Asymptotic Stability of a Rate Control System With Communication Delays

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Abstract—We study the issue of asymptotic stability of a family of rate control algorithms with communication delays between network elements and extend our earlier results: First, we derive delay-dependent stability conditions with a family of well-known utility and resource price functions when a finite upper bound is known on the feedback delay. These conditions are shown to be consistent with known stability conditions in two extreme cases—no delay or an arbitrarily large delay. Secondly, we provide a lower bound on the convergence rate with the same utility and resource price functions when delay-dependent stability conditions hold.

Index Terms—Asymptotic stability, communication system control, delay systems.

I. INTRODUCTION

Recently, there has been much interest in understanding the stability property of a family of rate control schemes, called primal algorithms, in the presence of communication delays [3], [5], [9], [12], [13], [17]. The stability of a similar algorithm proposed in [11] is studied in [16]. The primal algorithms, first proposed by Kelly et al. [7], are motivated by an optimization framework for a rate allocation with elastic traffic sources where the objective of the system is to maximize the aggregate utility of the users. Tan and Johari [5] studied the case where flows have the same round-trip delays and the same log utility functions, and provided local stability conditions in terms of users’ gain parameters and communication delays. Similar results have been obtained for single flow and single resource cases with more general utility functions in [3] and [9] and for single bottleneck with multiple heterogeneous users cases in [1].

In another set of work, Ranjan et al. [12] and Ying et al. [17] studied the stability of the rate control system in the presence of arbitrary fixed delays between network elements, e.g., network resources and end users, and arbitrary gain parameters of the end users. These approaches are consistent with the philosophy that, given the complexity and scale of the Internet, network protocols must be simple and robust. In addition, in large wireless networks, e.g., mobile ad-hoc networks (MANETs), round-trip delays are expected to be much larger than in their wireline counterparts. Portability of the network protocols will demand their stability in various environments in which they are expected to operate, including MANETs. The authors derived sufficient conditions on users’ utility and resource price functions for (global) asymptotic stability of the rate control system with arbitrary fixed communication delays and users’ gain parameters. We refer to these conditions as delay-independent stability conditions.

In this paper, we study the same class of rate control systems investigated in [12] and [17] and extend their results in two directions.

• We study a simple network with a single flow traversing a single resource and investigate its stability when a finite upper bound on the feedback delay is known in advance. We derive a stability condition that hints at how an increasing feedback delay affects the stability condition. We refer to this condition as a delay-dependent stability condition to distinguish it from the delay-independent stability conditions. The derived condition is consistent with the earlier stability conditions in two extreme cases; when there is no feedback delay, the system is always stable with the employed utility and resource price functions. When the delay is allowed to be arbitrarily large, we recover the delay-independent stability condition reported in [12] and [17].

• We derive a lower bound on the convergence rate of the rate control system under the delay-independent stability condition in [12] and [17]. The role of the maximum delay in feedback loop is highlighted. A closed-form expression of a lower bound on the local convergence rate is also provided.

This paper is organized as follows. Section II describes the optimization framework for rate control. Section III describes the rate control system model with fixed communication delays. The utility and resource price functions employed in this paper are explained in Section IV. We summarize earlier results in [12] and [17] in Section V. Delay-dependent stability conditions are obtained in Section VI. A lower bound on the convergence rate under the delay-independent stability conditions is derived in Section VII. We conclude in Section VIII.

II. BACKGROUND

In this section, we briefly describe the rate control problem in the proposed optimization framework. Consider a network with a set $\mathcal{L}$ of resources or links shared by a set $\mathcal{I}$ of users. Let $C_l$ denote the finite capacity of link $l \in \mathcal{L}$, each user has a fixed route $r_i$, which is a nonempty subset of $\mathcal{L}$. We define a zero-one matrix $A_i$, where $A_{i,j} = 1$ if link $l$ is in user $i$’s route $r_i$, and $A_{i,j} = 0$ otherwise. When the throughput of user $i$ is $x_i$, user $i$ receives utility $U_i(x_i)$. We take the view that the utility functions of the users are selected for a desired rate allocation among the users. The utility $U_i(x_i)$ is an increasing, strictly concave, and continuously differentiable function of $x_i$, over the range $x_i \geq 0$.

The rate allocation problem can be formulated as the following optimization problem [6]:

$$\begin{align*}
\max_{x \in \mathbb{R}_+^{|\mathcal{I}|}} & \sum_{i \in \mathcal{I}} U_i(x_i) \\
\text{subject to} & \begin{bmatrix} \mathbf{A}^\top \end{bmatrix} x \leq C, & x \geq 0
\end{align*}$$

(1)

where $C = (C_l; l \in \mathcal{L})$. The first constraint is the capacity constraint.

Assume that every user adopts rate-based flow control. Let $w_i(t)$ and $x_i(t)$ denote user $i$’s willingness to pay per unit time and rate at time $t$, respectively. Now suppose that at time $t$ each resource $l \in \mathcal{L}$ charges a price per unit flow of $p_l(t) = p_l(\sum_{i \in r_l} x_i(t))$, where $p_l(\cdot)$ is an increasing function of the total rate going through the resource. Consider the system of differential equations [7]

$$\frac{dx_i(t)}{dt} = \kappa_i \left( w_i(t) - x_i(t) \sum_{i \in r_l} p_l(t) \right)$$

(2)

where $w_i(t) = x_i(t) \cdot U_i'(x_i(t))$. For a further explanation of (2), see [7]. Since we assume that the utility functions of the users are selected to decide the rate allocation among the users, under (2) the design of

1All vectors are assumed to be column vectors.

2Throughout the rest of this paper, we refer to the willingness to pay per unit time as simply willingness to pay.

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rate control algorithms is equivalent to selecting users’ utility functions and the price functions of the resources in the network.

Kelly et al. [7] have shown that, under some conditions on \( p_i(\cdot), i \in \mathcal{L} \), the above system of differential equations converges to a point that maximizes the following expression:

\[
\mathcal{U}(x) = \sum_i U_i(x_i) - \sum_i \int_0^1 \sum_{i,j} x_{ij}^* p_i(y) dy.
\]

(3)

Note that the first term in (3) is the objective function in our SYSTEM\( (U, A, C) \) problem. Thus, the algorithm solves a relaxation of the SYSTEM\( (U, A, C) \) problem.

III. A NETWORK MODEL WITH FIXED DELAYS

In this section, we first describe the network model that was used in [12] to capture the delays between the network resources and end users under the assumption that the delays are constant. Although in practice the delays are time-varying due to time-varying queue sizes, we assume that the variation in the delays due to fluctuating queue sizes is not significant. For example, [AUTHOR: please define acronyms AQM, AVQ, and REM.]

We consider a set \( \mathcal{I} = \{1, \ldots, N\} \) of users that share a network with a set \( \mathcal{L} = \{1, \ldots, L\} \) of resources as described in Section II. Define \( I_r = \{i \in \mathcal{I} \mid r \in r_i\} \), i.e., the set of users traversing resource \( r \in \mathcal{L} \). We assume that the price functions \( p_i(\cdot), i \in \mathcal{L} \), are strictly increasing and continuous.

Both user rates from the senders to the resources and the feedback information from the resources to the users, which is typically carried by acknowledgments, are delayed due to link propagation and transmission delays. For all \( i \in \mathcal{I} \), \( r \in r_i \), let \( T_{i,r} \) denote the reverse delay from user \( i \) to resource \( r \), \( T_{i,r} \) the forward delay from resource \( r \) to user \( i \), and \( T_{i,r} = T_{i,r} + T_{i,r} \), \( r \in r_i \), \( i \in \mathcal{I} \), the delay from resource \( r \) to user \( i \). Suppose that the links in \( r_i = \{r_{i,1}, \ldots, r_{i,N_i}\} \) are arranged in the order user \( r \)’s packets visit, where \( r_i = \{r_i, r \} \) denotes the cardinality of \( r_i \)-time line \( T_r \) to be user \( i \)'s round-trip delay, i.e., the sum of forward and reverse delays \( T_{i,r} + T_{i,r} \), \( i \in \mathcal{I} \). Under this general model, the end user dynamics are given by

\[
\frac{dx_i(t)}{dt} = \kappa_i \left( x_i(t) U_i'(x_i(t)) - x_i(t - T_{i,r}) \left( \sum_{r \in r_i} \mu_i(t - T_{i,r}) \right) \right)
\]

(4)

where \( \mu_i(t - T_{i,r}) = p_i(\sum_{r \in r_i} x_i(t - T_{i,r} + T_{i,r})) \). Note that the price of resource \( r \) at time \( t \) depends on the rates of the users at time \( t - T_{i,r} \) due to the delay from the senders to the resource. The feedback signal generated by the resource is then delayed by \( T_{i,r} \) before sender \( i \) receives it.

