Optical Fibers and Waveguides

17

Fiber and Integrated Optics

In this chapter we shall discuss from both a ray and wave standpoint how light can be guided along by planar and cylindrical dielectric waveguides. We shall explain why optical fibers are important and useful in optical communication systems and discuss briefly how these fibers are manufactured. Some practical details of how fibers are used and how they are integrated with other important optical components will conclude the chapter.

17.1 Introduction

We saw in the previous chapter that a Gaussian beam can propagate without beam expansion in an optical medium whose refractive index varies in an appropriate manner in the radial direction. This is a rather specific example of how an optical medium can guide light energy. However, we can discuss this phenomenon in more general terms. By specifying the spatial variation in the refractive index, and through the use of the wave equation with appropriate boundary conditions, we can show that dielectric waveguides will support certain "modes" of propagation. However, it is helpful initially to see what can be learned about this phenomenon from ray optics.

Fig. 17.1. Meridional ray entering fiber and being
guided by total internal reflection

Fig. 17.2. Skew rays

17.2 Ray Theory of a Step-index Fiber

A step-index fiber has a central core of index n_1 surrounded by cladding
of index n_2 where $n_2 < n$. When a ray of light enters such a optical
fiber, as shown in Fig. (17.1), it will be guided along the core of the fiber
if the angle of incidence between core and cladding is greater than the
critical angle. Two distinct types of ray can travel along inside the fiber
in this way: *meridional* rays travel in a plane that contains the fiber
axis, *skew* rays travel in a non-planar zig-zag path and never cross the
fiber axis, as illustrated in Fig. (17.2).

For the meridional ray in Fig. (17.1), total internal reflection (TIR)
occurs within the core if $\theta_i > \theta_c$, or $\sin \theta_i > n_2/n_1$. From Snell's law,
applied to the ray entering the fiber

$$\sin \theta = \sin \theta_o / n_1, \qquad (17.1)$$

Fig. 17.3. Geometry for focusing light into fiber.

and since $\theta + \theta_i = 90°$, the condition for total internal reflection is

$$\sin\theta_0 < \sqrt{n_1^2 - n_2^2}. \tag{17.2}$$

For most optical fibers the relative difference between the index of core and cladding is $\Delta = (n_1 - n_2)/n_1$ is small, so Eq. (17.2) can be written

$$\sin\theta_0 < \sqrt{(n_1 - n_2)(n_1 + n_2)}, \tag{17.3}$$

which since $n_1 \simeq n_2$ can be written

$$\sin\theta_o < n_1\sqrt{2\Delta}. \tag{17.4}$$

If a lens is used to focus light from a point source into a fiber, as shown in Fig. (17.3), then there is a maximum aperture size D which can be used. When the end of the fiber is a distance d from the lens, light rays outside the cross-hatched region enter the fiber at angles too great to allow total internal reflection. The quantity $\sin\theta_o \simeq D/2d$ is called the numerical aperture $N.A.$ of the system. So, from Eq. (17.4)

$$N.A. = \sin\theta_o = \sqrt{n_1^2 - n_2^2} = n_1\sqrt{2\Delta} \tag{17.5}$$

If the point source in Fig. (17.3) is placed a great distance from the lens, then $d = f$. In this case

$$N.A. = \frac{D}{2f}. \tag{17.6}$$

If the lens is chosen to be no larger than necessary, then the lens diameter will be D. The ratio f/D is a measure of the focusing/light collecting properties of the lens, called the f/number. So to match a distant source to the fiber $2N.A. = 1/(f/\text{number})$.

Fig. 17.4. *P*-wave in a dielectric slab waveguide.

17.3 The Dielectric Slab Guide

Before considering the wave theory of a cylindrical fiber in detail we can gain some insight into the propagation characteristics of meridional rays by analyzing the two-dimensional problem of a dielectric slab waveguide.

A dielectric slab of dielectric constant ϵ_1 will guide rays of light by total internal reflection provided the medium in which it is immersed has dielectric constant $\epsilon_2 < \epsilon_1$. For light rays making angle θ with the axis as shown in Fig. (17.4) total internal reflection will occur provided $\sin \theta_i > \sqrt{\frac{\epsilon_2}{\epsilon_1}}$.

Although there might appear to be an infinite number of such rays, this is not so. As the ray makes its zig-zag path down the guide the associated wavefronts must remain in phase or the wave amplitude will decay because of destructive interference. We can analyze the slab as a Fabry-Perot resonator. Only those rays that satisfy the condition for maximum stored energy in the slab will propagate. This corresponds to the upward and downward components of the travelling wave constructively interfering. The component of the propagation constant perpendicular to the fiber axis for the ray shown in Fig. (17.1) is $k_1 \cos \theta_i$, so for constructive interference

$$4dk_1 \cos \theta_i = 2m\pi,$$

which gives the condition

$$\cos \theta_i = \frac{m\lambda_0}{4n_1 d}. \tag{17.7}$$

For simplicity we have neglected the phase shift that occurs when the wave totally internally reflects at the upper and lower core/cladding boundaries.

