H_q = \{\xi_Q^H\}, \tag{120}$$

where

$$\xi_Q^H = \gamma(\theta_{1,2}) \sin\theta_{1,2}\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \delta(\theta_{1,2})\frac{\partial}{\partial \theta_1} - \Delta(\theta_{1,2})\frac{\partial}{\partial \theta_{1,2}}. \tag{121}$$

Proof. Since dim $\mathcal{S}_q = 1$ and dim $\mathcal{D}_q = 2$, we should have dim $H_q = 1$. Consider an element $\xi_Q^H \in H_q$. As ξ_Q^H also belongs to $\mathcal{D}_q$, it can be written, as a function of the basis elements of $\mathcal{D}_q$, as

$$\xi_Q^H = \alpha_1 \xi_Q^1 + \alpha_2 \xi_Q^2, \tag{122}$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

(i) When $d_2 \neq 0$, we have from (23)

$$\xi_Q^H = \alpha_1 d_2\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \alpha_2 d_2 \frac{\partial}{\partial \theta_1} + [\alpha_1 \sin\theta_{1,2} - \alpha_2 r(\theta_{1,2})]\frac{\partial}{\partial \theta_{1,2}}. \tag{123}$$

The vector ξ_Q^H should be orthogonal to every $\xi_Q^q \in \mathcal{S}_q$, i.e. $\ll \xi_Q^q, \xi_Q^H \gg = 0$. This gives $\delta(\theta_{1,2})\alpha_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\alpha_2 = 0$. Choose $\alpha_1 = \gamma(\theta_{1,2})\sin\theta_{1,2}$ and $\alpha_2 = \delta(\theta_{1,2})$. Using (114) and dividing by d_2, we get (121).

(ii) In the case $d_2 = 0$, we have from (24):

$$\xi_Q^H = \alpha_1 d_1 \cos\theta_{1,2}\left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\right) + \alpha_1 \sin\theta_{1,2}\frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_{1,2}}. \tag{124}$$

From orthogonality to ξ_Q^q, we obtain $\alpha_1 = I_{z_2} \sin\theta_{1,2}$ and $\alpha_2 = [(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}]$, thus

$$\xi_Q^H = I_{z_2} d_1 \sin\theta_{1,2} \cos\theta_{1,2} \left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1} \right) + I_{z_2} \sin^2\theta_{1,2} \frac{\partial}{\partial \theta_1} - [(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}] \frac{\partial}{\partial \theta_{1,2}}. \tag{125}$$

Note that, when $d_2 = 0$, this expression is the same as (121). □

Proposition 13. *The orthogonal complement $\mathcal{U}_q$ of the subspace $\mathcal{S}_q$ with respect to the subspace $T_q \operatorname{Orb}(q)$, that is*

$$\mathcal{S}_q \oplus \mathcal{U}_q = T_q \operatorname{Orb}(q), \tag{126}$$

where orthogonality is defined with respect to the kinetic energy inner product $\ll, \gg$, is given by

$$\mathcal{U}_q = \{\xi_Q^{\mathcal{U}_1}, \xi_Q^{\mathcal{U}_2}\}, \tag{127}$$

where

$$\xi_Q^{\mathcal{U}_1} = -\sin\theta_1 \frac{\partial}{\partial x_1} + \cos\theta_1 \frac{\partial}{\partial y_1},$$

$$\xi_Q^{\mathcal{U}_2} = (I_{z_1} + I_{z_2}) \sin\theta_{1,2} \left(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1} \right) - m_1 (d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_1}. \tag{128}$$

Proof. Since dim $\mathcal{S}_q = 1$ and dim $T_q \operatorname{Orb}(q) = 3$, we have dim $\mathcal{U}_q = 2$. Let $\xi_Q^{\mathcal{U}_1}$ and $\xi_Q^{\mathcal{U}_2}$ be two basis elements of $\mathcal{U}_q$. Since the $\xi_Q^{\mathcal{U}_i}$, $i = 1, 2$ also belong to $T_q \operatorname{Orb}(q)$, they can be expressed as a function of its basis elements as

$$\xi_Q^{\mathcal{U}_i} = u_1^i \frac{\partial}{\partial x_1} + u_2^i \frac{\partial}{\partial y_1} + u_3^i \frac{\partial}{\partial \theta_1},$$

for some $u_j^i \in \mathbb{R}$. The $\xi_Q^{\mathcal{U}_i}$ need to be mutually linearly independent and orthogonal to $\xi_Q^q \in \mathcal{S}_q$. This last requirement gives

$$m_1 r(\theta_{1,2})(u_1^i \cos\theta_1 + u_2^i \sin\theta_1) + (I_{z_1} + I_{z_2}) \sin\theta_{1,2} u_3^i = 0, \ i = 1, 2.$$

Two linearly independent vectors that fulfil this condition are the ones given in (128). □

The configuration space for the Roller Racer is $Q = SE(2) \times S^1$. From left invariance of the system's kinematics, the tangent space $T_q Q$ to the configuration space is

$$T_q Q = \{(\dot{g}_1, \dot{\theta}_{1,2}) \mid g_1 \in SE(2), \theta_{1,2} \in S^1\} = \{(g_1 \xi_1, \dot{\theta}_{1,2}) \mid \xi_1 \in se(2), \dot{\theta}_{1,2} \in \mathbb{R}\}. \tag{129}$$

Consider, then, the configuration space $Q = SE(2) \times S^1$, the group $G = SE(2)$, the shape space $\mathcal{S} = S^1$ of the Roller Racer, which is the quotient space of Q by G, and the canonical projection

$$\pi : Q \longrightarrow \mathcal{S} : (g_1, \theta_{1,2}) \mapsto \theta_{1,2}. \tag{130}$$

The projection π is differentiable and its differential at $q = (g_1,\ \theta_{1,2}) \in Q$ is

$$T_q\pi : \ T_qQ \longrightarrow T_{\pi(q)}\mathcal{S} : \ (g_1\xi g_1, \dot{\theta}_{1,2}) \mapsto \dot{\theta}_{1,2}. \qquad (131)$$

The quadruple $(Q, \mathcal{S}, \pi, G)$, together with the left action Φ of G on Q defined by equation (28), is a (trivial) 'principal fibre bundle'. This bundle expresses the ultimate dependence of all configuration variables on the shape $\theta_{1,2}$.

By considering the Lagrangian dynamics in addition to the kinematic constraints, we can synthesize a principal connection for this system, which reflects the dependence of all configuration velocities on the shape variation $\dot{\theta}_{1,2}$.

Proposition 14 (non-holonomic connection). *The non-holonomic kinematic constraints and the system dynamics determine a connection on the principal fibre bundle* $(Q, \mathcal{S}, \pi, G)$. *The horizontal subspace of the connection is the subspace* H_q *defined in (120) and (121), that is the orthogonal complement of the subspace* $\mathcal{S}_q$ *with respect to the constraint distribution* $\mathcal{D}_q$, *with orthogonality defined with respect to the kinetic energy inner product. When* $d_1 \neq d_2$, *the horizontal subspace is*

$$\begin{aligned} H_q &\stackrel{\text{def}}{=} \{v \in T_qQ \mid \ v \in \{\xi Q^H\}\} \\ &= \Bigg\{ v = (\dot{x}_1,\ \dot{y}_1,\ \dot{\theta}_1,\ \dot{\theta}_{1,2}) \in T_qQ \mid \ \dot{x}_1 - \frac{\gamma(\theta_{1,2}) \sin\theta_{1,2}}{\Delta(\theta_{1,2})} \cos\theta_1\ \dot{\theta}_{1,2}, \\ &\qquad \dot{y}_1 = -\frac{\gamma(\theta_{1,2}) \sin\theta_{1,2}}{\Delta(\theta_{1,2})} \sin\theta_1\ \dot{\theta}_{1,2}, \dot{\theta}_1 = -\frac{\delta(\theta_{1,2})}{\Delta(\theta_{1,2})}\ \dot{\theta}_{1,2} \Bigg\}. \end{aligned} \qquad (132)$$

The vertical subspace of the connection is

$$V_q \stackrel{\text{def}}{=} \{v \in T_qQ \mid T_q\pi = 0\} = \{v = (\dot{x}_1,\ \dot{y}_1,\ \dot{\theta}_1,\ \dot{\theta}_{1,2}) \in T_qQ \mid \dot{\theta}_{1,2} = 0\}. \qquad (133)$$

Proof. It is easy to see that the horizontal subspace H_q defined in (132) is

$$H_q = \{(g_1\xi_1,\ \dot{\theta}_{1,2}) \mid g_1 \in SE(2),\ \ \xi_1 = -A_{\text{loc}}(\theta_{1,2})\ \dot{\theta}_{1,2}\}, \qquad (134)$$

where the local form A_{loc} of the connection (Bloch *et al.* 1996) is

$$A_{\text{loc}}(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{\delta(\theta_{1,2})}{\Delta(\theta_{1,2})} \mathcal{A}_1 + \frac{\gamma(\theta_{1,2}) \sin\theta_{1,2}}{\Delta(\theta_{1,2})} \mathcal{A}_2 \ \in se(2). \qquad (135)$$

To show property (1) of Definition 1, consider a non-zero vector $v \in V_q \cap H_q$. Since v is non-zero and belongs to H_q, we have from (134) that $\dot{\theta}_{1,2} \neq 0$. But then, because of (133), v cannot belong also to V_q, as we supposed. Thus, $V_q \cap H_q = \{0\}$. Moreover, dim V_q + dim $H_q = 3 + 1 = 4 =$ dim T_qQ. Thus, $V_q \oplus H_q = T_qQ$.

To show property (2) of Definition 1, consider a $g \in G$. From left-invariance

$$\begin{aligned} T_q\Phi_g \cdot H_q = g \cdot H_q &= g \cdot \{(\mathrm{g}_1\xi_1, \dot{\theta}_{1,2}) \mid \xi_1 = -A_{\text{loc}}(\theta_{1,2})\dot{\theta}_{1,2}\} \\ &\stackrel{\text{def}}{=} \{(g_1\xi_1, \dot{\theta}_{1,2}) \mid \ \xi_1 = -A_{\text{loc}}(\theta_{1,2})\dot{\theta}_{1,2}\} \end{aligned} \qquad (136)$$

and

$$H_{g\cdot q} = \{v \in T_{g\cdot q}Q \mid \ \xi_1 = -A_{\text{loc}}(\theta_{1,2})\dot{\theta}_{1,2}\} = \{(gg_1\xi_1, \dot{\theta}_{1,2}) \mid \xi_1 = -A_{\text{loc}}(\theta_{1,2})\dot{\theta}_{1,2}\}. \qquad (137)$$

Then, obviously, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.