IV. UTILITY AND RESOURCE PRICE FUNCTIONS

We are interested in isoelastic (or constant elasticity of substitution) utility functions and price functions [2]. The class of users’ utility functions is assumed to be of the form

\[
U_i(x) = -\frac{x_i^{a_i}}{a_i}, \quad a_i > 0.
\]

(5)

A slight variation of this class of utility function is given by

\[
U_i(x) = -\frac{x_i^{a_i - 1}}{a_i}, \quad a_i > 0.
\]

(6)

With these utility functions, the price elasticity of demand, which measures how responsive the demand is to a change in price and is defined to be the percent change in demand divided by the percent change in price [14], is given by

\[
\frac{p_i(x_i)}{x_i^{a_i}} \frac{dx_i^*(p)}{dp} = \frac{p}{p_i^{1 + a_i}} = \frac{1}{1 + a_i}
\]

(7)

where \( x_i^*(p) \) is the unique optimal rate that maximizes the net utility \( U_i(x) - x \cdot p \) given a price per unit flow \( p \). Note that the price elasticity of demand in (7) does not depend on the operating point \( x_i^*(p) \) and decreases with \( a_i \). This family of utility functions is also used to achieve a wide range of different fairness in [10] (called \( p, a_i \)-proportional fairness).

The class of resource price functions that we employ is of the form

\[
p_i(y) = c_i \left( \frac{y}{C_i} \right)^{b_i}
\]

(8)

where \( b_i > 0, c_i \) some positive constant and is assumed to be one without loss of generality and \( C_i \) is the capacity of resource \( r \in \mathcal{L} \). However, \( C_i \) can be replaced with any positive constant, e.g., virtual capacity in AVQ. The parameter \( b_i \) is used to change the shape of the price function.

V. A SUMMARY OF PREVIOUS RESULTS

As mentioned in Section I, depending on the network size, technologies (e.g., wireless versus wired), selected paths, etc., the round-trip delays of flows vary considerably. With the goal of ensuring the stability of the congestion control mechanism in different types of networks and simplifying the design of such mechanisms, we have investigated the stability issue of the system given by (12), (4) while allowing the delays \( T_{i,r} \) to be arbitrarily large. Define a map \( F : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N_+ \), where \( \mathbb{R}^N_+ = (0, \infty)^N \)

\[
F_i(y) = g_i^{-1}(y_i) \sum_{i \in r_i} \mu_i(t - T_{i,r}) \left( \sum_{j \in \mathcal{I}} g_j^{-1}(y_j) \right)
\]

where \( \overline{y} = (y_i; i \in \mathcal{I}) \), and \( g_i^{-1}(y_i) = y_i^{1/a_i} \). We showed that the delay differential system in (4) is asymptotically stable with a domain of attraction \( D = \prod_{i \in \mathcal{I}} \text{proj}_i(D) \), where \( \text{proj}_i(D) \) is the projection of the \( i \)-th component and \( \prod \) denotes a Cartesian product, if i) the discrete time map \( F \) has a stable fixed point in \( D \) and ii) the initial function lies in the Banach space of continuous functions mapping the interval \([-T_{\text{max}}, 0)\) to \( D \) with topology of uniform convergence, where

\[
T_{\text{max}} := \max_{i \in \mathcal{I}, r_i \in r_i} \left( \max_{i \in \mathcal{I}} T_{i,r} + T_{i,r} \right).
\]

(9)

An initial function \( \phi^\circ \) specifies \( x_i(s) = \phi_i^\circ(s) \) for all \( s \in [-T_{\text{max}}, 0] \).

We denote the Banach space of continuous functions from \([-T_{\text{max}}, 0]\) to a set \( D \) with topology of uniform convergence by \( C([-T_{\text{max}}, 0], D) \). Note that this stability condition depends on the delays \( T_{i,r} \) and \( T_{i,r} \) only through the initial function \( \phi^\circ \).

We applied this result to derive a sufficient condition for global asymptotic stability with the utility and resource price functions in (5) [or (6)] and (8), respectively. The system in (4) is shown to be globally

3When comparing the price elasticity, typically the absolute value of (7) is used.
asymptotically stable if $a_i > 1 + \max_{\xi \in I} b_i$ for all $i \in I$, starting with any continuous initial function in $C([-T_{\max}, 0], R_+)$. Similar results are reported in another independent study [17].

In the following sections, we extend the results obtained in [12] and [17] in two directions. First, we study a simpler case with a single flow and a single resource where a finite upper bound on feedback delay is known. We derive delay-dependent global asymptotic stability conditions for the utility and resource price functions of (5) and (8) (Section VI). This result complements the findings on delay-independent stability reported in [12] and [17] and hints at how the known finite upper bound on the delay affects the global stability. The derived condition is consistent with the known stability conditions in two extreme cases—either no delay or an arbitrary delay between the resource and the user. Secondly, we consider a general network described in Section III and derive a lower bound on the convergence rate of the system with the utility and price functions in (5) and (8), respectively, under the delay-independent stability condition, i.e., $a_i > 1 + \max_{\xi \in I} b_i$ for all $i \in I$ (Section VII).

VI. DELAY-DEPENDENT STABILITY CONDITION: A SINGLE FLOW, A SINGLE RESOURCE CASE

The stability results presented in [12] and [17] are concerned with the case where the delays $T_{\ell_1}$ and $T_{\ell_2}$ can be arbitrarily large. However, if a finite upper bound on the delays is known in advance, less stringent stability conditions may suffice to ensure stability. In this section, we consider the case where a single flow utilizes a single resource with feedback delay present only in the reverse path from the resource to the sender. Although we only consider the utility and price functions of the form (5) and (8), the basic idea used here can be applied to more general functions. We derive a delay-dependent stability condition that highlights the relation among the feedback delay, gain parameter, and stability of the system.

Denote the user utility function parameter and the resource price function parameter by $a$ and $b$, respectively. The feedback delay is given by $T > 0$. Assume that the initial function $\phi^o$ belongs to $C([-T, 0], R_+)$. Since we are interested in the case where the delay-independent stability condition does not hold, we assume $a < 1 + b$.

Theorem 1: Suppose that

$$
\frac{1 + b}{a} \left(1 - \exp(-\kappa \cdot \theta(T, \kappa))\right) < 1 \tag{10}
$$

where $\theta(T, \kappa) = a \cdot T \cdot v(T)^{\alpha_i+1}/\alpha_i$ and $v(t)$, $t \geq 0$, is the solution to the initial value problem of:

i) $v(0) = y^* = C^{-\alpha_i}/(1+b)$, where $C$ is the capacity of the source;

ii) $\frac{dv}{dt}(t) = K(v(t)) \cdot (y_v - v(t))$ with $y_v \equiv (C^{-\alpha_i}/(1+b)) + (1 + a) \kappa \cdot T^{-\alpha_i/(1+a)}$, where

$$
K(v) = -\kappa \cdot g'(y^{-1}(v)) = -\kappa \cdot a \cdot v^{(1+a)/a}.
$$

Then, $x(t; \phi^*)$ generated by (4) with an initial function $\phi^* \in C([-T, 0], R_+)$ satisfies $x(t; \phi^*) \rightarrow x^*$ as $t \rightarrow \infty$, i.e., the system is globally asymptotically stable.

Proof: The proof is provided in the Appendix.

Note that, for any fixed value of $\kappa > 0$, $\lim_{T \rightarrow \infty} \theta(T, \kappa) = 0$ and $\lim_{T \rightarrow \infty} \theta(T, \kappa) = \infty$. Therefore, one can see that (10) holds for all values of $a > 0$ and $b > 0$ in the absence of delay $T$ because $(1 + b)/a \times (1 - \exp(-\kappa \cdot \theta(T, \kappa))) = (1 + b)/a \times (1 - 1) = 0 < 1$.

Similarly, the condition holds for all values of $T \geq 0$ only if $(1 + b)/a < 1$, whence recovering the delay-independent stability condition in [12] and [17]. Thus, (10) in Theorem 1 hints at how the upper bound on the feedback delay affects the global stability condition. We plot the maximum delay $T$ that satisfies (10) as a function of the utility and price function parameters $a$ and $b$ in Fig. 2. Here we only plot the value for the cases where $a \leq 1 + b - 10^{-1}$ for numerical stability. The capacity $C = 10$ and the gain $\kappa = 0.2$. It is clear from the figure that as $a$ approaches $1+b$, the lower bound on the maximum stable delay increases rapidly.

