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# Sequential Sample Allocation for Multiple Attribute Selection Decisions

**Dennis D. Leber**
**National Institute of Standards and Technology**
**Gaithersburg, MD 20910, U.S.A.**

**Jeffrey W. Herrmann**
**University of Maryland**
**College Park, MD 20742, U.S.A.**

## Abstract

When faced with a limited budget to collect data in support of a multiple attribute selection decision, the decision-maker must decide how many samples to observe from each alternative and attribute. This allocation decision is of particular importance when the observation process is uncertain, such as with physical measurements. For example, when the U.S. Department of Homeland Security must decide upon a radiation detection system to acquire, a number of performance attributes are of interest and must be measured in order to characterize each of the considered systems. We developed and tested a sequential allocation scheme that uses Bayesian updating and maximizes the probability of selecting the true best alternative when the attribute value observations contain Gaussian measurement error. In this sequential approach, measurements are conducted one at a time. Prior to making a measurement the decision-maker's current knowledge of the attribute values is used to identify the attribute and alternative pair to sample next. We conducted a simulation study to compare the performance of the proposed sequential allocation scheme and a commonly used uniform allocation approach.

## Keywords
Ranking and Selection, Multiple Attribute Decision Making, Experimental Design, Sequential Sample Allocation, Measurement Uncertainty

## 1. Introduction

Procurement decisions are selection decisions that are often based on a large number of competing performance measures. In some cases, the value of the performance measures (attributes) for each alternative must be evaluated through experimentation. For example, when the U.S. Department of Homeland Security chooses a radiation detection system to install at U.S.-based international airports, the ability of the considered systems to identify an array of radiological and nuclear materials of interest must be evaluated in a laboratory setting before a system is selected.

We define a multiple attribute selection decision as a decision in which a decision-maker must select a single alternative from a finite set of alternatives, and each alternative is described by several attributes (characteristics important to the decision-maker). A common approach to the multiple attribute selection problem requires the decision-maker to determine a decision value for each alternative by considering the desirability of that alternative's important characteristics and choose the alternative that has the greatest decision value.

In some cases, the decision-maker does not know the true values of the attributes but must make observations (measurements, or samples) using measurement processes that have error. Each sample provides some information about a single attribute for a single alternative. To evaluate an alternative, the decision-maker must combine the information from the samples that have been collected.

We considered the case in which the decision-maker has a limited fixed budget for samples (measuring the attributes of the alternatives). A sample measures only one attribute for one alternative at a time. Moreover, the decision-maker wishes to maximize the probability of selecting the truly best alternative. Section 2 provides a brief review of the related literature. Section 3 formulates this multiple attribute selection problem precisely. We developed (Section 4) and tested (Sections 5 and 6) a sequential approach in which the results of the previous samples are used to select the attribute and alternative to sample next.

## 2. Literature Review

While our work is influenced by experimental design [1, 2], Bayesian inference [3, 4], and normative multiple attribute decision analysis [5-7], the most relevant practices are the ranking and selection methods that are used to compare a finite number of alternatives whose performance measures are generated by a stochastic process [8, 9]. Kim and Nelson [10] reviewed recent developments in ranking and selection with a focus on the indifferent zone (IZ) allocation procedure for selecting the alternative with the largest expected value. The IZ procedure, which has no fixed limit on the number of observations, determines how often each alternative is observed (simulated) while guaranteeing a specified probability of correct selection (*PCS*) if the true performance of the "best" alternative is sufficiently better than the others. The IZ allocation procedure is formulated based on a single performance measure (attribute).

Unlike the IZ allocation procedure, the Optimal Computing Budget Allocation (OCBA) procedure [11] derives a sample allocation based on a fixed budget with the goal of maximizing *PCS* under this constraint, but this procedure also considers the allocation across multiple alternatives with a single performance measure. While our work is based on a fixed budget with the goal of maximizing *PCS*, it is focused on the allocation across both the multiple alternatives and the multiple attributes. We adapted ideas from OCBA to develop the proposed sequential approach.

## 3. Multiple Attribute Selection Problem with Measurement Uncertainty

Under the multiple attribute value model [5], the decision-maker must estimate the unknown attribute values using the observed samples, derive decision values for each alternative by aggregating the estimated attribute values using the decision model and associated attributed weights, and finally select an alternative based upon a selection rule. This section describes the assumptions, estimation procedure, and selection rule that were used in developing the sequential allocation procedure.