The differentiability of H_q with respect to $q \in Q$ (property (3) of Definition 1) follows from the smooth dependence of $A_{\rm loc}$ on the shape $\theta_{1,2}$ and from the left-invariance of our system. □

Physically, V_q is the set of all possible rigid motions of the system on the plane that keep shape constant; these 'frozen-shape' motions do not need to satisfy the non-holonomic constraints. On the other hand, H_q is the set of all possible motions of the system on the plane that comply with the non-holonomic constraints. Observe that all such motions are due to shape variations.

Let the set of Lie algebra elements, whose infinitesimal generators belong to $\mathcal{S}_q$, be denoted as $\mathcal{G}^q$. From (40): $\mathcal{G}^q = {\rm sp}\{\xi^q\}$. The 'locked inertia tensor' $\mathbb{I}(q)$ relative to $\mathcal{G}^q$ is defined in Marsden and Ratiu (1994) and Bloch *et al.* (1996), as

$$\mathbb{I}(q) : \mathcal{G}^q \longrightarrow (\mathcal{G}^q)^\star : \xi^q \longmapsto \langle \mathbb{I}(q)\, \xi^q, \,\cdot\, \rangle, \tag{138}$$

where, for $\eta^q \in \mathcal{G}^q$, with corresponding infinitesimal generator $\eta^q_Q \in \mathcal{S}_q$, we define

$$\langle \mathbb{I}(q)\, \xi^q, \, \eta^q \rangle \overset{\rm def}{=} \ll \xi^q_Q, \, \eta^q_Q \gg, \tag{139}$$

and where $\ll \, , \, \gg$ is the kinetic energy inner product defined in (118).

It is easy to verify from (33) and (71), that for the Roller Racer

$$\langle \mathbb{I}(q)\, \xi^q, \, \xi^q \rangle = \ll \xi^q_Q, \, \xi^q_Q \gg = \Delta(\theta_{1,2}). \tag{140}$$

Since $\eta^q_Q = \beta\, \xi^q_Q$, for some $\beta \in \mathbb{R}$, we have

$$\langle \mathbb{I}(q)\, \xi^q, \, \eta^q \rangle = \ll \xi^q_Q, \, \eta^q_Q \gg = \beta \ll \xi^q_Q, \, \xi^q_Q \gg = \beta\, \Delta(\theta_{1,2}). \tag{141}$$

With the above definition of $A_{\rm loc}$ in equation (135), the reconstructed group trajectory equations (105), (106) and (107) take the form

$$\xi_1 = g_1^{-1} \dot{g}_1 = -A_{\rm loc}(\theta_{1,2})\, \dot{\theta}_{1,2} + \mathbb{I}^{-1}_{\rm loc}(\theta_{1,2})\, p, \tag{142}$$

where

$$\mathbb{I}^{-1}_{\rm loc}(\theta_{1,2}) = \frac{\sin\theta_{1,2}}{\Delta(\theta_{1,2})} \mathcal{A}_1 + \frac{r(\theta_{1,2})}{\Delta(\theta_{1,2})} \mathcal{A}_2 \tag{143}$$

is the local form of the inverse of the locked inertia tensor of the Roller Racer.

4.6. *The reduced dynamics*

A Lagrangian reduction procedure for systems with non-holonomic constraints is developed in Bloch *et al.* (1996). Its goal is to lower the dimension of the system's dynamics by passing to an appropriate quotient space. The reduced equations are composed of a set of Euler–Lagrange equations on the shape space of the system, where some of the forcing terms are due to the curvature of the non-holonomic connection, and a set of momentum equations. These reduced equations, together with the reconstruction ones and the non-holonomic constraints, give the full set of equations of motion of the system.

In the case of the Roller Racer, the reduced dynamics are composed of a Euler–Lagrange equation on the one-dimensional shape space $\mathcal{S}$ and of the momentum equation derived in section 4.2. The reduced Euler–Lagrange equation is a second-order equation on the shape variable $\theta_{1,2}$ and involves the 'reduced Lagrangian'

$$l_c(\theta_{1,2},\, \dot\theta_{1,2},\, p) \stackrel{\text{def}}{=} L(q, \dot q(q,\, \dot\theta_{1,2},\, p)) \stackrel{44,108}{=} \frac{1}{2}\frac{1}{\Delta(\theta_{1,2})}\, p^2 + \frac{1}{2}\frac{\Delta_1(\theta_{1,2})}{\Delta(\theta_{1,2})}\, \dot\theta_{1,2}^2, \qquad (144)$$

as described in Bloch *et al.* (1996). The quantity Δ_1 is defined in equation (145) below. The curvature of the non-holonomic connection of section 4.5 can be easily seen to be zero, since the shape space is one-dimensional, thus the corresponding forcing terms are zero.

In our derivation of the reduced dynamics of the Roller Racer, we take a short cut around Bloch *et al.* (1996), by employing directly the Lagrange–d'Alembert principle with a test vector horizontal with respect to the non-holonomic connection, but still in the constraint distribution, thus belonging to the orthogonal complement H_q of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$.

Let

$$\Delta_1(\theta_{1,2}) \stackrel{\text{def}}{=} I_{z_1} I_{z_2} \sin^2\theta_{1,2} + m_1(I_{z_1} d_2^2 + I_{z_2} d_1^2 \cos^2\theta_{1,2}). \qquad (145)$$

Observe that $\Delta_1(\theta_{1,2}) > 0,\ \forall q \in Q$.

Proposition 15 (reduced dynamics with external forces). *Consider external forcing to the system described by the 1-form* $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$*. The reduced dynamics of the Roller Racer take the form*

$$\begin{aligned} \ddot\theta_{1,2} &= B_1^4(\theta_{1,2})\dot\theta_{1,2} p + B_2^4(\theta_{1,2})\dot\theta_{1,2}^2 \\ &\quad + B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B_4^4(\theta_{1,2})F_{\theta_1} + B_5^4(\theta_{1,2})F_{\theta_{1,2}}, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= A_1^4(\theta_{1,2})\dot\theta_{1,2} p + A_2^4(\theta_{1,2})\dot\theta_{1,2}^2 + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2} F_{\theta_1}, \end{aligned} \qquad (146)$$

where

$$\begin{aligned} B_1^4(\theta_{1,2}) &\stackrel{\text{def}}{=} -\frac{A_2^4(\theta_{1,2})}{\Delta_1(\theta_{1,2})}, \\ B_2^4(\theta_{1,2}) &\stackrel{\text{def}}{=} -\frac{1}{2}\frac{\Delta}{\Delta_1}\frac{\partial}{\partial\theta_{1,2}}\left(\frac{\Delta_1}{\Delta}\right) \\ &= \frac{m_1\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})\Delta_1(\theta_{1,2})}[\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})], \\ B_3^4(\theta_{1,2}) &\stackrel{\text{def}}{=} -\frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta_1(\theta_{1,2})}, \quad B_4^4(\theta_{1,2}) \stackrel{\text{def}}{=} -\frac{\delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}, \\ B_5^4(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}. \end{aligned} \qquad (147)$$

For $d_1 \neq d_2$*, we have* $B_5^4(\theta_{1,2}) > 0$*, for all* $q \in Q$.

Proof. Consider the Lagrange–d'Alembert principle (53) with a test vector $u \in \mathcal{D}_q$, which we restrict to the orthogonal complement H_q of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$. Orthogonality is defined using the kinetic energy inner product $\ll\ ,\ \gg$ of equation (118). Without loss of generality, we choose u as

$$u = \xi_Q^H = \gamma(\theta_{1,2})\sin\theta_{1,2}\left(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}\right) + \delta(\theta_{1,2})\frac{\partial}{\partial\theta_1} - \Delta(\theta_{1,2})\frac{\partial}{\partial\theta_{1,2}}. \tag{148}$$

(i) Let $d_2 \neq 0$. From (53), with u given by (148) and $\dot{v}$ given by (57), and since $\dot{v}_4 \equiv \ddot{\theta}_{1,2}$, we have

$$\begin{aligned} & m_1 d_2\gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_1 + [I_{z_1}\delta(\theta_{1,2}) + I_{z_2}(\delta(\theta_{1,2}) \\ & - \Delta(\theta_{1,2}))]d_2\dot{\nu}_2 + I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})]\ddot{\theta}_{1,2} \\ & = \gamma(\theta_{1,2})\sin\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\delta(\theta_{1,2}) - F_{\theta_{1,2}}\Delta(\theta_{1,2}), \end{aligned} \tag{149}$$

for $\nu_1, \nu_2 \in \mathbb{R}$. Observe that $\Delta(\theta_{1,2}) - \delta(\theta_{1,2}) = I_{z_1}\sin^2\theta_{1,2} + m_1 d_1 \cos \theta_{1,2} r(\theta_{1,2})$, then $I_{z_1}\delta(\theta_{1,2}) + I_{z_2}\ [\delta(\theta_{1,2}) - \Delta(\theta_{1,2})] = -m_1\gamma(\theta_{1,2})r(\theta_{1,2})$. Then, the left-hand side of (149) becomes

$$m_1 d_2\gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_1 - m_1 d_2\gamma(\theta_{1,2})r(\theta_{1,2})\dot{\nu}_2 + I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})]\ddot{\theta}_{1,2}. \tag{150}$$

By differentiating (92), the terms $\dot{\nu}_1$ and $\dot{\nu}_2$ above can be expressed as functions of $\theta_{1,2},\ \dot{\theta}_{1,2},\ \ddot{\theta}_{1,2},\ p,\ \dot{p}$:

$$\begin{aligned} \dot{\nu}_1 = \frac{1}{d_2\Delta^2}\Bigg[& r(\theta_{1,2})\Delta(\theta_{1,2})\dot{p} + \left(\frac{\partial r}{\partial\theta_{1,2}}\Delta(\theta_{1,2}) - r(\theta_{1,2})\frac{\partial\Delta}{\partial\theta_{1,2}}\right)\dot{\theta}_{1,2}\, p \\ & - \gamma(\theta_{1,2})\Delta(\theta_{1,2})\sin\theta_{1,2}\ddot{\theta}_{1,2} + \Bigg(\gamma(\theta_{1,2})\sin\theta_{1,2}\frac{\partial\Delta}{\partial\theta_{1,2}} \\ & - \frac{\partial\gamma}{\partial\theta_{1,2}}\Delta(\theta_{1,2})\sin\theta_{1,2} - \gamma(\theta_{1,2})\Delta(\theta_{1,2})\cos\theta_{1,2}\Bigg)\dot{\theta}_{1,2}^2\Bigg], \\ \dot{\nu}_2 = \frac{1}{d_2\Delta^2}\Bigg[& \Delta(\theta_{1,2})\sin\theta_{1,2}\dot{p} + \left(\Delta(\theta_{1,2})\cos\theta_{1,2} - \frac{\partial\Delta}{\partial\theta_{1,2}}\sin\theta_{1,2}\right)\dot{\theta}_{1,2}\, p \\ & - \Delta(\theta_{1,2})\delta(\theta_{1,2})\ddot{\theta}_{1,2} + \left(\frac{\partial\Delta}{\partial\theta_{1,2}}\delta(\theta_{1,2}) - \frac{\partial\delta}{\partial\theta_{1,2}}\Delta(\theta_{1,2})\right)\dot{\theta}_{1,2}^2\Bigg]. \end{aligned} \tag{151}$$