Similar delay-dependent conditions are also reported in [3] and [9] in the same setting of a single flow and a single resource. The authors of [9] consider the case where $w(t) = w > 0$ for all $t \geq 0$. This represents the case with the utility function given by $w \cdot \log(x)$. They show that, under some conditions, the system is stable if $T \cdot \kappa \leq 1/4$. The stability conditions for more general utility and resource price functions are derived in [3, Theorem II.1]. The stability conditions derived in [3] depend on the size of the domain $\lambda_0 = [x_{\min}, x_{\max}]$ in which the user rate $x(t)$ lies after some finite time $t_0$, i.e., $x(t) \in \lambda_0$ for all $t \geq t_0$. It states that if $x: T \cdot \kappa < A(a, b, \lambda_0)/B(a, b, \lambda_0)$, where $A(\cdot) \cdot B(\cdot)$ are some functions, then the system is globally exponentially stable. However, with the utility and price functions of (5) and (8), respectively, the stability condition in [3, p. 1057] yields lower bounds on the maximum stable feedback delays much smaller than those shown in Fig. 2 over the same set of parameters.

VII. A LOWER BOUND ON CONVERGENCE RATE

In this section, we investigate the convergence rate of the delay differential system in (4) with the utility and resource price functions in (5) and (8) under the delay-independent stability condition given in Section V. A lower bound on the convergence rate is provided, and we explain how the maximum delay $T_{\max}$ in (9) affects the lower bound on the convergence rate. We also derive a closed-form expression for
a lower bound on the local convergence rate of the system around the equilibrium.

Let $\sigma$ be some negative constant such that $-1 < \sigma < -\Delta$, where $\Delta := \max_{v \in \mathcal{I}} (1 + b^v_{\max}) / a_v$ and $b^v_{\max} = \max_{v \in \mathcal{I}} b_v$. Note that under the delay-independent stability condition, $\Delta < 1$. Fix some $\alpha > 1$, and let $\beta$ be a constant that satisfies

$$\hat{\mathcal{F}}(\beta \cdot \bar{x}^* < \alpha \cdot \bar{x}^* \text{ and } \beta \cdot \bar{x}^* < \hat{\mathcal{F}}(\alpha \cdot \bar{x}^*)$$

where $\bar{x}^* = (x^*_i)_{i \in \mathcal{I}}$ is the unique solution to (3), $\hat{\mathcal{F}}(\bar{x}) := \mathcal{F}^{-1}(f(\bar{x})) = \mathcal{F}^{-1}(f_i(\bar{x})): i \in \mathcal{I}$, and

$$f_i(\bar{x}) := x_i \sum_{j \in \mathcal{N}_i} p_{ij} \left( \sum_{j \in \mathcal{N}_i} x_j \right).$$

It is shown [12, Lemma 1] that any $\beta$ such that $\alpha^{1/\gamma} < \beta < \alpha^{-\gamma}$ satisfies (12) and its existence is guaranteed under the delay-independent stability condition. Define

$$\mathcal{D}_k = \begin{cases} \prod_{v \in \mathcal{I}} [\alpha^{k} x^*_v, \beta^{k} x^*_v], & k \text{ odd} \\ \prod_{v \in \mathcal{I}} [\beta^{k} x^*_v, \alpha^{k} x^*_v], & k \text{ even} \end{cases}$$

Reference [12, Lemma 2] tells us that $\hat{\mathcal{F}}(\mathcal{D}_k) \subset \mathrm{int}(\mathcal{D}_{k+1}) \subset \mathcal{D}_{k+1} \subset \mathrm{int}(\mathcal{D}_{k})$, where $\mathrm{int}(\mathcal{D}_k)$ is the interior of $\mathcal{D}_k$ and $\cap_{k=0}^{\infty} \mathcal{D}_k = \{ \bar{x}^* \}$ under the assumed stability condition.

**Theorem 2:** Suppose $a_i > b^i_{\max} + 1$ for all $i \in \mathcal{I}$. If the initial function $\phi^\Delta \in \mathcal{C}([-T_{\max}, 0]; \mathcal{D}_0)$, then there exist $K := K(\mathcal{D}_0) > 0$ and $\gamma := \gamma(\mathcal{D}_0) > 0$ such that

$$\frac{|x_i(t) - x^*_i|}{x^*_i} \leq K \cdot \exp(\gamma \cdot t)$$

for all $i \in \mathcal{I}$ and for all $t \geq 0$. (14)

**Proof:** A proof is provided in [8] due to space constraints. 

In the proof of the theorem, we show that $K = K(\sigma) / |\sigma|^2$ and $\gamma = \psi(\sigma) / (|M^* + T_{\max}|)$ satisfy (14) with any $\sigma \in \mathcal{I}^{\Delta} := (-1, -\Delta)$, where $K(\sigma) = \exp(|\sigma| - 1) - 1$, $\psi(\sigma) = \log(-1/|\sigma|)$, and $M^* := M^*(\mathcal{D}_0, \sigma)$ is a constant that increases with the initial invariance set $\mathcal{D}_0$ and can be numerically computed. Hence

$$\sup_{\sigma \in \mathcal{I}^{\Delta}} \frac{\psi(\sigma)}{M^*(\mathcal{D}_0, \sigma) + T_{\max}} \leq \lim_{\kappa \to \infty} \frac{\log(-1/|\sigma|)}{M^*(\mathcal{D}_0, \sigma) + T_{\max}}$$

provides a lower bound on the convergence rate of the delay differential system in [12], which clearly depends on the initial invariance set $\mathcal{D}_0$ through $M^*(\mathcal{D}_0, \sigma)$. Furthermore, as the initial invariance set $\mathcal{D}_0$ becomes smaller (i.e., $\mathcal{D}_0 \downarrow \{ \bar{x}^* \}$), the constant $M^*(\mathcal{D}_0, \sigma)$ converges to

$$M := \min_{\sigma \in \mathcal{I}^{\Delta}} \frac{(\alpha^\Delta)^{1+\kappa_i} (1 - \Delta^2)}{\kappa_i \cdot \Delta}.$$ (15)

Note that (16) does not depend on the selected value of $\sigma \in \mathcal{I}^{\Delta}$. Hence, under the stability conditions in Theorem 2, as $\bar{x}(t)$ gets close to the solution $\bar{x}^*$, the (local) convergence rate approaches or becomes larger than $\log(\Delta^{-1}) / (M + T_{\max})$. This clearly highlights the dependence of the lower bound on $T_{\max}$.

Interestingly, the lower bound is monotonically increasing in users’ gain parameters $\kappa_i, i \in \mathcal{I}$, as $M$ is decreasing in users’ gain parameters and neither $\bar{x}^*$ nor $\Delta$ depend on them. It is clear that

$$\lim_{\kappa \to \infty} \frac{\log(\Delta^{-1})}{M + T_{\max}} = \frac{\log(\Delta^{-1})}{T_{\max}}.$$ (17)

In fact, for any fixed $\mathcal{D}_0$, $M^*(\mathcal{D}_0, \sigma)$ goes to zero as $\kappa, i \in \mathcal{I}$, go to $\infty$, and the limit of the lower bound in (15) is also given by $\log(\Delta^{-1}) / T_{\max}$.

Numerical examples of the bounds in (14) with the solutions of (4) are provided in [8]. They show that the accuracy of the lower bound generally improves with users’ gain parameters. Moreover, for all sufficiently large users’ gain parameters, the behavior of the solutions is similar, and the bounds in (17) provide close bounds.

We plot the lower bound $\log(\Delta^{-1}) / (M + T_{\max})$ on the convergence rate around the equilibrium in a simple case of a single flow traversing a single resource in Fig. 3. The gain parameter $\kappa$ is set to
0.5, the round-trip delay of the flow is 100, and the capacity \( C = 10 \). We vary the utility parameter \( a \) and the price function parameter \( b \) to see how these parameters affect the lower bound on the convergence rate. As expected, for a fixed value of \( b \), the lower bound increases with \( a \), whereas it decreases with \( b \) for a fixed value of \( a \).

VIII. CONCLUSION

We studied the stability of a family of rate control schemes; we first studied the delay-dependent stability in a simple network with a single flow and a single resource. We derived a sufficient condition for the asymptotic stability when there is a finite upper bound on a communication delay. Secondly, we provided a lower bound on the convergence rate of a system when the delay-independent stability conditions hold.

APPENDIX

Define \( y(t) := g(x(t)) = 1/x(t)^{a} \) to be the user’s willingness to pay at time \( t \). We rewrite the delay differential equation

\[
\frac{d}{dt}x(t) = \kappa(1/x(t)^{a} - x(t - T)p(x(t - T)))
\]

in terms of \( y(t) \) as follows:

\[
\frac{d}{dt}y(t) = K(y(t))(F(y(t)) - y(t))
\]

where \( K(y) \) is defined in (11) and \( F(y) = y^{-1}(y)p(y^{-1}(y)) = C^{-1}y^{1+a} \) is the total price of the user when its willingness to pay is known by the resource is \( y \).