### 3.1. Assumptions

We make the following assumptions: The set of *m* distinct alternatives, $\{a_1,\ldots,a_m\}$, is provided, where *m* is a finite positive integer such that all alternatives can be assessed. Each alternative is described by $k \geq 2$ attributes. The true but unknown value of attribute *j* of alternative $a_i$ is $\mu_{ij}$. Separate and independent measurement processes are used in obtaining measurement data (samples) for each attribute, and for a given attribute *j*, the measurement process is the same for all $i=1,\ldots,m$ alternatives. $n_{ij}$ is the number of samples observed for attribute *j* of alternative $a_i$. The $l^{\text{th}}$ outcome of the measurement process for attribute *j* of alternative $a_i$ is described by the random variable $X_{ijl} = \mu_{ij} + \varepsilon_{ijl}$ with realized measurements $x_{ij1},\ldots,x_{ijn_{ij}}$. The random measurement errors are independent and normally distributed, $\varepsilon_{ijl} \sim N\left(0,\sigma_j^2\right), i=1,\ldots,m; l=1,\ldots,n_{ij}$, and the $\sigma_j^2$ are known. Note that in the field of metrology, it is not uncommon for the error associated with a continuous measurand to be modeled with a normal distribution [12] and for the variance of this distribution to be well characterized and assumed known [13]. Also provided is a decision model, $\xi_i = f\left(\mu_{i1},\ldots,\mu_{ik}\right) = \sum_{j=1}^{k} \lambda_j v_j\left(\mu_{ij}\right)$, that reflects the decision-maker's preference structure and combines the multiple attribute values to produce a decision value, $\xi_i$, for each alternative, $a_i$. The individual value functions are defined as $v_j\left(\mu_{ij}\right) = \mu_{ij}$. The attribute weights, $\lambda_j$, are defined such that $\sum_{j=1}^{k} \lambda_j = 1$. The experimental budget in terms of number of sample measurements, *B*, shall not be exceeded, and the cost of each measurement is equivalent. Thus, *B* is the upper bound on the number of measurements that can be performed. The decision-maker seeks to maximize the probability of selecting the most preferred alternative, i.e., maximize *PCS*.

### 3.2. Bayesian Estimation

To support the selection decision, the decision-maker uses the limited number of samples, $x_{ij1},\ldots,x_{ijn_{ij}}$, to estimate the true value, $\mu_{ij}$, of attribute *j* of alternative $a_i$. We used Bayesian inference to define a posterior distribution to describe the decision-maker's knowledge of the true attribute value. Before any samples are collected, the decision-maker's knowledge of $\mu_{ij}$ can be described by the conjugate normal prior distribution $N\left(\mu_{0ij}, \tau_{0ij}^2\right)$ and *a priori*, the $\mu_{ij}$ are independent. After observing the samples, the decision-maker's knowledge of $\mu_{ij}$ is updated and represented by the normal posterior distribution [4] in Equation (1).

$$\mu_{ij} \mid x_{ij1}, \ldots, x_{ijn_{ij}} \sim N\left(\frac{\sigma_j^2 \mu_{0ij} + n_{ij}\tau_{0ij}^2 \overline{x}_{ij}}{\sigma_j^2 + n_{ij}\tau_{0ij}^2}, \frac{\sigma_j^2 \tau_{0ij}^2}{\sigma_j^2 + n_{ij}\tau_{0ij}^2}\right) \quad i = 1, \ldots, m,\ j = 1, \ldots, k \tag{1}$$

The decision-maker combines the attribute values for each alternative $a_i$ per the decision model $\xi_i = \sum_{j=1}^{k} \lambda_j \mu_{ij}$. Thus, after obtaining $n_{ij}$ samples for each attribute $j = 1, \ldots, k$ of alternative $a_i$, the decision-maker's knowledge of the true decision value, $\xi_i$, can be described by the posterior distribution in Equation (2).

$$\xi_i \mid x_{i11}, \ldots x_{i1n_{i1}}, x_{i21}, \ldots x_{i2n_{i2}}, \ldots, x_{ik1}, \ldots x_{ikn_{ik}} \sim N\left(\sum_{j=i}^{k} \lambda_j \frac{\sigma_j^2 \mu_{0ij} + n_{ij}\tau_{0ij}^2 \overline{x}_{ij}}{\sigma_j^2 + n_{ij}\tau_{0ij}^2}, \sum_{j=1}^{k} \lambda_j^2 \frac{\sigma_j^2 \tau_{0ij}^2}{\sigma_j^2 + n_{ij}\tau_{0ij}^2}\right) \tag{2}$$

This posterior distribution is used in Sections 3.3 and 4 to derive a selection rule and an allocation rule to maximize the probability that the decision-maker makes a correct selection.