Thus, the left-hand side of (149) becomes, after some calculations using (150) and (151):

$$\begin{aligned} & \left[m_1 d_2\gamma(\theta_{1,2}) + I_{z_2}(\delta(\theta_{1,2}) - \Delta(\theta_{1,2}))\right]\ddot{\theta}_{1,2} \\ & \quad + \frac{m_1\gamma(\theta_{1,2})}{\Delta(\theta_{1,2})}\left(\frac{\partial r}{\partial\theta_{1,2}}\sin\theta_{1,2} - r(\theta_{1,2})\cos\theta_{1,2}\right)\dot{\theta}_{1,2}\, p \\ & \quad - \frac{m_1\gamma(\theta_{1,2})}{\Delta(\theta_{1,2})}\Bigg(d_2\frac{\partial\Delta}{\partial\theta_{1,2}} + \frac{\partial\gamma}{\partial\theta_{1,2}}\sin^2\theta_{1,2} \\ & \quad + \gamma(\theta_{1,2})\sin\theta_{1,2}\cos\theta_{1,2} - r(\theta_{1,2})\frac{\partial\delta}{\partial\theta_{1,2}}\Bigg)\dot{\theta}_{1,2}^2. \end{aligned} \tag{152}$$

The expression in parentheses in the second term above can be shown to be equal to $-\lambda(\theta_{1,2})$, while the expression in parentheses in the third term can be shown to be equal to $-[\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})]\sin\theta_{1,2}$. Thus, equation (146) follows.

(ii) Let $d_2 = 0$. From (53), with u given by (148) and $\dot{v}$ given by (64), we have

$$\begin{aligned}
&[(I_{z_1} + I_{z_2})\delta(\theta_{1,2}) - I_{z_2}\Delta(\theta_{1,2}) + m_1 d_1\gamma(\theta_{1,2})\cos\theta_{1,2}]\sin\theta_{1,2}\dot{\nu}_1 + I_{z_2}[\delta(\theta_{1,2})\\
&\qquad - \Delta(\theta_{1,2})]\dot{\nu}_2 + [(I_{z_1} + I_{z_2})\delta(\theta_{1,2})\cos\theta_{1,2} - I_{z_2}\Delta(\theta_{1,2})\cos\theta_{1,2}\\
&\qquad - m_1 d_1\gamma(\theta_{1,2})\sin^2\theta_{1,2}]\dot{\theta}_{1,2}\,\nu_1\\
&\quad = \gamma(\theta_{1,2})\sin\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \delta(\theta_{1,2})F_{\theta_1} - \Delta(\theta_{1,2})F_{\theta_{1,2}},
\end{aligned} \tag{153}$$

for $\nu_1, \nu_2 \in \mathbb{R}$. Observe that the coefficient of $\dot{\nu}_1$ is zero, while $I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})] = -\Delta_1(\theta_{1,2})$ and $(I_{z_1} + I_{z_2})\delta(\theta_{1,2})\cos\theta_{1,2} - I_{z_2}\Delta(\theta_{1,2})\cos\theta_{1,2} - m_1 d_1\gamma(\theta_{1,2})\sin^2\theta_{1,2} = -m_1 I_{z_2} d_1^2\cos\theta_{1,2}$. Notice that from (64), we have $\ddot{\theta}_{1,2} \equiv \dot{v}_4 = \dot{\nu}_2$. Using this in (153) and rearranging terms, the result follows. □

The following two results are special cases of Proposition 15.

Proposition 16 (reduced dynamics without external forces). *In the absence of external forces or torques, other than the torque $\tau_{1,2}$ applied to the joint $O_{1,2}$, the reduced dynamics of the Roller Racer, take the form*

$$\begin{aligned}
\ddot{\theta}_{1,2} &= B_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 + B_5^4(\theta_{1,2})\tau_{1,2},\\
\frac{\mathrm{d}p}{\mathrm{d}t} &= A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2.
\end{aligned} \tag{154}$$

Proposition 17 (reduced dynamics with friction). *In the presence of friction, the reduced dynamics of the Roller Racer take the form*

$$\begin{aligned}
\ddot{\theta}_{1,2} &= [B_1^4(\theta_{1,2})\dot{\theta}_{1,2} + B_6^4(\theta_{1,2})]\,p + [B_2^4(\theta_{1,2})\dot{\theta}_{1,2} - B_7^4(\theta_{1,2})]\,\dot{\theta}_{1,2} + B_5^4(\theta_{1,2})\,\tau_{1,2},\\
\frac{\mathrm{d}p}{\mathrm{d}t} &= [A_1^4(\theta_{1,2})\dot{\theta}_{1,2} - A_1^5(\theta_{1,2})]\,p + [A_2^4(\theta_{1,2})\dot{\theta}_{1,2} + A_2^5(\theta_{1,2})]\,\dot{\theta}_{1,2},
\end{aligned} \tag{155}$$

where B_1^4, B_2^4 and B_5^4 were defined previously in (147) and where

$$B_6^4(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{A_2^5(\theta_{1,2})}{\Delta_1(\theta_{1,2})}, \qquad B_7^4(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{A_3^5(\theta_{1,2})}{\Delta_1(\theta_{1,2})}, \tag{156}$$

with A_2^5 as defined in (99), with Δ_1 as defined in (145) and with

$$A_3^5(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}\left[\eta_3(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2} + \eta_4(\theta_{1,2})\delta(\theta_{1,2}) + \eta_5(\theta_{1,2})\Delta(\theta_{1,2})\right], \tag{157}$$

where

$$\eta_3(\theta_{1,2}) \overset{\text{def}}{=} 2\left[\left(\frac{k_1}{R_1^2}+\frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)\gamma(\theta_{1,2})+\frac{k_2}{R_2^2}d_1\delta(\theta_{1,2})\cos\theta_{1,2}\right]\sin\theta_{1,2},$$

$$\eta_4(\theta_{1,2}) \overset{\text{def}}{=} 2\frac{k_2}{R_2^2}d_1\gamma(\theta_{1,2})\sin^2\theta_{1,2}\cos\theta_{1,2}$$

$$+2\left(\frac{k_1}{R_1^2}\frac{L_1^2}{4}+\frac{k_2}{R_2^2}\frac{L_2^2}{4}+\frac{k_2}{R_2^2}d_1^2\sin^2\theta_{1,2}\right)\delta(\theta_{1,2}),$$

$$\eta_5(\theta_{1,2}) \overset{\text{def}}{=} 2\frac{k_2}{R_2^2}\frac{L_2^2}{4}\Delta(\theta_{1,2}). \tag{158}$$

If $d_1 \neq d_2$, *then* $A_3^5(\theta_{1,2}) > 0$, *for all* $\theta_{1,2}$.

Proof. From the reduced dynamics of the Roller Racer with external forces given by equation (146) and from the external force 1-form due to friction given by equation (68), we have for the last three terms of (146), using the definitions of (147):

$$B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1+F_{y_1}\sin\theta_1)+B_4^4(\theta_{1,2})F_{\theta_1}+B_5^4(\theta_{1,2})F_{\theta_{1,2}}$$
$$=\frac{1}{\Delta_1(\theta_{1,2})}\eta_3(\theta_{1,2})(\dot{x}_1\cos\theta_1+\dot{y}_1\sin\theta_1)+\frac{1}{\Delta_1(\theta_{1,2})}\eta_4(\theta_{1,2})\dot{\theta}_1$$
$$-\frac{1}{\Delta_1(\theta_{1,2})}\eta_5(\theta_{1,2})\dot{\theta}_{1,2}+\frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\tau_{1,2}, \tag{159}$$

where η_3, η_4 and η_5 are defined in (157).

(i) Let $d_2 \neq 0$. For $v = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \in \mathcal{D}_q$, we have from (55)

$$\dot{x}_1\cos\theta_1+\dot{y}_1\sin\theta_1=\nu_1 d_2 \quad\text{and}\quad \dot{\theta}_1=\nu_2 d_2,$$

for $\nu_1, \nu_2 \in \mathbb{R}$. From this and from (92) we get

$$B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1+F_{y_1}\sin\theta_1)+B_4^4(\theta_{1,2})F_{\theta_1}+B_5^4(\theta_{1,2})F_{\theta_{1,2}}$$
$$=B_6^4(\theta_{1,2})\,p-B_7^4(\theta_{1,2})\,\dot{\theta}_{1,2}+B_5^4(\theta_{1,2})\,\tau_{1,2}. \tag{160}$$

It is an easy calculation to show that

$$B_6^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{1}{\Delta_1(\theta_{1,2})}\frac{1}{\Delta(\theta_{1,2})}[\eta_3(\theta_{1,2})r(\theta_{1,2})+\eta_4(\theta_{1,2})\sin\theta_{1,2}]=\frac{1}{\Delta_1(\theta_{1,2})}A_2^5(\theta_{1,2}),$$

with $A_2^5(\theta_{1,2})$ as defined in (99).

(ii) Let $d_2 = 0$. From equations (62) and (96), we get, similarly to case (i), the desired result. □

Remark 2. Consider the unforced Roller Racer dynamics ($\alpha_e = 0$). From equation (146), the dynamics of the shape variable $\theta_{1,2}$ are $\ddot{\theta}_{1,2} = B_1^4(\theta_{1,2})\,\dot{\theta}_{1,2}\,p + B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2$. It can be easily seen that the reduced Lagrangian l_c, defined in equation (144), is 'conserved' on the trajectories of this system.