A given value of m in Eq. (17.1) corresponds to a phase shift of $2m\pi$

per round trip between the upper and lower boundaries of the slab: Such a *mode* will not propagate if θ is greater than the critical angle. In other words the cut-off condition for the mth mode is

$$\frac{d}{\lambda_o} = \frac{m}{4\sqrt{n_1^2 - n_2^2}} \qquad (17.8)$$

This condition is identical for waves polarized in the xz plane (P-waves) or polarized in the y direction (S-waves). Only the lowest mode, $m = 0$, has no cutoff frequency. A dielectric waveguide that is designed to allow the propagation of only the lowest mode is called a *single-mode* waveguide. The thickness of the guiding layer needs to be quite small to accomplish this. For example: at an operating wavelength of 1.55 μm and indices $n_1 = 1.5$, $n_2 = 1.485$ the maximum guide width that will support single-mode operation is 3.66 μm. Such waveguides are important because, in a simple sense, there is only one possible ray path in the guide. A short pulse of light injected into such a guide will emerge as a single pulse at its far end. In a multimode guide, a single pulse can travel along more than one path, and can emerge as multiple pulses. This is undesirable in a digital optical communication link. We can estimate when this would become a problem by using Eq. (17.7)

For the $m = 0$ mode in a guide of total length L the path length is L. For the $m = 1$ mode, the total path length along the guide is $L + \Delta L$, where

$$L + \Delta L = L/(1 - (\lambda_o/4n_1d)^2)^{\frac{1}{2}} \qquad (17.8a)$$

A single, short optical pulse injected simultaneously into these two modes will emerge from the fiber as two pulses separated in time by an interval $\Delta\tau$ where

$$\Delta\tau = \frac{n_1\Delta L}{c_o} = \frac{n_1L}{c_o(1 - (\lambda_o/4n_1d)^2)^{\frac{1}{2}}} - \frac{n_1L}{c_o} \qquad (17.8b)$$

This effect is called *group delay*. For example, if $L = 1$km, $n_1 = 1.5$, and $d/\lambda_o = 5$ then $\Delta\tau = 2.78$ns. Clearly, communication rates in excess of about 200MHz would be impossible in this case. As the number of modes increases, the bandwidth of the system decreases further.

17.4 The Goos-Hänchen Shift

We can examine the discussion of the previous section in a little more detail by including the phase shift that occurs when a wave undergoes total internal reflection. For a *S*-wave striking the core/cladding bound-

ary the reflection coefficient is

$$\rho = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, \tag{17.a}$$

where we have made use of the effective impedance for S-waves at angle of incidence, θ_1, or refraction, θ_2, at the boundary, namely

$$Z'_{\text{core}} = \frac{Z_0}{n_1 \cos\theta_1},$$

$$Z'_{\text{cladding}} = \frac{Z_0}{n_2 \cos\theta_2} \tag{17.b}$$

For angles of incidence greater than the critical angle, $\cos\theta_2$ becomes imaginary and can be written as

$$\cos\theta_2 = i\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1 - 1\right]^{1/2}. \tag{17.c}$$

In terms of the critical angle, θ_c,

$$\cos\theta_2 = i\frac{n_1}{n_2}(\sin^2\theta_1 - \sin^2\theta_c)^{1/2}$$

$$= i\frac{n_1}{n_2}X \tag{17.d}$$

If we write the phase shift on reflection for the S-wave as ϕ_s, then from Eqs. (17.a) and (17.d)

$$\rho = |\rho|e^{i\phi_s} = \frac{\cos\theta_1 - iX}{\cos\theta_1 + iX}. \tag{17.e}$$

It is easy to show that $|\rho| = 1$: all the wave energy is reflected as a result of total internal reflection. Therefore, if we write

$$\cos\theta_1 - iX = e^{i\phi_s/2},$$

then

$$\tan(\phi_s/2) = -\frac{(\sin^2\theta_1 - \sin^2\theta_c)^{1/2}}{\cos\theta_1}. \tag{17.f}$$

For P-waves the phase shift on reflection, ϕ_p, satisfies

$$\tan(\phi_p/2) = -\frac{(\sin^2\theta_1 - \sin^2\theta_c)^{1/2}}{\left(\frac{n_2}{n_1}\right)\cos\theta_1} \tag{17.g}$$