Reference [4, Corollary 3.2] states that the delay differential system in (19) will either converge to the equilibrium \( y^{*} = g(x^{*}) \) or oscillate around \( y^{*} \). Define

\[
m := \liminf_{t \to \infty} y(t) \quad \text{and} \quad \bar{m} := \limsup_{t \to \infty} y(t).
\]

Let \( y_{+} : \mathbb{R}_{+} \to C([-T, 0], \mathbb{R}_{+}) \), where \( y_{+}(s) = y(t + s), s \in [-T, 0] \). Define \( \Omega(y) := \{y_{+}, y \geq t \} \subset C([-T, 0], \mathbb{R}_{+}) \). Then, for every \( y \in \Omega(y) \), there exists a solution \( y(s; \phi) : \mathbb{R}_{+} \to \mathbb{R}_{+} \) such that \( \phi(s) = y(t; \phi), s \in [-T, 0] \). Moreover, for all \( y \in \Omega(y) \), we have

\[
\liminf_{t \to \infty} y(t) = m \leq y(t; \phi) \leq \bar{m} = \limsup_{t \to \infty} y(t).
\]

In addition, there exist \( \gamma, \xi \in \Omega(y) \), such that

\[
\bar{m} = y(t_{1}; \gamma) \quad \text{and} \quad m = y(t_{2}; \xi)
\]

for some \( t_{1}, t_{2} > 0 \). Reference [4, Lemma 3.3] tells us that if \( y(t_{1}) \leq \bar{m} \), then \( y(t_{1} - T) \leq y^{*} \). Similarly, if \( y(t_{2}) \geq m \), then \( y(t_{2} - T) \geq y^{*} \). Therefore, without loss of generality, we can assume that

\[
y(0; \gamma) = y(0; \xi) = y^{*} \quad \text{and} \quad t_{1} \leq T, t_{2} \leq T
\]

with \( y(t; \gamma) \in (y^{*}, \bar{m}) \) for all \( t \in (0, t_{1}) \) and \( y(t; \xi) \in (m, \bar{m}) \) for all \( t \in (0, t_{2}) \).

First, consider \( x(t) = x(t; \gamma) = y^{-1}(y(t; \gamma)) \) constructed by (18), where \( \gamma(s) = y^{-1}(\gamma(s)) \) for all \( s \in [-T, 0] \). Let \( z(t) \) be the solution to the following initial value problem: \( z(0) = x^{*} \) and

\[
\frac{d}{dt}z(t) = \kappa \cdot z(t) \cdot U'(z(t))
\]

Then, since \( (x(t)U'(x(t)) - x(t - T)p(x(t - T))) \leq x(t)U'(x(t)) \), we have \( x(t) \leq z(t) \) for all \( t \geq 0 \) [15, Theorem 5.III]. Substituting \( U'(x) = x^{-(1+a)} \), after a little algebra, one can show that

\[
z(t) = (x^{*1+a} + (1 + a)\kappa \cdot t^{1/(1+a)})
\]

\[
z(T) = (x^{*1+a} + (1 + a)\kappa \cdot T^{1/(1+a)}) \geq z(t_{1})
\]

\[
\geq x(t_{1}) = \limsup_{t \to \infty} x(t; \gamma)
\]

\[
= y^{-1}(\liminf_{t \to \infty} y(t; \gamma))
\]

where the last equality follows from the fact that \( g(x) = 1/x^{a} \) is a decreasing function of \( x \). Therefore, \( z(T) \) provides an upper bound on \( \limsup_{t \to \infty} x(t; \gamma) \). Furthermore, this also tells us

\[
\liminf_{t \to \infty} y(t; \gamma) \geq z(T) =: y^{*}.
\]

Let \( v(t) \) be the solution to the following initial value problem:

\[
v(0) = y^{*} \quad \text{and} \quad \frac{d}{dt}v(t) = K(v(t))(F(v(t)) - v(t)).
\]

Then, since \( F(y) \geq F(y(t - T) - y(t)) \) for all \( t \geq 0 \), we have \( F(y(t - T) - y(t)) \geq F(y(t)) - y(t) \) and, hence, \( v(t) \geq y(t) \) for all \( t \in (0, t_{1}) \) [15, Theorem 5.III]. Therefore, \( v(T) \geq y^{*} \) provides an upper bound on \( \limsup_{t \to \infty} x(t; \gamma) \). Now let

\[
\Theta(T, \kappa) := \frac{K(v(T))}{\kappa} \cdot T = a \cdot T \cdot \Theta(T)^{1+a}.
\]

Define a map

\[
G(y) := (1 - \exp(-\kappa \cdot \Theta(T, \kappa)))F(y)
\]

\[
+ \left[y^{*} - (1 - \exp(-\kappa \cdot \Theta(T, \kappa)))F(y^{*})\right]
\]

\[
= (1 - \exp(-\kappa \cdot \Theta(T, \kappa)))F(y)
\]

\[
+ y^{*} \cdot \exp(-\kappa \cdot \Theta(T, \kappa)).
\]

We first state a lemma that will be used to complete the proof of the theorem. The proof of the lemma is provided in [8] due to space constraints.

Lemma 1: The map \( G(y) \) has a globally attracting fixed point \( y^{*} \) under the condition in Theorem 1.

We continue with the proof of Theorem 1. Take an arbitrary solution \( y(t) \) of (19) and let

\[
0 < m = \liminf_{t \to \infty} y(t) \leq y(t) \leq \bar{m} = \limsup_{t \to \infty} y(t).
\]

Suppose that \( \gamma \) and \( \xi \) are initial functions satisfying (20) and (21). Define

\[
F_{+}(q(t), m, \bar{m}) := K(q(t))(F(m) - q(t))
\]

\[
F_{-}(q(t), m, \bar{m}) := K(q(t))(F(\bar{m}) - q(t)).
\]

Let \( z(t) \) be the solution to the initial value problem of:

\[
j = y^{*};
\]

\[
\int_{t_{0}}^{t} \frac{dz}{F_{+}(z, m, \bar{m})} = t_{1} \leq T.
\]

Note that \( \Psi := \Psi(m, \bar{m}) \geq z(t_{1}, m, \bar{m}) \) satisfies

\[
\int_{t_{0}}^{t} \frac{dz}{F_{+}(z, m, \bar{m})} = t_{1} \leq T.
\]

*The existence of such \( \Psi \) can be shown.
Hence, if we define $\Phi_+ : = \Phi_+(z, m) > y^*$ as the unique solution of $F_+(z, m) > y^*$, then we have $\Phi_+ \geq \Psi \geq \Phi_-'$. Also, $\Phi_-' \leq \nu(T)$ from the construction of $\nu(T)$ because $y^* \leq m$ (hence $F(y^*) \geq F(m)$). An analogous study of $y(t, x)$ tells us that $\Phi_-' : = \Phi_-'(z, m) \leq m$, where $\Phi_-'$ is defined by $\Theta_-(z, m) \leq z$.

Note that

$$T = \int_{y^*}^{\Phi_+} \frac{dz}{F_+(z, m)} = \int_{y^*}^{\Phi_+} \frac{dz}{\Phi_-(z, m) - z} \leq \int_{y^*}^{\Phi_+} \frac{dz}{\Phi_-(z, m) - z},$$

where the inequality follows from the fact that $\nu(T) \geq \Phi_+'$ and $\Phi_-(y) = \mu \cdot \kappa \cdot y^{1+1/\mu}$ is an increasing function of $y$. Integrating this last equation, we obtain

$$T \geq \frac{-1}{K} \ln \left( \frac{\Phi_+ - \Phi_-(m)}{y^* - \Phi_-(m)} \right).$$

This, with $K(\nu(T)) = \mu \cdot \kappa \cdot \Theta(T, \kappa)$, yields

$$\Phi_+ \leq \Phi_-(m) + \exp(-\mu \cdot \kappa \cdot \Theta(T, \kappa))(y^* - \Phi_-(m))$$

$$= \Phi_-(m)(1 - \exp(-\mu \cdot \kappa \cdot \Theta(T, \kappa)))$$

$$+ y^* \exp(-\mu \cdot \kappa \cdot \Theta(T, \kappa))$$

$$= \max_{y \in [m, \bar{m}]} G(y). \tag{22}$$

Following the same steps, one can show that

$$\Phi_-' \geq \min_{y \in [m, \bar{m}]} G(y). \tag{23}$$

Equations (22) and (23) imply that $[m, \bar{m}] \subseteq G([m, \bar{m}])$ because $\Phi_-' \leq m \leq \bar{m} \leq \Phi_+$. However, Lemma 1 tells us that the map $G(y)$ has a globally attracting fixed point $y^*$ and thus $G([m, \bar{m}]) \subseteq [m, \bar{m}]$. This implies $m = y^* = \bar{m}$, proving that the system is asymptotically globally stable.