5. Controllability and motion control of the Roller Racer

We are interested in controlling a nonlinear system where the number of controls is less than the dimension of its state space and whose tangent linearization is uncontrollable. Tools from nonlinear control theory are, then, necessary to analyse it. The discussion in this section follows Nijmeijer and van der Schaft (1990), unless otherwise noted.

Consider the smooth affine nonlinear control system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j, \tag{161}$$

where x are local coordinates for the smooth manifold M with $\dim M = n$ and $u : [0, T] \to U \subset \mathbb{R}^m$ is the set of admissible controls. The unique solution of (161) at time $t \geq t_0$ with initial condition $x(t_0) = x_0$ and input function $u(\cdot)$ is denoted $x(t, t_0, x_0, u)$ or simply $x(t)$.

The 'reachable set' $R^V(x_0, T)$ is the set of points in M which are reachable from $x_0 \in M$ at exactly time $T > 0$, following system trajectories which, for $t \leq T$, remain in the neighbourhood V of x_0. Consider also $R^V_T(x_0) \stackrel{\text{def}}{=} \bigcup_{t \leq T} R^V(x_0, t)$, the set of points in M reachable from x_0 at time less or equal to T.

The system (161) is locally 'accessible' from $x_0 \in M$, if, for any neighbourhood V of x_0 and all $T > 0$, the set $R^V_T(x_0)$ contains a non-empty open set. If the system is locally accessible from any $x_0 \in M$, then it is locally accessible. The system (161) is locally 'strongly accessible' from $x_0 \in M$, if, for any neighbourhood V of x_0 and for any $T > 0$ sufficiently small, the set $R^V(x_0, T)$ contains a non-empty open set.

The 'strong accessibility algebra' $\mathcal{C}_0$ is the smallest subalgebra of the Lie algebra of smooth vector fields on M containing the control vector fields $g_1, \ldots, g_m$, which is invariant under the drift vector field f, that is $[f, X] \in \mathcal{C}_0$, $\forall X \in \mathcal{C}_0$. The 'strong accessibility distribution' C_0 is the corresponding involutive distribution $C_0(x) = \{X(x) \,|\, X \in \mathcal{C}_0\}$. Every element of the algebra $\mathcal{C}_0$ is a linear combination of repeated Lie brackets of the form $[X_k, [X_{k-1}, [\ldots, [X_1, g_j] \ldots]]]$, for $j \in \{1, \ldots, m\}$ and where X_i, $i \in \{1, \ldots, k\}$, $k = 0, 1, \ldots$ belongs to $\{f, g_1, \ldots, g_m\}$. Observe that the drift vector field f is not contained explicitly in these expressions.

Proposition 18. *If the Strong Accessibility Rank Condition at $x_0 \in M$ is satisfied, that is if*

$$\dim C_0(x_0) = n, \tag{162}$$

then the system (161) is locally strongly accessible from x_0. If the Strong Accessibility Rank Condition is satisfied at every $x \in M$, then the system is locally strongly accessible. If the system (161) is locally strongly accessible, then $\dim C_0(x) = n$, *for x in an open and dense subset of M.*

The system (161) is 'controllable', if, for every $x_1, x_2 \in M$, there exists a finite time $T > 0$ and an admissible control $u : [0, T] \to U$ such that $x(T, 0, x_1, u) = x_2$.

For systems without drift (i.e. where $f = 0$ in (161)), accessibility is equivalent to controllability. However, this is no longer true for systems with drift and various notions of controllability have been developed. Below we consider the notion of small-time local controllability, for which relatively simple verification tests have been established, as well as links to the closed-loop control of non-holonomic systems (Sussmann 1983, 1987, Coron 1995).

The system (161) is 'small-time locally controllable (STLC)' from $x_0 \in M$, if, for any neighbourhood V of x_0 and any $T > 0$, x_0 is an interior point of the set $R_T^V(x_0)$, that is a whole neighbourhood of x_0 is reachable from x_0 at arbitrarily small time (Sussmann 1987).

In Sussmann (1983), a condition for 'lack' of STLC is given for single-input systems (see also the discussion on single-input systems in Sussmann (1987)).

Proposition 19. *Consider an analytic affine nonlinear system with a single input, of the form*

$$\dot{x} = f(x) + g(x)\, u, \tag{163}$$

with $|u| \leq 1$, $f(x_0) = 0$ *and* $g(x_0) \neq 0$, *for some* $x_0 \in M$. *Assume that the bracket* $[g, [g, f]](x_0)$ *does not belong to the linear span of the vector fields* $\{\mathrm{ad}_f^j\, g\,(x_0),\ j = 0, 1, \ldots\}$. *Then, the system 'is not' STLC from* x_0.

Certain nonlinear systems can be transformed, at least locally, into a linear controllable system, via a state coordinate transformation and static state feedback. This process is called 'static feedback linearization'. Other nonlinear systems can be transformed into a linear controllable system via dynamic state feedback and a coordinate transformation involving the extended state of the system. This process is called 'dynamic feedback linearization'. As Pomet (1995) remarks, dynamic feedback linearization, as defined above, is equivalent to the concept of 'differential flatness' introduced by Fliess *et al.* (1995).

Remark 3. Charlet *et al.* (1989) and Pomet (1995) show that differential flatness is equivalent, for single-input systems, to static feedback linearization, a necessary and sufficient condition for which is provided in Nijmeijer and van der Schaft (1990).

Proposition 20. *Consider system (161) with* $f(x_0) = 0$. *Assume that the strong accessibility rank condition holds at* x_0. *This system is static feedback linearizable if and only if the distributions* $D_1, \ldots, D_n$ *defined by*

$$D_k(x) = \mathrm{sp}\{\mathrm{ad}_f^r g_1(x), \ldots, \mathrm{ad}_f^r g_m(x) \mid r = 0, 1, \ldots, k-1\},\ k = 1, 2, \ldots \tag{164}$$

are all involutive and constant dimensional in a neighbourhood of x_0.

Assume further that this is a single-input system. This system is static feedback linearizable around x_0 *if and only if* $\dim D_n(x_0) = n$ *and* D_{n-1} *is involutive around* x_0.

5.1. *The reduced dynamics*

In this section we only consider the reduced dynamics for the Roller Racer model without external forces, other than the torque $\tau_{1,2}$ applied to the rotary joint (equations (154)).

For control purposes, we assume that p, $\theta_{1,2}$ and $\dot{\theta}_{1,2}$ are available from proprioceptive sensors. The dynamics of the base variable $\theta_{1,2}$ for the system without external forces (first of equations (154)) can be transformed into the form of a double integrator by the nonlinear static state feedback

$$\tau_{1,2} = \frac{1}{B_5^4(\theta_{1,2})} [u - B_1^4(\theta_{1,2})\dot{\theta}_{1,2} p - B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2]. \tag{165}$$

Note that for $d_1 \neq d_2$, we have $B_5^4(\theta_{1,2}) > 0$, for all $q \in Q$. Thus, after feedback linearization, the reduced dynamics take the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}\, p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2, \quad \frac{\mathrm{d}\theta_{1,2}}{\mathrm{d}t} = \dot{\theta}_{1,2}, \quad \frac{\mathrm{d}\dot{\theta}_{1,2}}{\mathrm{d}t} = u. \tag{166}$$

Defining the state vector $z \stackrel{\text{def}}{=} (p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^{\mathrm{T}} \in M$, where $M \stackrel{\text{def}}{=} \mathbb{R}^2 \times S^1$, the reduced dynamics (166) take the form of an affine nonlinear system with a single control $u \in \mathbb{R}$:

$$\dot{z} = f(z) + g(z)\, u, \tag{167}$$

with

$$f(z) \stackrel{\text{def}}{=} \begin{pmatrix} A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix} \quad \text{and} \quad g(z) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{168}$$

The equilibria of this system are states $z_e \in M$ where $f(z_e) = 0$. It can be easily seen that these are of the form $z_e = (p_e,\ \theta_{1,2_e},\ 0)^{\mathrm{T}} \in M$, with $p_e \in \mathbb{R}$ and $\theta_{1,2_e} \in S^1$. In particular, the origin $z_0 = (0,\ 0,\ 0)^{\mathrm{T}} \in M$ is an equilibrium.

The tangent linearization of the system (167) is 'not' controllable at equilibria, since the matrix

$$\left[g \,\middle|\, \frac{\partial f}{\partial z} g \,\middle|\, \left(\frac{\partial f}{\partial z}\right)^2 g \right]\Bigg|_{z_e} = \begin{pmatrix} 0 & A_1^4(\theta_{1,2_e})p_e & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is singular.

Define

$$A_3^4(\theta_{1,2}) \stackrel{\text{def}}{=} A_1^4(\theta_{1,2})A_2^4(\theta_{1,2}) - \frac{\partial A_2^4(\theta_{1,2})}{\partial \theta_{1,2}} \tag{169}$$

and, iteratively

$$A_{i+1}^4(\theta_{1,2}) \stackrel{\text{def}}{=} -A_1^4(\theta_{1,2})A_i^4(\theta_{1,2}) + \frac{\partial A_i^4(\theta_{1,2})}{\partial \theta_{1,2}}, \quad \text{for } i = 3, 4, \ldots \tag{170}$$

When $d_1 > d_2$, the roots $\theta_{1,2}^\star$ of $A_2^4(\theta_{1,2})$ correspond to the solutions of $\gamma(\theta_{1,2}) = 0$, that is to

$$\theta_{1,2}^\star = \cos^{-1}\frac{I_{z_1}d_2}{I_{z_2}d_1}.$$

Notice that at roots of $A_2^4(\theta_{1,2})$ such that $I_{z_1}d_2 \neq I_{z_2}d_1$, we have $\theta_{1,2}^\star \neq 0, \pi$ and

$$\frac{\partial A_2^4(\theta_{1,2}^\star)}{\partial \theta_{1,2}} = -\frac{m_1}{\Delta^2} I_{z_2} d_1 \lambda \sin\theta_{1,2}^\star \neq 0.$$

Thus, when $A_2^4(\theta_{1,2}) = 0$ and $I_{z_1}d_2 \neq I_{z_2}d_1$, we have from (169) that $A_3^4(\theta_{1,2}) \neq 0$.