### 3.3. Multinomial Selection

The alternative with the largest decision value is the decision-maker's preferred alternative [5]. Given the decision-maker's knowledge of each decision value (the posterior distributions described in Equation (2)), Equation (3) gives the probability, $p_i$, that alternative $a_i$ has the largest decision value:

$$p_i = P\left(\xi_i > \xi_r, \forall r = 1, \ldots, m; r \neq i\right) \tag{3}$$

If the decision-maker selects alternative $a_s$, let the *probability of correct selection* (*PCS*) be the probability that $a_s$ has the largest decision value (Equation (4)).

$$PCS = p_s = P\left(\xi_s > \xi_r, \forall r = 1, \ldots, m; r \neq s\right) \tag{4}$$

The decision-maker, who wants to maximize *PCS*, will select $a_s$ where $s = \arg\max_i p_i$. We refer to this procedure as *multinomial selection* because it is consistent with existing multinomial selection procedures [10, 14]. Note that when developing OCBA, Chen and Lee [11] defined *PCS* in a manner similar to Equation (4) but suggested that the alternative be selected based upon its expected decision value.

## 4. Sequential Sample Allocation

Previous work [15] has considered single-stage sample allocation plans for multiple attribute selection decisions where the complete allocation plan is determined before any samples are collected. Here we consider a sequential allocation approach where the experimental effort is divided into stages, each consisting of a single sample. Within each stage, the decision-maker determines, based on his current knowledge, which single alternative and attribute pair to sample next. This process is repeated until the experimental budget is exhausted and a selection decision is made (Figure 1).

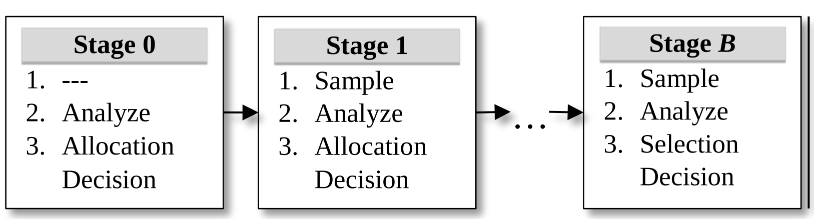


Figure 1: Sequential allocation procedure

In stage $t = 0, \ldots, B-1$, the approach analyzes the available information and identifies the alternative and attribute to sample in stage $t+1$ (that is, the next sample is allocated to that alternative and attribute). Let $n_{ij}(t)$ be the number of samples and $\mathbf{x}_{ij}(t) = x_{ij1}, \ldots x_{ijn_{i1}(t)}$ be the data collected in stages $1, \ldots, t$ for alternative $a_i$ and attribute $j$ (note

$n_{ij}(0)=0\ \forall i,j$ ). Let $\mathbf{x}_i(t)=\mathbf{x}_{i1}(t),\ldots,\mathbf{x}_{ik}(t)$ and $\mathbf{X}(t)=\mathbf{x}_1(t),\ldots,\mathbf{x}_m(t)$. In stage *B*, the final sample is collected, $\sum_{i=1}^{m}\sum_{j=1}^{k}n_{ij}(B)=B$, and the selection decision made.

We note that in any stage $t=0,\ldots,B$, the probability, $p_i$, that alternative $a_i$ has the largest decision value can be calculated (Equation (3)) and the alternative $a_s$ where $s=\arg\max_i p_i$ identified. Thus, the *PCS* at stage *t* is as described in Equation (5).

$$PCS(t)=P\left(\xi_s>\xi_r,\forall r=1,\ldots,m;r\neq s\mid\mathbf{X}(t)\right)=\int\underset{\xi_s>\xi_r,\forall r=1,\ldots m;r\neq s}{\cdots}\int p\left(\xi_1,\ldots,\xi_m\mid\mathbf{X}(t)\right)d\xi_1\ldots d\xi_m \tag{5}$$

The joint posterior probability distribution of $\xi_1,\ldots,\xi_m$ at stage *t*, $p\left(\xi_1,\ldots,\xi_m\mid\mathbf{X}(t)\right)$, is the product of the individual marginal distributions (probability density of Equation (2) given $\mathbf{x}_i(t)$ ). This follows from the conditional independence of the measurements, $X_{ijl}$, and the prior independence of the true attribute values, $\mu_{ij}$.