**REFERENCES**


Asymptotic Stability of a Rate Control System With Communication Delays

Richard J. La and Priya Ranjan

Abstract—We study the issue of asymptotic stability of a family of rate control algorithms with communication delays between network elements and extend our earlier results: First, we derive delay-dependent stability conditions with a family of well-known utility and resource price functions when a finite upper bound is known on the feedback delay. These conditions are shown to be consistent with known stability conditions in two extreme cases—no delay or an arbitrarily large delay. Secondly, we provide a lower bound on the convergence rate with the same utility and resource price functions when delay-independent stability conditions hold.

Index Terms—Asymptotic stability, communication system control, delay systems.

I. INTRODUCTION

Recently, there has been much interest in understanding the stability property of a family of rate control schemes, called primal algorithms, in the presence of communication delays [3], [5], [9], [12], [13], [17]. The stability of a similar algorithm proposed in [11] is studied in [16]. The primal algorithms, first proposed by Kelly et al. [7], are motivated by an optimization framework for a rate allocation with elastic traffic sources where the objective of the system is to maximize the aggregate utility of the users. Tan and Johari [5] studied the case where flows have the same round-trip delays and the same log utility functions, and provided local stability conditions in terms of users’ gain parameters and communication delays. Similar results have been obtained for single flow and single resource cases with more general utility functions in [3] and [9] and for single bottleneck with multiple heterogeneous users cases in [1].

In another set of work, Ranjan et al. [12] and Ying et al. [17] studied the stability of the rate control system in the presence of arbitrary fixed delays between network elements, e.g., network resources and end users, and arbitrary gain parameters of the end users. These approaches are consistent with the philosophy that, given the complexity and scale of the Internet, network protocols must be simple and robust. In addition, in large wireless networks, e.g., mobile ad-hoc networks (MANETs), round-trip delays are expected to be much larger than in their wireline counterparts. Portability of the network protocols will demand their stability in various environments in which they are expected to operate, including MANETs. The authors derived sufficient conditions on users’ utility and resource price functions for (global) asymptotic stability of the rate control system with arbitrary fixed communication delays and users’ gain parameters. We refer to these conditions as delay-independent stability conditions.

In this paper, we study the same class of rate control systems investigated in [12] and [17] and extend their results in two directions.

• We study a simple network with a single flow traversing a single resource and investigate its stability when a finite upper bound on the feedback delay is known in advance. We derive a stability condition that hints at how an increasing feedback delay affects the stability condition. We refer to this condition as a delay-dependent stability condition to distinguish it from the delay-independent stability conditions. The derived condition is consistent with the earlier stability conditions in two extreme cases; when there is no feedback delay, the system is always stable with the employed utility and resource price functions. When the delay is allowed to be arbitrarily large, we recover the delay-independent stability condition reported in [12] and [17].

• We derive a lower bound on the convergence rate of the rate control system under the delay-independent stability condition in [12] and [17]. The role of the maximum delay in feedback loop is highlighted. A closed-form expression of a lower bound on the local convergence rate is also provided.

This paper is organized as follows. Section II describes the optimization framework for rate control. Section III describes the rate control system model with fixed communication delays. The utility and resource price functions employed in this paper are explained in Section IV. We summarize earlier results in [12] and [17] in Section V. Delay-dependent stability conditions are obtained in Section VI. A lower bound on the convergence rate under the delay-independent stability conditions is derived in Section VII. We conclude in Section VIII.

II. BACKGROUND

In this section, we briefly describe the rate control problem in the proposed optimization framework. Consider a network with a set \( \mathcal{L} \) of resources or links shared by a set \( \mathcal{I} \) of users. Let \( C_i \) denote the finite capacity of link \( i \in \mathcal{L} \). Each user has a fixed route \( r_i \), which is a nonempty subset of \( \mathcal{L} \). We define a zero-one matrix \( A_i \), where \( A_{i,j} = 1 \) if link \( j \) is in user \( i \)’s route \( r_i \) and \( A_{i,j} = 0 \) otherwise. When the throughput of user \( i \) is \( x_i \), user \( i \) receives utility \( U_i(x_i) \). We take the view that the utility functions of the users are selected for a desired rate allocation among the users. The utility \( U_i(x_i) \) is an increasing, strictly concave, and continuously differentiable function of \( x_i \) over the range \( x_i \geq 0 \). The rate allocation problem can be formulated as the following optimization problem [6]:

\[
\text{maximize } \sum_{i \in \mathcal{I}} U_i(x_i) \\
\text{subject to } A^T x \leq C, \quad x \geq 0
\]

where \( C = (C_i; i \in \mathcal{I}) \).

The first constraint is the capacity constraint. Assume that every user adopts rate-based flow control. Let \( w_i(t) \) and \( x_i(t) \) denote user \( i \)’s willingness to pay per unit time and rate at time \( t \), respectively. Now suppose that at time \( t \) each resource \( i \in \mathcal{L} \) charges a price per unit flow of \( \mu_i(t) = p_i(\sum_{l \in r_i} x_l(t)) \), where \( p_i(\cdot) \) is an increasing function of the total rate going through the resource.

Consider the system of differential equations [7]

\[
\frac{d}{dt} x_i(t) = \kappa_i \left( w_i(t) - x_i(t) \sum_{l \in r_i} \mu_l(t) \right)
\]

where \( w_i(t) = x_i(t) \cdot U_i'(x_i(t)) \). For a further explanation of (2), see [7]. Since we assume that the utility functions of the users are selected to decide the rate allocation among the users, under (2) the design of

\[1\] All vectors are assumed to be column vectors.

\[2\] Throughout the rest of this paper, we refer to the willingness to pay per unit time as simply willingness to pay.
rate control algorithms is equivalent to selecting users’ utility functions and
the price functions of the resources in the network.

Kelly et al. [7] have shown that, under some conditions on $p_i(\cdot)$, $l \in \mathcal{L}$, the above system of differential equations converges to a point that maximizes the following expression:

$$\mathcal{U}(x) = \sum_i U_i(x_i) - \sum_i \int_0^\infty \sum_j x_i^j \mu_i(t - T_{ij}) dt.$$  

(3)

Note that the first term in (3) is the objective function in our SYSTEM($U, A, C$) problem. Thus, the algorithm solves a relaxation of the SYSTEM($U, A, C$) problem.

III. A NETWORK MODEL WITH FIXED DELAYS

In this section, we first describe the network model that was used in [12] to capture the delays between the network resources and end users under the assumption that the delays are constant. Although in practice the delays are time-varying due to time-varying queue sizes, we assume that the variation in the delays due to fluctuating queue sizes is not significant. For example, [AUTHOR: please define acronyms AQM, AVQ, and REM] AQM mechanisms that attempt to maintain very small queue sizes, e.g., AVQ, or keep the queue sizes around some target queue sizes, e.g., REM, may be well approximated by our model. (The definitions of the variables introduced throughout the paper are provided in of [8, Table 1].)

We consider a set $\mathcal{I} = \{1, \ldots, n\}$ of users that share a network with a set $\mathcal{L} = \{1, \ldots, L\}$ of resources as described in Section II. Define $\mathcal{I}_l := \{i \in \mathcal{I} | l \in r_i\}$, i.e., the set of users traversing resource $l \in \mathcal{L}$. We assume that the price functions $p_i(\cdot), l \in \mathcal{L}$, are strictly increasing and continuous.

Both user rates from the senders to the resources and the feedback information from the resources to the users, which is typically carried by acknowledgments, are delayed due to link propagation and transmission delays. For all $i \in \mathcal{I}$ and $l \in r_i$, let $T_{il}^f$ denote the forward delay from sender $i$ to resource $l$ and $T_{il}^r$, the reverse delay of the feedback signal from resource $l$ to sender $i$. If $l \notin r_i$, we assume that $T_{il}^f = T_{il}^r = 0$. Suppose that the links in $r_i = \{l_{i0}, \ldots, l_{ik}\}$ are arranged in the order user $i$’s packets visit, where $R_i = |r_i|$ denotes the cardinality of $r_i$. Define $T_i$ to be user $i$’s round-trip delay, i.e., the sum of forward and reverse delays $T_{i1}^f + T_{i1}^r$, $k = 1, \ldots, R_i$. Under this general model, the end user dynamics are given by

$$\frac{dx_i(t)}{dt} = \mu_i\left(\sum_{j \in r_i} x_i(t)[x_j(t) - T_{ij}]\right)$$  

(4)

where $\mu_i(t - T_{ij}) = p_i(\sum_{j \in r_i} x_j(t) - (T_{ij} + T_{ij}^r))$. Note that the price of resource $l$ at time $t$ depends on the rates of the users at time $t - T_{ij}^r$ due to the delay from the senders to the resource. The feedback signal generated by the resource is then delayed by $T_{ij}^r$ before sender $i$ receives it.