Proposition 21. *Assume $d_1 > d_2$ and $I_{z_1}d_2 \neq I_{z_2}d_1$. The reduced dynamics (167) are locally strongly accessible from equilibria $z_e = (p_e,\ \theta_{1,2_e},\ 0)^{\mathrm{T}}$.*

Proof. If $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) \neq 0$, then $\mathrm{sp}\{g,\ [f,g],\ [[f,g],g]\}(z_e) = \mathbb{R}^3$. If $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) = 0$ and $I_{z_1}d_2 \neq I_{z_2}d_1$, then $\mathrm{sp}\{g,\ [f,g],\ \big[[f,g],$

$[[f, g], g]\big]\}(z_e) = \mathbb{R}^3$. In both cases, the system satisfies the strong accessibility rank condition at z_e. □

However, since system (167) is a system with drift, its accessibility does not imply its controllability. In particular, it is possible to show that, under the parametric condition which occurs most frequently in practice, the reduced dynamics are not STLC from all equilibria

Proposition 22. *Assume $d_1 > d_2$.*

(a) Let $I_{z_1} d_2 > I_{z_2} d_1$. The reduced dynamics (167) are not STLC from all equilibria $z_e = (p_e,\ \theta_{1,2_e},\ 0)^{\mathrm{T}}$.
(b) Let $I_{z_1} d_2 \leq I_{z_2} d_1$. The reduced dynamics (167) are not STLC from all equilibria $z_e = (p_e,\ \theta_{1,2_e},\ 0)^{\mathrm{T}}$ such that $\cos(\theta_{1,2_e}) \neq (I_{z_1} d_2 / I_{z_2} d_1)$.

Proof. Observe that at equilibria z_e we have $g(z_e) \neq 0$,

$$[g, [g,f]](z_e) = \begin{pmatrix} 2A_2^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathrm{sp}\{\mathrm{ad}_f^j\, g(z_e),\ j = 0, 1, \ldots\} = \mathrm{sp}\{g(z_e),\ [f, g]\ (z_e)\} = \mathrm{sp}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} A_1^4(\theta_{1,2_e}) p_e \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Obviously, the bracket $[g, [g,f]]\ (z_e)$ does not belong to $\mathrm{sp}\{\mathrm{ad}_f^j\, g(z_e),\ j = 0, 1, \ldots\}$ when $A_2^4(\theta_{1,2_e}) \neq 0$. In this case, the lack of STLC of the reduced dynamics follows from Proposition 19.

When $d_1 > d_2$ and $I_{z_1} d_2 > I_{z_2} d_1$, then $A_2^4(\theta_{1,2_e}) \neq 0$, for all $q \in Q$.

When $d_1 > d_2$ and $I_{z_1} d_2 \leq I_{z_2} d_1$, then there are at most two discrete values of $\theta_{1,2_e}$, specified above, such that $A_2^4(\theta_{1,2_e}) = 0$. □

Proposition 23. *The reduced dynamics (167) are not static feedback linearizable around equilibria $z_e = (p_e, \theta_{1,2_e}, 0)^{\mathrm{T}}$.*

Proof. The dimension of the state space is $n = 3$. At the equilibrium z_e, the distribution D_n is

$$D_n(z_e) = \mathrm{sp}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} A_1^4(\theta_{1,2_e}) p_e \\ 1 \\ 0 \end{pmatrix} \right\}$$

and its dimension is strictly less than n. Thus, the result follows from Proposition 20. □

Remark 4. In view of Remark 3, the reduced dynamics (167) are neither dynamic feedback linearizable, nor differentially flat.

5.2. *The full dynamics*

The full dynamics of the Roller Racer (model without external forces other than the torque actuating the rotary joint) are given by the reduced dynamics (equations (154)) and by the group equations $\dot{g}_1 = g_1\xi_1$, where $1 \in SE(2)$, $\xi_1 \in se(2)$ and ξ_1 is given by (105). Consider local coordinates (x_1, y_1, θ_1) for $g_1 \in SE(2)$. Letting $z \overset{\text{def}}{=} (\theta_1,\ x_1,\ y_1,\ p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^{\mathrm{T}} \in M = \mathbb{R}^6$ and after feedback linearization of the dynamics of the base variable (cf. equation (165)), the dynamics take the form of an affine nonlinear control system, with the shape acceleration $\ddot{\theta}_{1,2}$ being the single control of the system:

$$\dot{z} = f(z) + g(z)\, u, \tag{171}$$

with $u \in \mathbb{R}$ and

$$f(z) \overset{\text{def}}{=} \begin{pmatrix} \dfrac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\, \dot{\theta}_{1,2}] \\ \dfrac{\cos\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2}) \sin\theta_{1,2}\, \dot{\theta}_{1,2}] \\ \dfrac{\sin\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2}) \sin\theta_{1,2}\, \dot{\theta}_{1,2}] \\ A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix} \quad \text{and} \quad g(z) \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{172}$$

The 'equilibria' of the system are states $z_e \in M$ of the form $z_e = (\theta_{1_e},\ x_{1_e},\ y_{1_e},\ 0,\ \theta_{1,2_e},\ 0)^{\mathrm{T}} \in M$, that is states where, not only the shape $\theta_{1,2}$ is constant, but also the non-holonomic momentum p is zero.

Remark 5. Based in part on the results for the reduced dynamics in section 5.1, it can be shown that for $d_1 > d_2$, the full dynamics of the Roller Racer (equation (171)) are locally accessible from equilibria z_e, but are not STLC from equilibria where $A_2^4(\theta_{1,2_e}) \neq 0$ and are not differentially flat. See details in Krishnaprasad and Tsakiris (1998).

Remark 6. The above properties still hold for $d_2 = 0$. This is due to the non-zero inertia I_{z_2} of the second platform.

Other undulatory locomotors, like the snakeboard, are known to be STLC (Ostrowski and Burdick 1995). The non-trivial second term in the momentum equation (76) of the Roller Racer $(A_2^4(\theta_{1,2})\ \dot{\theta}_{1,2}^2)$ plays a crucial role in its property of being accessible, but not being STLC.

It is interesting to observe that, even though the Roller Racer resembles a unicycle with one trailer which is hitched to a point displaced from the centre of the unicycle's wheel axis (also referred to as kingpin hitch) and this last system has been shown to be differentially flat (Rouchon *et al.* 1993), the peculiar actuation scheme of the Roller Racer makes it non-flat.

6. Simulation and experimental results

A computer-controlled prototype of the Roller Race was built at the Intelligent Servosystems Laboratory (ISL) of the University of Maryland (figure 5). The assumption of our models that the only feature of the body motion of a Roller Racer rider which is crucial to the propulsion of this mechanism is the swinging of the steering arm around the pivot axis, was verified using this and other similar prototypes.

The models of the dynamics of the Roller Racer, which were developed in the previous sections, were used in computer simulations of the system on Silicon Graphics workstations and in Mathematica and Simparc (Astraudo and Borrelly 1992) simulations on SUN SPARCstations.

A periodic shape trajectory of period $T_{1,2}$ of the form

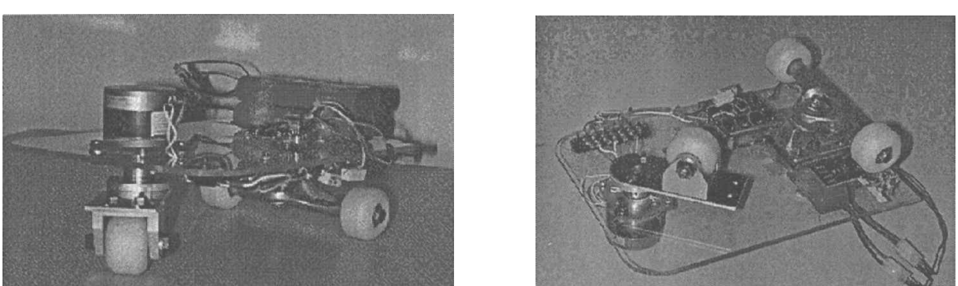
$$\theta_{1,2}(t) = \theta_{1,2}(0) + \alpha_{1,2}\sin(\omega_{1,2}t + \phi_{1,2}), \tag{173}$$

with $\omega_{1,2} = 2\pi/T_{1,2}$ is used in the simulations. The average value of $\theta_{1,2}$ is $\theta_{1,2}(0)$. Setting this average to π, as in figure 6, generates a 'straight-line' motion. Setting $\theta_{1,2}(0)$ to a value other than π or zero, as in figure 9 (where $\theta_{1,2}(0) = 1.772\,154\,2$ rad), generates a rotation around the point where the axes of the platforms intersect when the system is in the configuration corresponding to this average value. Once momentum has built up through periodic shape variations, we can stop varying the shape periodically and use $\theta_{1,2}$ just to steer the system. In what follows, we give only a sampling of our simulation results. For further details, see Krishnaprasad and Tsakiris (1998). Movies of experiments with Roller Racer prototypes can be seen on the home page of the Intelligent Servosystems Laboratory (URL: http://www.isr.umd.edu/Labs/ISL/isl.html) and on the second author's home page at FORTH (URL: http://www.ics.forth.gr/~tsakiris).

6.1. *Gaits*

Consider the Roller Racer model without friction or external forces, except for the joint torque needed to actuate $\theta_{1,2}$. The model parameters used in these simulations are $m_1 = 1$, $d_1 = 5$, $d_2 = 1$, $I_{z_1} = 10$, $I_{z_2} = 1$ $(I_{z_1}d_2 > I_{z_2}d_1)$.

In all the (x_1, y_1) plots that follow, the system starts at (0,0) and is initially oriented towards the positive x_1-axis $(\theta_1 = 0)$.



(a) (b)

Figure 5. Roller Racer prototype.