To make the sample allocation decision at stage *t*, we note that the next sample, $x_{ijn_{ij}(t+1)}$, observed from alternative and attribute pair $(a_i,j)$, will lead to a new *PCS* value. Although the value of the sample and the subsequent new *PCS* cannot be known until the observation is made, the probability distribution of each can be described based upon the decision-maker's current knowledge. The distribution of the new observation is described by its posterior predictive distribution [3, 16] with density $p\left(x_{ijn_{ij}(t+1)}\mid\mathbf{x}_{ij}(t)\right)=\int_{-\infty}^{\infty}p\left(x_{ijn_{ij}(t+1)}\mid\mu_{ij}\right)p\left(\mu_{ij}\mid\mathbf{x}_{ij}(t)\right)d\mu_{ij}$. Given our normality assumptions and Bayesian framework, the predictive distribution of $x_{ijn_{ij}(t+1)}$ is provided by the normal distribution [13] in Equation (6).

$$x_{ijn_{ij}(t+1)}\mid\mathbf{x}_{ij}(t)\sim N\left(\frac{\sigma_j^2\mu_{0ij}+n_{ij}(t)\tau_{0ij}^2\overline{x}_{ij}(t)}{\sigma_j^2+n_{ij}(t)\tau_{0ij}^2},\sigma_j^2+\frac{\sigma_j^2\tau_{0ij}^2}{\sigma_j^2+n_{ij}(t)\tau_{0ij}^2}\right) \tag{6}$$

For each of the *mk* alternative and attribute pairs, assuming that the selection is to be made using the multinomial selection approach, the expected *PCS* in stage $t+1$ if attribute *j* of alternative $a_i$ is sampled can be calculated according to Equation (7).

$$E\left(PCS_{ij}(t+1)\right)=\int_{-\infty}^{\infty}\left(\max_q\int\underset{\substack{\xi_q>\xi_r,r\neq q\\ \forall q,r=1,\ldots,m}}{\cdots}\int p\left(\xi_1,\ldots,\xi_m\mid\mathbf{X}(t),x_{ijn_{ij}(t+1)}\right)d\xi_1\ldots d\xi_m\right)p\left(x_{ijn_{ij}(t+1)}\mid\mathbf{x}_{ij}(t)\right)dx_{ijn_{ij}(t+1)} \tag{7}$$

The sequential allocation approach allocates the sample in stage $t+1$ to the alternative and attribute pair that yields the maximum $E\left(PCS_{ij}(t+1)\right)$.

Upon collecting the final observation in stage *B*, the approach calculates the probability, $p_i$, that alternative $a_i$ has the largest decision value according to Equation (3) and identifies the selected alternative, $a_s$, where $s=\arg\max_i p_i$.

## 5. Numerical Experiments

We conducted a numerical study that gauged the performance of (1) the proposed sequential sample allocation approach and (2) a uniform allocation approach where the sample allocation to each attribute and alternative is equal (a common allocation tactic, consistent with the principle of balance in the design of experiments discipline). For both allocation approaches we used Bayesian estimation and multinomial selection as described in Section 3, and common random numbers when appropriate. Because the decision-maker wishes to maximize *PCS*, we estimated the *PCS* with the frequency of correct selection (*fcs*), and used this estimate as the allocation performance metric.

We generated decision cases with $m = 5$ alternatives and $k = 2$ attributes whose true attribute values formed concave efficient frontiers. To generalize our results over this class of decision cases, we sought to estimate the marginal mean proportion of correct selection over a random sample of such decision cases. Per Fleiss *et al*. [17], the variance of this estimator is minimized by a single evaluation of a large number of decision cases. We generated 50,000 random concave efficient frontiers (decision cases) using a rejection algorithm with the true values of the attributes, $\mu_{ij}$, randomly assigned from the domain [100, 200]. For each decision case, the standard deviation of the measurement error for each attribute, $\sigma_j$, was randomly assigned from a uniform distribution with parameters $min = 1$ and $max = 30$. We considered 19 different decision models defined by $(\lambda_1, \lambda_2) = (0.05, 0.95), (0.1, 0.9), \ldots, (0.95, 0.05)$ and an overall sample budget of $B = 50$ (thus the uniform allocation assigned 5 samples to each attribute of each alternative). The prior distribution $N(150, 35^2)$ was used for all $\mu_{ij}$ for all decision cases. Note that, for each decision case and decision model, there is a true best alternative (the one with the greatest value of $\xi$).

For each decision case and decision model, the measurements were simulated by randomly sampling from normal probability distributions with means defined by the true attribute values and variances defined by the measurement errors. Based on the simulated measurements, an alternative was selected, and we determined the proportion of decision cases where the selected alternative was the true best alternative. In this way we established, as an estimate of the probability of correct selection, the *fcs* for each decision model and allocation approach. Due to the large number of decision cases and decision models in this evaluation study, and the computational effort needed to calculate $E(PCS_{ij}(t+1))$ precisely at every stage in the sequential sample allocation approach, this experiment was performed using the University of Maryland's High Performance Computing cluster *Deepthought2*.