In practical optical waveguides usually $n_1 \simeq n_2$ so $\phi_s \simeq \phi_p = \phi$. We can put Eq. (17.f) in a more convenient form by using angles measured with respect to the fiber axis. We define

$$\theta_z = \pi/2 - \theta_1$$

$$\theta_a = \pi/2 - \theta_c$$

In a fiber with $n_1 \simeq n_2$, both θ_z and θ_a will be small angles. With these

definitions

$$\phi = -2\frac{(\theta_a^2 - \theta_z^2)^{1/2}}{\theta_z}, \tag{17.h}$$

If we represent the electric field of the guided wave in the slab as

$$E_i(z) = E_1 e^{i\beta_1 z}$$

then after total internal reflection the wave becomes

$$E_r(z) = E_1 e^{(\beta_1 z + \phi(\theta_1))}$$

where $\beta_1 = k_1 \sin\theta_1$ is the propagation constant parallel to the z axis of the slab. Because this wave is laterally confined within the core we can include the effects of diffraction by considering the wave as a group of rays near the value θ_1. The phase shift $\phi(\beta_1)$ can be expanded as a Taylor series for propagation constants near β_1:

$$\phi(\beta) = \phi(\beta_1) + (\beta - \beta_1)\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_1} \tag{17.i}$$

Therefore, the reflected wave can be written as

$$E_r(z) = E_1 \exp\left[i(\beta z + \phi(\beta_1) + (\beta - \beta_1)\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_1}\right]$$
$$= E_0 \exp\left[i(\beta z + \beta\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_1}\right] \tag{17.j}$$

where all the terms independent of angle have been incorporated into the new complex amplitude E_0. The additional phase factor $\beta\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_1}$ can be interpreted as a shift, Z_s, in the axial position of the wave as it reflects at the boundary, as shown in Fig. (17.5).

This shift in axial position Z_s is called the Goos-Hänchen shift [A.L. Snyder and J.D. Love, "The Goos-Hänchen shift," Appl. Opt., 15, 236-8, 1976]

$$Z_s = -\left(\frac{\partial\phi}{\partial\beta}\right)_{\beta_1} \tag{17.k}$$

which can be evaluated as

$$Z_s = -\left(\frac{\partial\phi}{\partial\theta_z}\frac{\partial\theta_z}{\partial\beta}\right)_{\beta_1} \tag{17.l}$$

where

$$\frac{\partial\theta_z}{\partial\beta} = -\frac{1}{k\sin\theta_z} \tag{17.m}$$

By differentiating Eq. (17.h) and assuming that the angle θ_z and θ_a are

small, we get

$$Z_s \simeq \frac{\lambda}{\pi n_1 \theta_z} \frac{1}{(\theta_a^2 - \theta_z^2)}^{1/2} \qquad (17.n)$$

This shift in axial position makes the distance travelled by the ray in propagating a distance ℓ along the fiber shorter than it would be without the shift. However, the shift is very small unless the ray angle is close to the critical angle, in which case the evanescent portion of the associated wave penetrates very far into the cladding.

17.5 Wave Theory of the Dielectric Slab Guide

When a given mode propagates in a slab, light energy is guided along in the slab medium, or core. There are, however, finite field amplitudes that decay exponentially into the external medium. The propagating waves in this case, because they are guided along by the slab, but are not totally confined within it, are often called *surface waves*. This distinguishes them from the types of wave that propagate inside, and are totally confined by, hollow conducting structures – such as rectangular or cylindrical microwave waveguides. We can find the electromagnetic field profiles in the one-dimensional slab guide shown in Fig. (17.4) by using Maxwell's equations. We look for a propagating solution with fields that vary like $e^{i\omega t - \gamma z}$, where γ is the propagation constant.

There are both P and S-wave solutions to the wave equation. Clearly, as can be seen from Fig. (17.4), the P wave has an E_z component of its electric field so it could also be called a transverse magnetic field (TM) wave. Similarly the S-wave is a transverse electric field (TE) wave. From the curl equations, written in the form

$$\text{curl } \mathbf{E} = -i\omega\mu\mathbf{H}$$
$$\text{curl } \mathbf{H} = i\omega\epsilon\mathbf{E} \qquad (17.9)$$

the following equations result

$$\frac{\partial E_z}{\partial y} + \gamma E_y = -i\omega\mu H_x \qquad (17.10a)$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = i\omega\epsilon E_y \qquad (17.10b)$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -i\omega\mu H_y \qquad (17.10c)$$

$$\frac{\partial H_z}{\partial y} + \gamma H_y = i\omega\epsilon E_x \qquad (17.10d)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z \qquad (17.10e)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\epsilon E_z \qquad (17.10f)$$