IV. UTILITY AND RESOURCE PRICE FUNCTIONS

We are interested in isoelastic (or constant elasticity of substitution) utility functions and resource price functions [2]. The class of users’ utility functions is assumed to be of the form

$$U_i(x) = -\frac{1}{a_i \cdot x_i^{a_i}}, \quad a_i > 0.$$  

(5)

A slight variation of this class of utility function is given by

$$U_i(x) = -\frac{x_i^{-a_i} - 1}{a_i}, \quad a_i > 0.$$  

(6)

With these utility functions, the price elasticity of demand, which measures how responsive the demand is to a change in price and is defined to be the percent change in demand divided by the percent change in price [14], is given by

$$\frac{d x_i^*(p)}{dp} = \frac{p}{p - \frac{1}{1 + a_i}} - \frac{1}{1 + a_i} = \frac{p}{p - \frac{1}{1 + a_i}}.$$  

(7)

where $x_i^*(p)$ is the unique optimal rate that maximizes the net utility $U_i(x) - x \cdot p$ given a price per unit flow $p$. Note that the price elasticity of demand in (7) does not depend on the operating point $x_i^*(p)$ and decreases with $a_i$. This family of utility functions is also used to achieve a wide range of different fairness in [10] (called $(p, a_i)$-proportional fairness).

The class of resource price functions that we employ is of the form

$$p_i(y) = c_i \cdot \left(\frac{y}{C_i}\right)^{b_i}$$  

(8)

where $b_i > 0$, $c_i$ is some positive constant and is assumed to be one without loss of generality and $C_i$ is the capacity of resource $i \in \mathcal{L}$. However, $C_i$ can be replaced with any positive constant, e.g., virtual capacity in AVQ. The parameter $b_i$ is used to change the shape of the price function.

V. A SUMMARY OF PREVIOUS RESULTS

As mentioned in Section I, depending on the network size, technologies (e.g., wireless versus wireline), selected paths, etc., the round-trip delays of flows vary considerably. With the goal of ensuring the stability of the congestion control mechanism in different types of networks and simplifying the design of such mechanisms, we investigated the stability issue of the system given by [12, (4)] while allowing the delays $T_{ij}$ to be arbitrarily large. Define a map $F : \mathbb{R}_+^n \to \mathbb{R}_+^n$, where $\mathbb{R}_+ := (0, \infty)$.

$$F_i(y) = g_i^{-1}(y_i) \sum_{j \in r_i} x_j(t) \cdot \left(\sum_{i \in r_j} g_i^{-1}(y_j)\right)$$  

where $g_i^{-1}(y_i) := y_i^{-1/a_i}$. We showed that the delay differential system in (4) is asymptotically stable with a domain of attraction $D = \prod_{i \in \mathcal{I}} \text{proj}_i(D)$, where $\text{proj}_i(D)$ is the projection of the $i$th component and denotes a Cartesian product, if i) the discrete time map $F$ has a stable fixed point in $D$ and ii) the initial function lies in the Banach space of continuous functions mapping the interval $[-T_{\max}, 0]$ to $D$ with topology of uniform convergence, where

$$T_{\max} := \max_{i \in \mathcal{I}_l, j \in r_i} \left(\max_{i \in \mathcal{I}_l} T_{ij}^r + T_{ij}^f\right).$$  

(9)

An initial function $\phi^s$ specifies $x_i(s) = \phi^s_i(s)$ for all $s \in [-T_{\max}, 0]$. We showed the Banach space of continuous functions from $[-T_{\max}, 0]$ to a set $D$ with topology of uniform convergence by $C([-T_{\max}, 0], D)$. Note that this stability condition depends on the delays $T_{ij}^r$ and $T_{ij}^f$ only through the initial function $\phi^s$.

We applied this result to derive a sufficient condition for global asymptotic stability with the utility and resource price functions in (5) or (6) and (8), respectively. The system in (4) is shown to be globally

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Footnotes:

3When comparing the price elasticity, typically the absolute value of (7) is used.
asymptotically stable if \( a_i > 1 + \max_{i \in I} b_i \) for all \( i \in I \), starting with any continuous initial function in \( C([-T_{\text{max}}, 0], R_+) \). Similar results are reported in another independent study [17].

In the following sections, we extend the results obtained in [12] and [17] in two directions. First, we study a simpler case with a single flow and a single resource where a finite upper bound on feedback delay is known. We derive delay-dependent global asymptotic stability conditions for the utility and resource price functions of (5) and (8) (Section VI). This result complements the findings on delay-independent stability reported in [12] and [17] and hints at how the known finite upper bound on the delay affects the global stability. The derived condition is consistent with the known stability conditions in two extreme cases—either no delay or an arbitrary delay between the resource and the user. Secondly, we consider a general network described in Section III and derive a lower bound on the convergence rate of the system with the utility and price functions in (5) and (8), respectively, under the delay-independent stability condition, i.e., \( a_i > 1 + \max_{i \in I} b_i \) for all \( i \in I \) (Section VII).

VI. DELAY-DEPENDENT STABILITY CONDITION: A SINGLE FLOW, A SINGLE RESOURCE CASE

The stability results presented in [12] and [17] are concerned with the case where the delays \( T_{\text{up}} \) and \( T_{\text{down}} \) can be arbitrarily large. However, if a finite upper bound on the delays is known in advance, less stringent stability conditions may suffice to ensure stability. In this section, we consider the case where a single flow utilizes a single resource with feedback delay present only in the reverse path from the resource to the sender. Although we only consider the utility and price functions of the form (5) and (8), the basic idea used here can be applied to more general functions. We derive a delay-dependent stability condition that highlights the relation among the feedback delay, gain parameter, and stability of the system.

Denote the user utility function parameter and the resource price function parameter by \( a \) and \( b \), respectively. The feedback delay is given by \( \tau > 0 \). Assume that the initial function \( \phi^0 \) belongs to \( C([-T, 0], R_+) \). Since we are interested in the case where the delay-independent stability condition does not hold, we assume \( a < 1 + b \).

**Theorem 1:** Suppose that

\[
1 + \frac{b}{a}(1 - \exp(-\kappa \cdot \Theta(T, \kappa))) < 1
\]

where \( \Theta(T, \kappa) = a \cdot T \cdot v(T)^{1+a} \) and \( v(t), t \geq 0 \), is the solution to the initial value problem of:

i) \( v(0) = y^* = C^{-a/(1+a+b)} \), where \( C \) is the capacity of the resource;

ii) \( (d/y)/dt)v(t) = K(v(t))F(y, \kappa \cdot v(t)) \) with \( y = (C^{(1+a)/(1+a+b)} + (1 + a) \kappa \cdot T)^{-a/(1+a)} \), where

\[
K(v) = -\kappa \cdot g'(y^{-1}(v)) = -\kappa \cdot a \cdot v^{(1+a)/a}. \tag{11}
\]

Then, \( x(t; \phi^0) \) generated by (4) with an initial function \( \phi^0 \in C([-T, 0], R_+) \) satisfies \( x(t; \phi^0) \to x^* \) as \( t \to \infty \), i.e., the system is globally asymptotically stable.

**Proof:** The proof is provided in the Appendix.

Note that for any fixed value of \( \kappa > 0 \), \( \lim_{\tau \to \infty} \Theta(T, \kappa) = 0 \) and \( \lim_{\tau \to \infty} \Theta(T, \kappa) = \infty \). Therefore, one can see that (10) holds for all values of \( a > 0 \) and \( b > 0 \) in the absence of delay \( T = 0 \). This is because \( (1 + b)/a \times (1 - \exp(-\kappa \cdot \Theta(T, \kappa))) = (1 + b)/a \times (1 - 1) = 0 < 1 \). Similarly, the condition holds for all values of \( T \geq 0 \) only if \( (1 + b)/a < 1 \), whence recovering the delay-independent stability condition in [12] and [17]. Thus, (10) in Theorem 1 hints at how the upper bound on the feedback delay affects the global stability condition. We plot the maximum delay \( T \) that satisfies (10) as a function of the utility and price function parameters \( a \) and \( b \) in Fig. 2. Here we only plot the value for the cases where \( a \leq 1 + b - 10^{-1} \) for numerical stability. The capacity \( C = 10 \) and the gain \( \kappa = 0.2 \). It is clear from the figure that as \( a \) approaches \( 1 + b \), the lower bound on the maximum stable delay increases rapidly.