## 6. Results

The left panel of Figure 2 displays the estimated *PCS* values for the sequential and uniform allocation approaches for each decision model ($\lambda_1$ value) investigated. The shaded areas in Figure 2 represent the 95 % pointwise confidence bounds. These results show that the performance of both allocation procedures varied in the same way as the weights varied: the *PCS* values were lowest when the decision weights were nearly equal ($\lambda_1 \approx \lambda_2 \approx 0.5$) and increased when one weight was much larger than the other ($\lambda_1$ or $\lambda_2$ near 1). This phenomenon may occur because, when the weights are nearly equal, there are multiple alternatives with similar decision values.

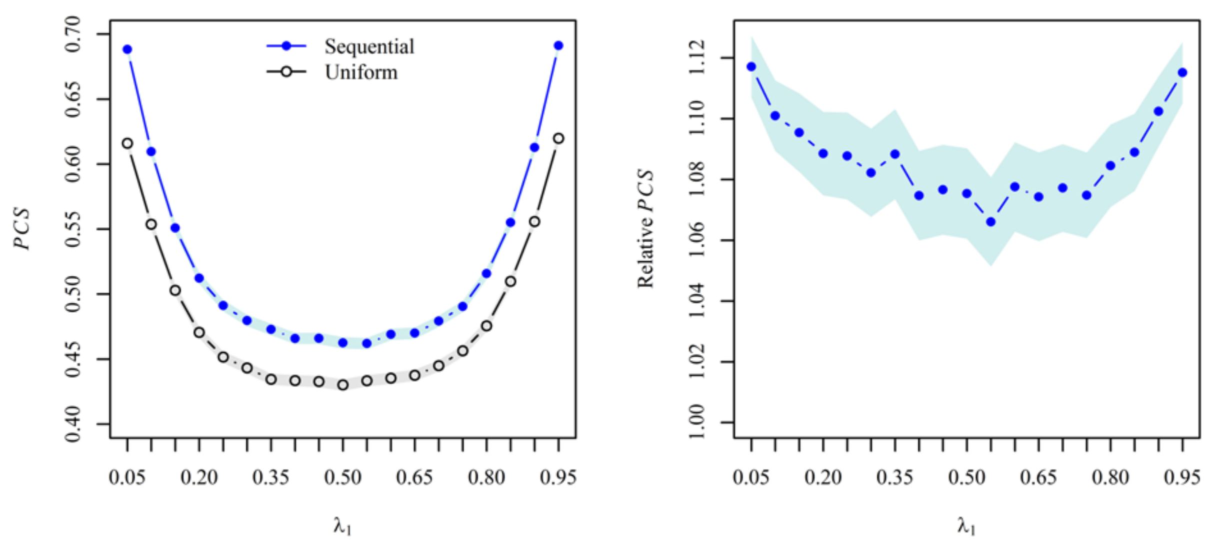


Figure 2: Estimated *PCS* values for uniform and sequential allocation approaches (left) and sequential allocation *PCS* relative to uniform allocation *PCS* (right); shaded area represents 95 % pointwise confidence bounds

As shown in the left panel of Figure 2, the sequential allocation approach provided *PCS* values that are significantly larger than the *PCS* values provided by the uniform allocation approach (i.e., the confidence intervals do not overlap). The sequential allocation *PCS* relative to uniform allocation *PCS* across the $\lambda_1$ values are displayed in the right panel of Figure 2. The sequential allocation approach provided *PCS* values that are approximately 10 % larger (1.09 average relative *PCS*) than those provided by the uniform allocation approach across all decision models. Although the lower 95 % confidence bound on the relative *PCS* value was as low as 1.05, these results show that the sequential allocation produced significantly better results than the uniform allocation.

## 7. Conclusions and Future Work

We considered the problem of maximizing the probability of correct selection in a multiple attribute selection decision where the attribute values are estimated using measurements that contain Gaussian measurement error. We developed a sequential sample allocation approach to allocate a fixed measurement budget and performed a numerical experiment to compare this approach and a standard uniform allocation. The results show that allocating the fixed measurement budget by the sequential allocation approach increased the frequency of correct selection by approximately 10 % over an allocation of the same budget using a common uniform allocation approach. Because these results are specific to the class of decisions tested, we will attempt to generalize these performance characteristics by considering scenarios with smaller and larger measurement budgets in future work and situations that have other error distributions (e.g., binomial).

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