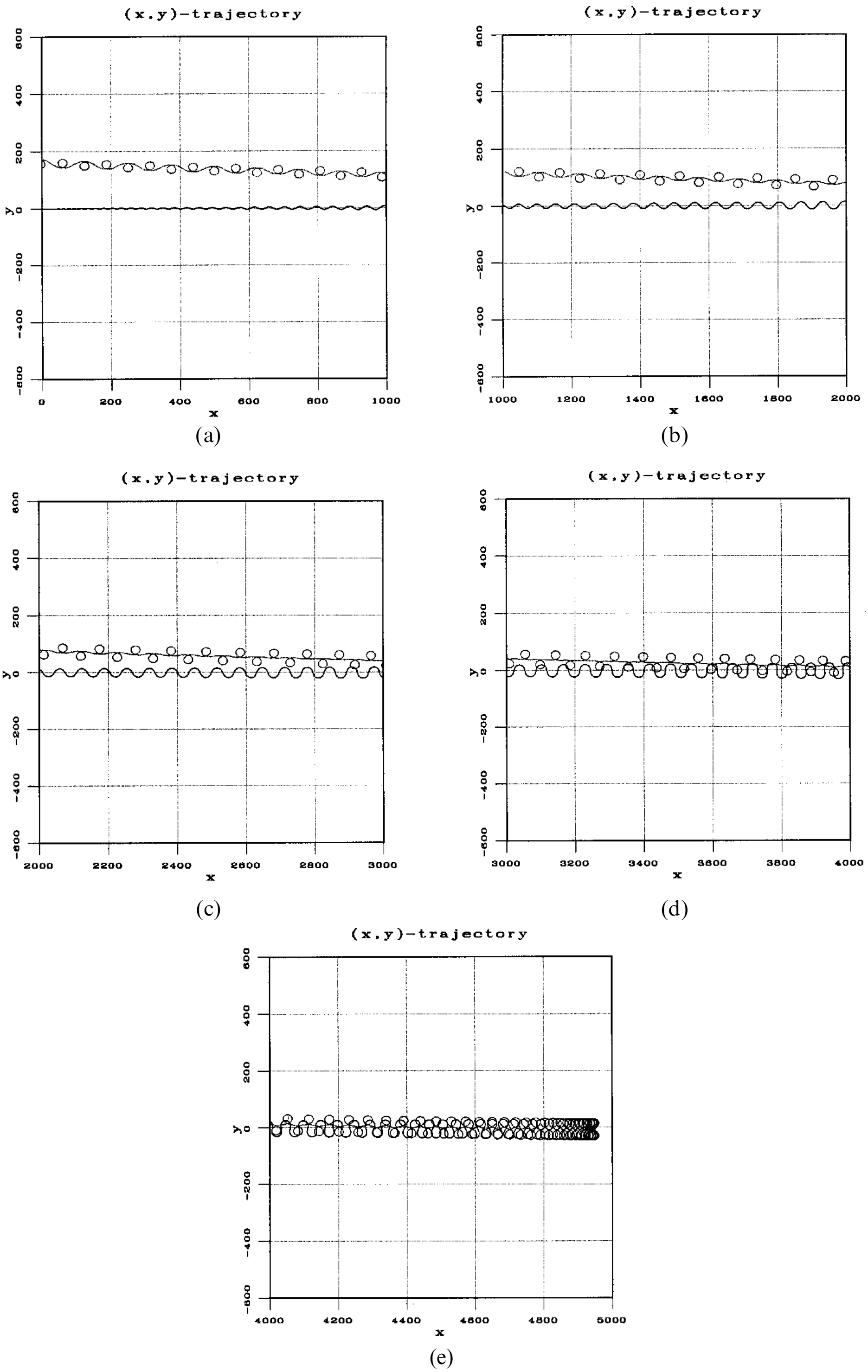


Figure 6. Forward translation: (a) $x_1 \in [0, 1000]$; (b) $x_1 \in [1000, 2000]$; (c) $x_1 \in [2000, 3000]$; (d) $x_1 \in [3000, 4000]$; (e) $x_1 \in [4000, 5000]$.

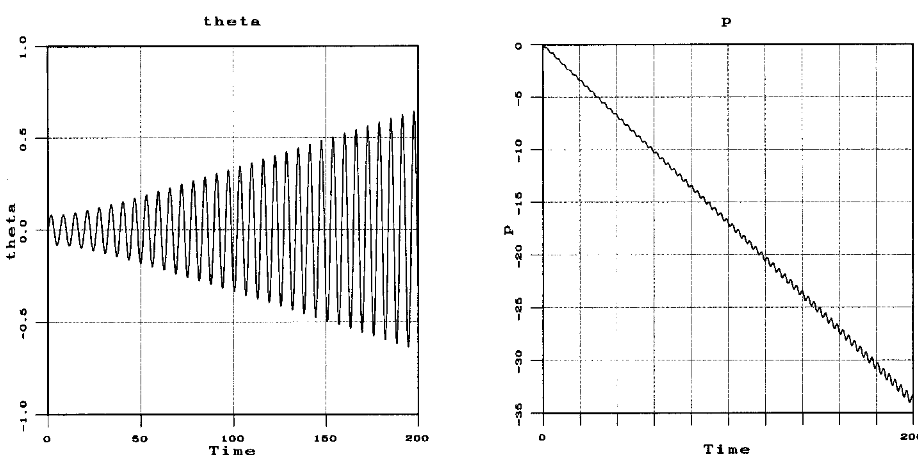
6.1.1. *Forward translation*. When the initial shape angle $\theta_{1,2}(0)$ is equal to π, the system translates forward. In the present simulations, the system starts at $(x_1, y_1) = (0, 0)$ pointing towards the positive x_1-axis and the shape control has amplitude of oscillation $\alpha_{1,2} = 0.3$ and frequency $\omega_{1,2} = 1$.

In figure 6, the (x_1, y_1)-trajectory is shown. Figure 6(a) shows the initial part of the trajectory: the system initially translates to the right, while oscillating about the x_1-axis. These oscillations become more pronounced as the momentum increases, giving rise to elastica-like trajectory segments (cf. section 3.2.6. of Tsakiris 1995), which at some point reverse direction and the system starts moving to the left, creating the upper branch of the trajectory of figure 6. As we move from figure 6(a) to (e), we are moving to the right of the x_1-axis. The lower branch of the trajectory, the one that corresponds to a translation of the Roller Racer to the right, is shown as a solid line. The upper branch of the trajectory, the one that corresponds to a translation to the left, is shown as a dotted line.

The group variable θ_1 is shown in figure 7(a), showing that the system oscillates with increasing amplitude as the non-holonomic momentum increases, but that the average of this oscillation is zero. Thus, the system translates on a more or less straight-line trajectory. The corresponding non-holonomic momentum p, which increases on the average, is shown in figure 7(b).

6.1.2. *Backward translation*. When $\theta_{1,2}(0) = 0$, the system translates backwards. In the present simulations, the system starts at $(x_1, y_1) = (0, 0)$ pointing towards the positive x_1-axis and the shape oscillation has amplitude $\alpha_{1,2} = 0.1$ and frequency $\omega_{1,2} = 1$. The corresponding group variables (x_1, y_1, θ_1) are shown in figure 8. Observe that y_1 and θ_1 merely oscillate around zero, while the magnitude of x_1 increases.

6.1.3. *Pure rotation*. When the instantaneous centre of rotation of the system is, on the average, at the middle of the rear wheel axis, that is when $\theta_{1,2}(0)$ is a root of $r(\theta_{1,2}) = 0$, which we denote as $\theta_{1,2}^{r=0}$, the Roller Racer rotates without translat-



(a) (b)

Figure 7. Forward translation: (a) angle θ_1; (b) non-holonomic momentum p.

ing (on the average). This can be seen in figure 9 for a clockwise rotation with $\theta_{1,2}(0) = \theta_{1,2}^{r=0} = 1.772\,154\,2$ rad.

When the average of the shape oscillations $(\theta_{12,}(0))$ is not set to 0, π or $\pm\,\theta_{1,2}^{r=0}$, the system rotates around the average position of the instantaneous centre of rotation.

6.2. *Geometric and dynamic phase*

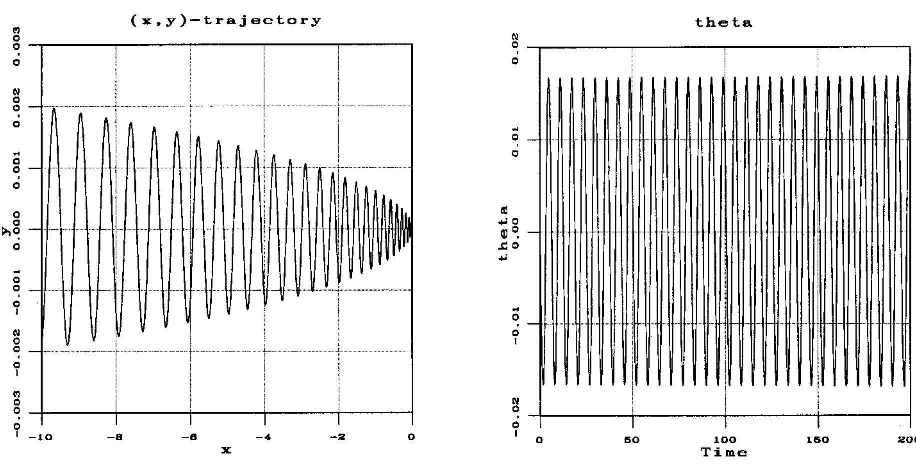
Consider a periodic shape variation of the type of equation (173) corresponding to forward translation of the system with $\theta_{1,2}(0) = \pi$, $\alpha_{1,2} = 0.1$ and $\omega_{1,2} = 1$. We consider the momentum equation without friction (equation (76)).

The notions of 'geometric' and 'dynamic phase' (Marsden *et al.* 1990, Bloch *et al.* 1996) describe how much the system moved after one period of the oscillatory controls. The group velocity, given by the reconstructed group motion equations (142)

$$\xi_1 = g_1^{-1}\dot{g}_1 = -A_{\mathrm{loc}}(\theta_{1,2})\dot{\theta}_{1,2} + \mathbb{I}_{\mathrm{loc}}^{-1}(\theta_{1,2})p \tag{174}$$

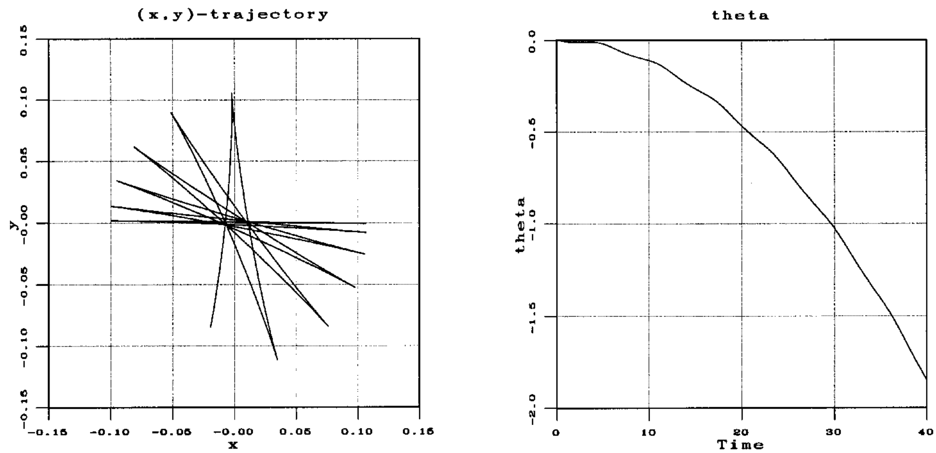
is composed of two parts: the system motion due to the first term $-A_{\mathrm{loc}}(\theta_{1,2})\dot{\theta}_{1,2}$ (where the non-holonomic momentum plays no role) is called the 'geometric phase', while the system motion due to the second term $\mathbb{I}_{\mathrm{loc}}^{-1}(\theta_{1,2})p$ is called the 'dynamic phase'. Thus, the geometric phase is $\int_0^{T_{1,2}} \dot{g}_1(t)\,\mathrm{d}t$, with $\dot{g}_1(t) = -g_1(t)A_{\mathrm{loc}}(\theta_{1,2}(t))\,\dot{\theta}_{1,2}(t)$ and the dynamic phase is $\int_0^{T_{1,2}} \dot{g}_1(t)\mathrm{d}t$, with $\dot{g}_1(t) = g_1(t)\mathbb{I}_{\mathrm{loc}}^{-1}(\theta_{1,2}(t))p(t)$. The dynamic phase obviously depends on the initial value of the momentum $p(0)$. In the simulation results presented below, we suppose that the system starts at rest, that is $p(0) = 0$.