Similar delay-dependent conditions are also reported in [3] and [9] in the same setting of a single flow and a single resource. The authors of [9] consider the case where \( w(t) = w > 0 \) for all \( t \geq 0 \). This represents the case with the utility function given by \( w \cdot \log(x) \). They show that, under some conditions, the system is stable if \( T \cdot \kappa \leq 1/4 \). The stability conditions for more general utility and resource price functions are derived in [3, Theorem II.1]. The stability conditions derived in [3] depend on the size of the domain \( \lambda_0 = [\mathbb{R}, \mathbb{R}] \) in which the user rate \( x(t) \) lies after some finite time \( t_0 \), i.e., \( x(t) \in \lambda_0 \) for all \( t \geq t_0 \). It states that if \( \kappa \cdot T < A(a, b, \lambda_0)/B(a, b, \lambda_0) \), where \( A(\cdot) \) and \( B(\cdot) \) are some functions, then the system is globally exponentially stable. However, with the utility and price functions of (5) and (8), respectively, the stability condition in [3, p. 1057] yields lower bounds on the maximum stable feedback delays much smaller than those shown in Fig. 2 over the same set of parameters.

VII. A LOWER BOUND ON CONVERGENCE RATE

In this section, we investigate the convergence rate of the delay differential system in (4) with the utility and resource price functions in (5) and (8) under the delay-independent stability condition given in Section V. A lower bound on the convergence rate is provided, and we explain how the maximum delay \( T_{\text{max}} \) in (9) affects the lower bound on the convergence rate. We also derive a closed-form expression for
a lower bound on the local convergence rate of the system around the equilibrium.

Let \( \sigma \) be some negative constant such that \(-1 < \sigma < -\Delta\), where 
\[
\Delta := \max_{i \in I} \frac{1}{a_i} (1 + b_{i,\max}^a), \quad \text{and} \quad b_{i,\max}^a := \max_{i \in I} b_i.
\]
Note that under the delay-independent stability condition, \( \Delta < 1 \). Fix some \( \alpha > 1 \), and let \( \beta \) be a constant that satisfies
\[
\hat{F}(\beta \cdot \bar{x}^*) < \alpha \cdot \bar{x}^* \quad \text{and} \quad \beta \cdot \bar{x}^* < \hat{F}(\alpha \cdot \bar{x}^*),
\]
where \( \bar{x}^* = (x_i^*; i \in I) \) is the unique solution to (3), \( \hat{F}(\bar{x}) := g^{-1}(f(\bar{x})) = (g_i^{-1}(f_i(\bar{x})); i \in I) \), and 
\[
f_i(\bar{x}) := x_i \sum_{i \in I} p_i \left( \sum_{j \in \mathcal{I}} x_j \right).
\]
It is shown [12, Lemma 1] that any \( \beta \) such that \( \alpha^{-1}/\sigma < \beta < \alpha^\sigma \) satisfies (12) and its existence is guaranteed under the delay-independent stability condition. Define
\[
\mathcal{D}_k := \begin{cases} 
\prod_{i=1}^\infty [\alpha_i^{-\kappa_i} x_i^+, \beta_i^{\kappa_i} x_i^+], & \text{if } k \text{ odd} \\
\prod_{i=1}^\infty [\beta_i^{-\kappa_i} x_i^+, \alpha_i^{\kappa_i} x_i^+], & \text{if } k \text{ even}, 
\end{cases}
\]
(13)
Reference [12, Lemma 2] tells us that \( \hat{F}(\mathcal{D}_k) \subset \text{int}(\mathcal{D}_{k+1}) \subset \mathcal{D}_k \), where \( \text{int}(\mathcal{D}_k) \) is the interior of \( \mathcal{D}_k \) and \( \cap_{k=0}^\infty \mathcal{D}_k = \{ \bar{x}^* \} \) under the assumed stability condition.

Theorem 2: Suppose \( a_i > b_{i,\max} + 1 \) for all \( i \in I \). If the initial function \( \phi \in \mathcal{C}([-T_{\max}, 0]; \mathcal{D}_0) \), then there exist \( K := K(\mathcal{D}_0) > 0 \) and \( \gamma := \gamma(\mathcal{D}_0) > 0 \) such that 
\[
\frac{|x_i(t) - x_i^*|}{x_i^*} \leq K \cdot \exp(-\gamma \cdot t)
\]
for all \( i \in I \) and all \( t \geq 0 \). (14)

Proof: A proof is provided in [8] due to space constraints. ■

In the proof of the theorem, we show that \( K = K(\sigma)/\sigma^2 \) and \( \gamma = \psi(\sigma)/(M + T_{\max}) \) satisfy (14) with any \( \sigma \in \mathcal{I}^* := (-1, -\Delta) \), where \( K(\sigma) = \exp(\alpha^{-1}/\sigma) - 1, \psi(\sigma) = \log(-1/\sigma), \) and \( M^* := M^*(\mathcal{D}_0, \sigma) \) is a constant that increases with the initial invariance set \( \mathcal{D}_0 \) and can be numerically computed. Hence
\[
\sup_{\sigma \in \mathcal{I}^*} \frac{\psi(\sigma)}{M^*(\mathcal{D}_0, \sigma) + T_{\max}} = \sup_{\sigma \in \mathcal{I}^*} \frac{\log(-1/\sigma)}{M^*(\mathcal{D}_0, \sigma) + T_{\max}}.
\]
(15)
provides a lower bound on the convergence rate of the delay differential system in [12], which clearly depends on the initial invariance set \( \mathcal{D}_0 \) through \( M^*(\mathcal{D}_0, \sigma) \). Furthermore, as the initial invariance set \( \mathcal{D}_0 \) becomes smaller (i.e., \( \mathcal{D}_0 \downarrow \{ \bar{x}^* \} \)), the constant \( M^*(\mathcal{D}_0, \sigma) \) converges to
\[
M := \min_{\sigma \in \mathcal{I}^*} \frac{(\sigma^*)^{1+\kappa_i}(1 - \Delta^2)}{\kappa_i \cdot \Delta}.
\]
(16)
Note that (16) does not depend on the selected value of \( \sigma \in \mathcal{I}^* \). Hence, under the stability conditions in Theorem 2, as \( \bar{x}(t) \) gets close to the solution \( \bar{x}^* \), the (local) convergence rate approaches or becomes larger than \( \log(\Delta^{-1})/(M + T_{\max}) \). This clearly highlights the dependence of the lower bound on \( T_{\max} \).

Interestingly, the lower bound is monotonically increasing in users’ gain parameters \( \kappa_i, i \in \mathcal{I} \), as \( M \) is decreasing in users’ gain parameters and neither \( \Delta \) nor \( \Delta \) depends on them. It is clear that
\[
\lim_{\sigma \in \mathcal{I}^*} \frac{\log(\Delta^{-1})}{M + T_{\max}} = \frac{\log(\Delta^{-1})}{T_{\max}}.
\]
(17)
In fact, for any fixed \( \mathcal{D}_0 \), \( M^*(\mathcal{D}_0, \sigma) \) goes to zero as \( \kappa_i, i \in \mathcal{I} \), go to \( \infty \), and the limit of the lower bound in (15) is also given by \( \log(\Delta^{-1})/T_{\max} \).

Numerical examples of the bounds in (14) with the solutions of (4) are provided in [8]. They show that the accuracy of the lower bound generally improves with users’ gain parameters. Moreover, for all sufficiently large users’ gain parameters, the behavior of the solutions is similar, and the bounds in (17) provide close bounds.

We plot the lower bound \( \log(\Delta^{-1})/(M + T_{\max}) \) on the convergence rate around the equilibrium in a simple case of a single flow traversing a single resource in Fig. 3. The gain parameter \( \kappa \) is set to
0.5, the round-trip delay of the flow is 100, and the capacity \( C = 10 \). We vary the utility parameter \( a \) and the price function parameter \( b \) to see how these parameters affect the lower bound on the convergence rate. As expected, for a fixed value of \( b \), the lower bound increases with \( a \), whereas it decreases with \( b \) for a fixed value of \( a \).

**VIII. CONCLUSION**

We studied the stability of a family of rate control schemes; we first studied the delay-dependent stability in a simple network with a single flow and a single resource. We derived a sufficient condition for the asymptotic stability when there is a finite upper bound on a communication delay. Secondly, we provided a lower bound on the convergence rate of a system when the delay-independent stability conditions hold.

**APPENDIX**

Define \( y(t) := g(x(t)) = 1/x(t)^* \) to be the user’s willingness to pay at time \( t \). We rewrite the delay differential equation

\[
\frac{d}{dt}x(t) = \kappa \left( 1/x(t)^* - x(t - T)p(x(t - T)) \right)
\]

(18)

in terms of \( y(t) \) as follows:

\[
\frac{d}{dt}y(t) = \mathbf{K}(y(t))(\mathbf{F}(y(t - T)) - y(t))
\]

(19)

where \( \mathbf{K}(y) \) is defined in (11) and \( \mathbf{F}(y) = g^{-1}(y)p(g^{-1}(y)) = C^{-1}y^{1+1/a} \) is the total price of the user when its willingness to pay is seen by the resource is \( y \).