The components of x_1, y_1 and θ_1, that are due to each of the above two terms, are shown in figure 10. It is easy to see from this figure that the geometric phase, over one period of the periodic shape controls, is zero (this is shown by the curves marked '(x, y) geom' and 'th geom' in the figures). In figure 10(a), the contribution of '(x, y) geom' is the swallowtail to the left of point (0,0). In figure 10(b), the curve 'th geom', an oscillation around zero, initially overlaps the curve 'th total'. These simulation



(a) (b)

Figure 8. Backward translation: (a) angle (x_1, y_1)-trajectory; (b) θ_1-trajectory.



(a) (b)

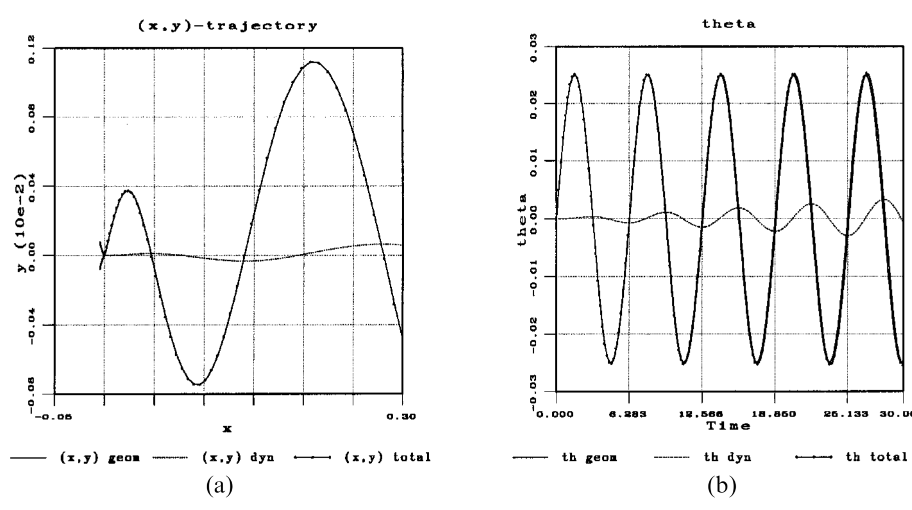
Figure 9. Clockwise rotation by $\pi/2$: (a) (x_1, y_1)-trajectory; (b) θ_1-trajectory.

results show that the geometric phase in this case is zero. However, the dynamic phase is not zero. In figure 10(a), the curve '(x, y) dyn' has an evident non-zero component in the x_1-direction, while the motions in the y_1 and θ_1 directions are, again, oscillations around zero.

6.3. *Parametric study of the system*

In the present section, we study the dependence of the motion of the Roller Racer, for the model without external forces, on the amplitude and the frequency of the sinusoidal shape controls (equation (173)).

Figure 11 shows the evolution of the group variables x_1, y_1, θ_1 for a forward translation of the system $(\theta_{1,2} = \pi, \omega_{1,2} = 1.0)$ and for control oscillation amplitude $\alpha_{1,2}$ varying from 0.1 to 1.0. Figure 11(a) shows the evolution of x_1, figure 11(c)



(a) (b)

Figure 10. Geometric and dynamic phase: (a) (x_1, y_1)-trajectory; (b) θ_1-trajectory.

shows this of y_1 and figure 11(d) shows this of θ_1 all for a time duration of four time periods of the controls, while figure 11(b) shows the evolution of x_1 for a bigger time duration of 20 time periods of the controls. It is obvious that y_1 and θ_1 merely oscillate around zero. This is not the case for x_1. Figure 11(a) shows that for short times (of 1–2 periods of the controls), the bigger $\alpha_{1,2}$ is, the bigger the system's forward motion. However, as can be seen in figure 11(b), this is no longer true for longer time periods. From the above it appears that small-amplitude motion gives forward translation without too much oscillation in the group variables, which closely approximates a straight-line motion.

Consider now the effect of the frequency $\omega_{1,2}$ on the group variables x_1, y_1, θ_1. We vary the frequency from 0.1 to 1.0, while $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 0.1$.

Figure 12 shows the (x_1, y_1)-trajectory of the system for $\omega_{1,2} = 0.1$, superimposed to the corresponding trajectory for $\omega_{1,2} = 1.0$, for a time duration of $2T_{1,2}$ (40π and

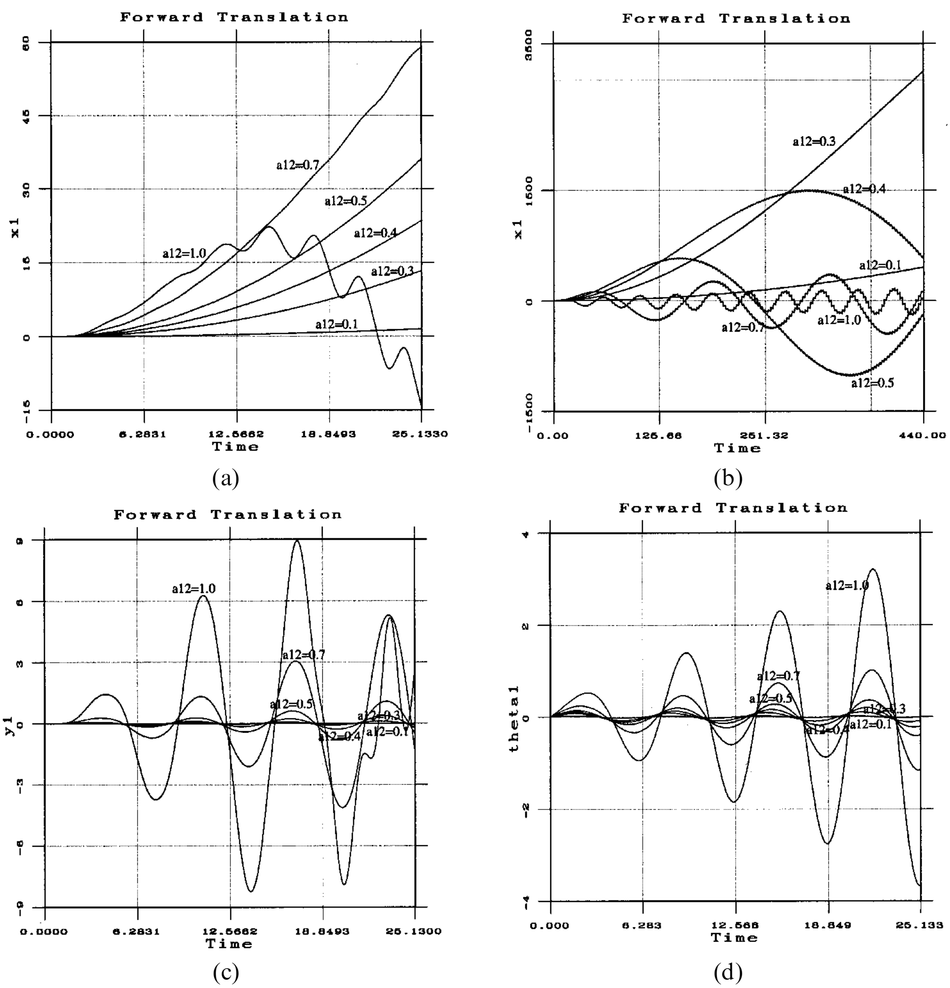
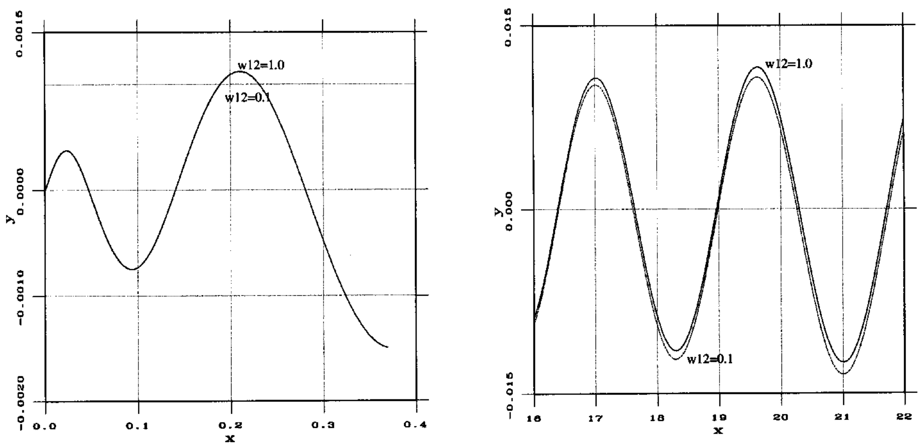


Figure 11. Effect of the amplitude $\alpha_{1,2}$ on x_1, y_1, θ_1: (a) x_1 for a duration of $4T_{1,2}$; (b) x_1 for a duration of $20T_{1,2}$; (c) y_1 for a duration of $4T_{1,2}$; (d) θ_1 for a duration of $4T_{1,2}$.



(a) (b)

Figure 12. Effect of the frequency $\omega_{1,2}$ on (x_1, y_1)-trajectory: (a) low p; (b) high p.

4π seconds respectively). Figure 12(a) shows the initial part of the trajectory, where non-holonomic momentum is low, and figure 12(b) shows a later part of it, where non-holonomic momentum is higher. The trajectory corresponding to $\omega_{1,2} = 1.0$ appears as a solid line, while the one for $\omega_{1,2} = 0.1$ appears as a dotted line. When non-holonomic momentum is low, the trajectories for $\omega_{1,2} = 1.0$ and $\omega_{1,2} = 0.1$ are geometrically almost the same (figure 12(a)); it is the time traversal of the trajectory that becomes faster as $\omega_{1,2}$ increases. However, as non-holonomic momentum increases, both the geometry of the trajectory and its time traversal become different (figure 12(b)).

6.4. *Model with friction*

6.4.1. *Forward translation*. We consider the Roller Racer model with friction (momentum equation (98)) with the following parameters (in addition to the ones mentioned earlier): $k_1 = k_2 = 0.01$, $R_1 = R_2 = 0.5$, $L_1 = 1$, $L_2 = 0.25$. The control input (173) is considered with $\theta_{1,2}(0) = \pi$, $\alpha_{1,2} = 0.1$ and $\omega_{1,2} = 1.0$. The non-holonomic momentum corresponding to this control is shown in figure 13. Comparing this with figure 7(b), we observe that, contrary to the continuously increasing, on the average, momentum p of figure 7(b), here, each shape oscillation pumps just enough energy into the system to overcome friction. This is similar to the real system's behaviour observed by the prototypes built at ISL.

6.4.2. *Parallel parking*. We now set the friction coefficients to $k_1 = k_2 = 0.1$, leaving the rest of the parameters as before. In order to create a 'parallel parking' behaviour, the idea is to generate a motion in the Lie-bracket direction by first translating forward, then rotating clockwise, then translating backwards and finally rotating counter-clockwise. The first step corresponds to a shape oscillation with average π, for a few periods, the second step corresponds to a shape oscillation with average $\theta_{1,2}^{r=0}$, the third step corresponds to a shape oscillation with average 0 and the final step corresponds to a shape oscillation with average $-\theta_{1,2}^{r=0}$. This sequence of shape controls is shown in figure 14(b), where, starting with a

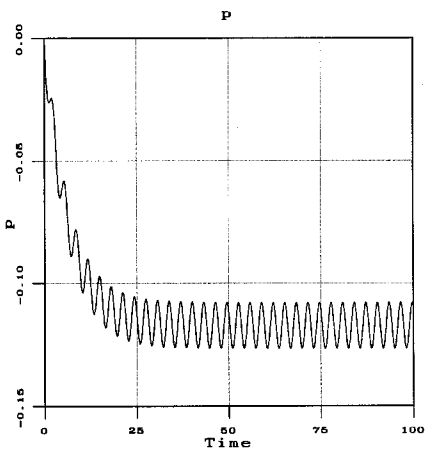
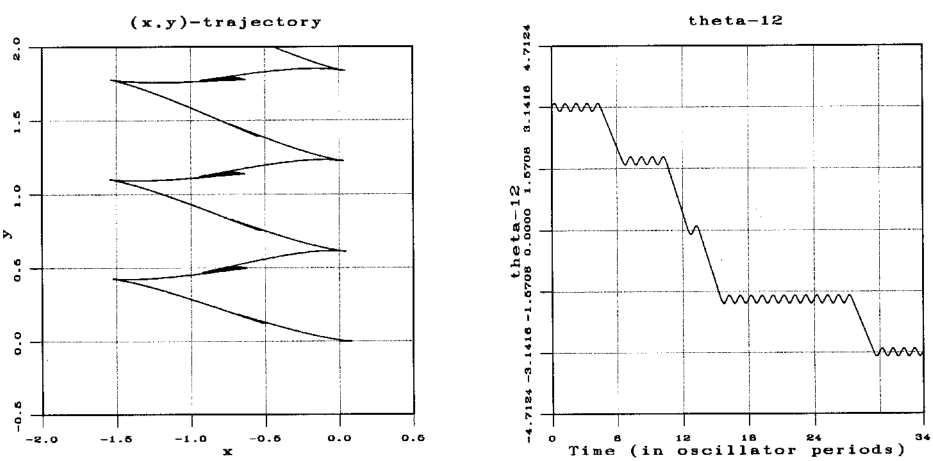


Figure 13. Model with friction: forward translation: non-holonomic momentum p.

basic shape oscillation of the type of equation (173) with amplitude $\alpha_{1,2} = 0.1$, frequency $\omega_{1,2} = 1.0$ and period $T_{1,2} = 2\pi/\omega_{1,2}$, we reset its average as was described above. The whole cycle lasts $30T_{1,2}$, after which we restart at π (shown as $-\pi$ in figure 14(b)). The corresponding (x_1, y_1)-trajectory is shown in figure 14(a).

7. Conclusions

The present work is aimed at revealing some of the rich mathematical and physical structure associated with a specific mechanical system that is underactuated. Part of our fascination with this system derives from the drive to understand how it works at all! As shown in this paper, the interplay between the symmetries and the constraints is crucial to this understanding. Additionally, Lie algebraic analysis reveals both the



(a) (b)

Figure 14. Model with friction: parallel parking manoeuvre: (a) (x_1, y_1)-trajectory; (b) $\theta_{1,2}$-trajectory.

capabilities and the limitations of such an underactuated system. The first draft of this paper was provided to the organizers of a workshop at the IEEE Conference on Decision and Control in Kobe in December 1996. (After the first draft of this paper was completed, in the summer of 1997, we received a preprint of Zenkov *et al.* (1998), which investigates the stability of relative equilibria of the 'unforced' Roller Racer as an application of a general theory of stability of non-holonomic systems.)

The present paper also explores via simulation certain motion control questions: specifically, controls for generating translational and curved motions, as well as parking manoeuvres. The influence of dissipation is also considered in some detail. Much remains to be done to understand the problem of constructive control for the Roller Racer. Models of the type used here may prove to be of interest in understanding problems of locomotion in biology and in bio-mimetic robotic systems.

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Appendix. List of symbols

$\mathcal{A}_i$	Basis element of Lie algebra $\mathcal{G}$
$\mathcal{A}_i^\flat$	Basis element of dual space $\mathcal{G}^*$ of Lie algebra $\mathcal{G}$
$A_{\rm loc}(\theta_{1,2})$	Local form of non-holonomic connection
$A_i^4(\theta_{1,2})$	Auxiliary function ($i = 1, 2, 3, \ldots$)
$A_i^5(\theta_{1,2})$	Auxiliary function ($i = 1, 2, 3$)
$B_i^4(\theta_{1,2})$	Auxiliary function ($i = 1, \ldots, 5$)
$\mathcal{D}_q$	Constraint distribution

$D_k(x)$	Distributions related to static feedback linearization
d_i	Distance of centre of platform i from the rotary joint
$(F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$	Components of external forcing one-form α_e
$f(x)$	Drift vector field of affine nonlinear control system
G	Lie group
$\mathcal{G}$	Lie algebra of Lie group G
$\mathcal{G}^*$	Dual space of Lie algebra $\mathcal{G}$
$\mathcal{G}^q$	Set of Lie algebra elements with infinitesimal generators in $\mathcal{S}_q$
g	Element of Lie group G
$g_j(x)$	Control vector field of affine nonlinear control system
H_q	Horizontal subspace of principal bundle $(Q, \mathcal{S}, \pi, G)$
H_q	Orthogonal complement of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$
$\mathrm{I}(q)$	Locked inertia tensor relative to $\mathcal{G}^q$
I_{z_i}	Moment of inertia of platform i
k_i	Friction coefficient of wheels of platform i
L	Lagrangian
L_i	Length of wheel axis for platform i
m_i	Mass of platform i
$\mathrm{Orb}(q)$	Orbit of $q \in Q$ under action Φ
p	Non-holonomic momentum
Q	Configuration space
q	Element of configuration space Q
$(Q, \mathcal{S}, \pi, G)$	Principal fibre bundle
$\mathcal{R}$	Rayleigh dissipation function
R_i	Wheel radius for platform i
$r(\theta_{1,2})$	Auxiliary function
$\mathcal{S}$	Shape space
$\mathcal{S}_q$	Intersection of constraint distribution $\mathcal{D}_q$ with $T_q\ \mathrm{Orb}(q)$
$SE(2)$	Special Euclidean group of rigid planar motions
$se(2)$	Lie algebra of the Special Euclidean group $SE(2)$
T_qQ	Tangent space to $q \in Q$
$T_{1,2}$	Period of shape control
$\mathcal{U}_q$	Orthogonal complement of $\mathcal{S}_q$ with respect to $T_q\ \mathrm{Orb}(q)$
V_q	Vertical subspace of principal bundle $(Q, \mathcal{S}, \pi, G)$
(x, y, θ)	Coordinates of $g \in SE(2)$
$(x_1, y_1, \theta_1, \theta_{1,2})$	Coordinates of Roller Racer configuration space $Q = SE(2) \times S^1$
z	State of affine nonlinear control system
z_e	Equilibrium of affine nonlinear control system

Greek symbols

$\alpha_{1,2}$	Amplitude of shape control
α_e	External forcing 1-form
$\beta(\theta_{1,2})$	Auxiliary function
$\Gamma_{i,j}^k$	Structure constants of Lie algebra
$\gamma(\theta_{1,2})$	Auxiliary function
$\Delta(\theta_{1,2})$	Auxiliary function
$\Delta_1(\theta_{1,2})$	Auxiliary function
$\delta(\theta_{1,2})$	Auxiliary function

$\eta_i(\theta_{1,2})$	Auxiliary function ($i = 1, \ldots, 5$)
$\theta_{1,2}$	Angle of Roller Racer rotary joint
$\lambda(\theta_{1,2})$	Auxiliary function
ν_i	Auxiliary velocity ($i = 1, 2$)
ξ	Element of Lie algebra $\mathcal{G}$
$\xi_Q(q)$	Infinitesimal generator of action Φ corresponding to $\xi \in \mathcal{G}$
ξQ^i	Basis element of $\mathcal{D}_q$
ξ_Q^H	Basis element of H_q
$\xi_Q^{\mathcal{U}_i}$	Basis element of $\mathcal{U}_q$
ξQ^q	Basis element of $\mathcal{S}_q$
$\pi : Q \to \mathcal{S} = Q/G$	Canonical bundle projection
$\tau_{1,2}$	Torque applied by motor to the rotary joint
Φ	Action of a Lie group on a manifold
$\Phi(t, t_0)$	State transition matrix
$\phi_{j,i}$	Angle of jth wheel of platform i
$\omega_{1,2}$	Frequency of shape control
ω_q^i	Non-holonomic constraint 1-forms

Miscellaneous symbols

$\square$	End of proof
[,]	Lie bracket of Lie algebra $\mathcal{G}$
$\ll , \gg$	Kinetic energy inner product

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