Reference [4, Corollary 3.2] states that the delay differential system in (19) will either converge to the equilibrium \( y^* = g(x^*) \) or oscillate around \( y^* \). Define

\[
m := \liminf_{t \to \infty} y(t) \quad \text{and} \quad \bar{m} := \limsup_{t \to \infty} y(t).
\]

Let \( y_t : \mathbb{R}_+ \to C([-T, 0], \mathbb{R}_+) \), where \( y_t(s) = y(t + s), s \in [-T, 0] \). Define \( \Omega(y) := \{ y \in \mathbb{R}_+ : y \neq \infty \} \). Then, for every \( \varphi \in \Omega(y) \), there exists a solution \( y(s; \varphi) : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \varphi = y(s; \varphi) \), \( s \in [-T, 0] \). Moreover, for all \( \varphi \in \Omega(y) \), we have

\[
\liminf_{t \to \infty} y(t) = m \leq y(s; \varphi) \leq \bar{m} = \limsup_{t \to \infty} y(t).
\]

In addition, there exist \( \gamma, \xi \in \Omega(y) \), such that

\[
m = y(t_1; \gamma) \quad \text{and} \quad \bar{m} = y(t_2; \xi)
\]

(20)

for some \( t_1, t_2 > 0 \). Reference [4, Lemma 3.3] tells us that if \( y(t_1) = \bar{m} \), then \( y(t_1 - T) \leq y^* \). Similarly, if \( y(t_2) = m \), then \( y(t_2 - T) \geq y^* \). Therefore, without loss of generality, we can assume that

\[
y(0; \gamma) = y(0; \xi) = y^* \quad \text{and} \quad t_i \leq T, i = 1, 2
\]

(21)

with \( y(t; \gamma) \in (y^*, \bar{m}) \) for all \( t \in (0, t_1) \) and \( y(t; \xi) \in (m, \bar{m}) \) for all \( t \in (0, t_2) \).

First, consider \( x(t) = x(t; \gamma) = g^{-1}(y(t; \gamma)) \) constructed by (18), where \( \gamma(s) \in g^{-1}(\gamma(s)) \) for all \( s \in [-T, 0] \). Let \( z(t) \) be the solution to the following initial value problem: \( z(0) = x^* \) and

\[
\frac{d}{dt}z(t) = \kappa \cdot z(t) \cdot U'(z(t)).
\]

Then, since \( x^*(t)U'(x^*) - x(t - T)p(x(t - T)) \leq x(t)U'(x(t)) \), we have \( x(t) \leq z(t) \) for all \( t \geq 0 \) [15, Theorem 5.3]. Substituting \( U'(x) = x^{1+1/a} \), after a little algebra, one can show that

\[
z(t) = (x^{1+1/a} + (1 + a)\kappa \cdot t)^{1/(1+a)}
\]

\[
z(T) = (x^{1+1/a} + (1 + a)\kappa \cdot T)^{1/(1+a)} \geq z(t_1)
\]

\[
\geq x(t_1) = \limsup_{t \to \infty} x(t; \gamma)
\]

\[
= y^{-1}(\liminf_{t \to \infty} y(t; \gamma))
\]

where the last equality follows from the fact that \( g(x) = 1/x^a \) is a decreasing function of \( x \). Therefore, \( z(T) \) provides an upper bound on \( \limsup_{t \to \infty} x(t; \gamma) \). Furthermore, this also tells us

\[
\liminf_{t \to \infty} y(t; \gamma) \geq g(z(T)) =: y^*.
\]

Let \( v(t) \) be the solution to the following initial value problem:

\[
v(0) = y^*
\]

\[
\frac{d}{dt}v(t) = \mathbf{K}(v(t))(\mathbf{F}(v(t)) - v(t)).
\]

Then, since \( \mathbf{F}(y) \geq \mathbf{F}(y(t - T)) \) for all \( t \geq 0 \), we have \( \mathbf{F}(y(t - T)) - y(t) \leq \mathbf{F}(y) - y(t) \) and, hence, \( v(t) \geq y(t) \) for all \( t \in (0, t_1) \) [15, Theorem 5.3]. Therefore, \( v(T) \geq y^* \) provides an upper bound on \( \limsup_{t \to \infty} y(t) \). Now let

\[
\Theta(T, \kappa) := \mathbf{K}(v(T))/\kappa = a \cdot T \cdot \Theta(T)^{1+1/a}.
\]

Define a map

\[
\mathbf{G}(y) := (1 - \exp(-\kappa \cdot \Theta(T, \kappa))) \mathbf{F}(y)
\]

\[
+ [y^* - (1 - \exp(-\kappa \cdot \Theta(T, \kappa))) \mathbf{F}(y^*)]
\]

\[
= (1 - \exp(-\kappa \cdot \Theta(T, \kappa))) \mathbf{F}(y)
\]

\[
+ y^* \cdot \exp(-\kappa \cdot \Theta(T, \kappa)).
\]

We first state a lemma that will be used to complete the proof of the theorem. The proof of the lemma is provided in [8] due to space constraints.

**Lemma 1**: The map \( \mathbf{G}(y) \) has a globally attracting fixed point \( y^* \) under the condition in Theorem 1.

We continue with the proof of Theorem 1. Take an arbitrary solution \( y(t) \) of (19) and let

\[
0 < m = \liminf_{t \to \infty} y(t) \leq y(t) \leq \bar{m} = \limsup_{t \to \infty} y(t).
\]

Suppose that \( \gamma \) and \( \xi \) are initial functions satisfying (20) and (21). Define

\[
\mathbf{F}_+(q(t), m; \bar{m}) := \mathbf{K}(q(t))(\mathbf{F}(m) - q(t))
\]

\[
\mathbf{F}_-(q(t), m; \bar{m}) := \mathbf{K}(q(t))(\mathbf{F}(\bar{m}) - q(t)).
\]

Let \( z(t) \) be the solution to the initial value problem of:

i) \( z(0) = y^* \);

ii) \( (d/dt)z(t) = \mathbf{F}_+(z(t), m; \bar{m}) \).

Note that \( \Psi := \Psi(m, \bar{m}) = z(t, m, \bar{m}) \geq \bar{m} \) satisfies

\[
\int_{z(t, m, \bar{m})}^{\bar{m}} \mathbf{F}_+(z; m, \bar{m}) = t_1 \leq T.
\]

*The existence of such \( \Psi \) can be shown.*
Hence, if we define \( \Phi_+ := \Phi_+ (m, \bar{m}) > y^* \) as the unique solution of 
\[
\int_{y^*}^{\bar{m}} \frac{dz}{(F_+ (z, \bar{m}) - F_+ (m, \bar{m}))} = T,
\]
then we have \( \Phi_+ \geq \Psi \geq \bar{m} \). Also, \( \Phi_+ \leq v(T) \) from the construction of \( v(T) \) because \( y^* \leq m \) (hence \( F(y^*) \geq F(m) \)). An analogous study of \( y(t, \xi) \) tells us that \( \Phi_- := \Phi_- (m, \bar{m}) \leq m \), where \( \Phi_- \) is defined by 
\[
\int_{y^*}^{\bar{m}} \frac{dz}{(F_- (z, \bar{m}) - F_- (m, \bar{m}))} = T.
\]
Note that 
\[
T = \int_{y^*}^{\bar{m}} \frac{dz}{F_+ (z, \bar{m}) - F_+ (m, \bar{m})} \geq \int_{y^*}^{\bar{m}} \frac{dz}{F(v(T)) (F(m) - z)}
\]
where the inequality follows from the fact that \( v(T) \geq \Phi_+ \) and 
\[
K(y) = a \cdot \kappa \cdot y^{1+1/s} \text{ is an increasing function of } y.
\]
Integrating this last equation, we obtain 
\[
T \geq \frac{-1}{K(v(T))} \ln \left( \frac{\Phi_+ - F(m)}{y^* - F(m)} \right).
\]
This, with \( K(v(T)) = \kappa \cdot \Theta(T, \kappa) / T \), yields 
\[
\Phi_+ \leq F(m) + \exp(-\kappa \cdot \Theta(T, \kappa))(y^* - F(m)) = F(m) + y^* \exp(-\kappa \cdot \Theta(T, \kappa)) + y^* \exp(-\kappa \cdot \Theta(T, \kappa))
\]
\[
= \max_{\bar{m} \in [m, \bar{m}]} G(y). \tag{22}
\]
Following the same steps, one can show that 
\[
\Phi_- \geq \min_{\bar{m} \in [m, \bar{m}]} G(y). \tag{23}
\]
Equations (22) and (23) imply that \([m, \bar{m}] \subset G([m, \bar{m}]) \) because \( \Phi_- \leq m \leq \bar{m} \leq \Phi_+ \). However, Lemma 1 tells us that the map \( G(y) \) has a globally attracting fixed point \( y^* \) and thus \( G([m, \bar{m}]) \subset [m, \bar{m}] \). This implies \( m = y^* = \bar{m} \), proving that the system is asymptotically globally stable.

REFERENCES