An expression for H_x can be found by eliminating E_y from Eqs. (17.10a) and (17.10b), or vice-versa. E_x and H_y can be found from Eqs. (17.10c) and (17.10d).

$$E_x = \frac{-1}{\gamma^2 + k^2}(\gamma\frac{\partial E_z}{\partial x} + i\omega\mu\frac{\partial H_z}{\partial y}) \qquad (17.11)$$

$$H_y = \frac{-1}{\gamma^2 + k^2}(i\omega\epsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y}) \qquad (17.12)$$

$$E_y = \frac{1}{\gamma^2 + k^2}(-\gamma\frac{\partial E_z}{\partial y} + i\omega\mu\frac{\partial H_z}{\partial x}) \qquad (17.13)$$

$$H_x = \frac{1}{\gamma^2 + k^2}(i\omega\epsilon\frac{\partial E_z}{\partial y} + \gamma\frac{\partial H_z}{\partial x}) \qquad (17.14)$$

where $k = \omega\sqrt{\mu\epsilon}$.

From the Helmholtz equation (Eq. (16.4))

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0; \quad \nabla^2\mathbf{H} + k^2\mathbf{H} = 0 \qquad (17.15)$$

Since $\partial^2/\partial z^2$ is equivalent to multiplication by $-\gamma^2$ Eq. (17.15) can be written as

$$\frac{\partial^2\mathbf{E}}{\partial x^2} + \frac{\partial^2\mathbf{E}}{\partial y^2} = -(\gamma^2 + k^2)\mathbf{E} \qquad (17.16)$$

$$\frac{\partial^2\mathbf{H}}{\partial x^2} + \frac{\partial^2\mathbf{H}}{\partial y^2} = -(\gamma^2 + k^2)\mathbf{H} \qquad (17.17)$$

17.6 P-waves in the Slab Guide

For P-waves in the slab guide $E_y = 0, \partial/\partial y \equiv 0$, so from Eq. (17.12)

$$H_y = -\frac{-i\omega\epsilon}{\gamma^2 + k^2}\frac{\partial E_z}{\partial x} \qquad (17.18)$$

and from Eq. (17.16)

$$\frac{\partial^2 E_z}{\partial x^2} = -(\gamma^2 + k^2)E_z \qquad (17.19)$$

The solutions to Eq. (17.19) above are sine or cosine functions, or exponentials. If we wish the waves to be confined to the slab we must choose the solution in medium 2 where the field amplitudes decay exponentially in the positive (or negative) x direction. So, for example,

using Eqs. (17.18) and (17.19) the x dependence of E_z and H_y will be

$$E_{z_2} = Be^{-\alpha_x x}; \quad H_{y_2} = -\frac{i\omega\epsilon_2}{\alpha_x}Be^{-\alpha_x x} \quad x \geq d \qquad (17.20)$$

$$E_{z_2} = Be^{\alpha_x}x; \quad H_{y_2} = \frac{i\omega\epsilon_2}{\alpha_x}Be^{\alpha_x x} \quad x \leq -d \qquad (17.21)$$

Inside the slab the fields are

$$E_{z_1} = A\sin k_x x; \quad H_{y_1} = \frac{-i\omega\epsilon_1}{k_x}A\cos k_x x \qquad (17.22)$$

or

$$E_{z_1} = A\cos k_x x; \quad H_{y_1} = \frac{i\omega\epsilon_1}{k_x}A\sin k_x x \qquad (17.23)$$

where we have taken note that in medium 2

$$\alpha_x^2 = -(\gamma^2 + k_2^2) \qquad (17.24)$$

where $\alpha_x > 0$, and in medium 1

$$k_x^2 = \gamma^2 + k_1^2 \qquad (17.25)$$

where $k_x > 0$.

A propagating wave exists in the slab provided γ is imaginary. If γ were to be real then the $e^{-\gamma z}$ variation of the fields would represent an attenuated, non-propagating *evanescent* wave. For γ to be imaginary, from Eq. (17.25), $k_1^2 > k_x^2$; therefore, $k_x^2 < \omega^2\mu_0\epsilon_1$.

To find the relation between α_x and k_x we use the boundary condition that at $x = d$ the E_z and H_y fields (which are the tangential fields at the boundary between the 2 media) must be continuous. Therefore,

$$\left(\frac{E_{z_1}}{H_{y_1}}\right)_{x=d} = \left(\frac{E_{z_2}}{H_{y_2}}\right)_{x=d} \qquad (17.26)$$

and for the modes with odd E_z field symmetry in the slab, (the sine solution above)

$$\frac{-k_x}{i\omega\epsilon_1}\tan k_x d = \frac{-\alpha_x}{i\omega\epsilon_2}, \qquad (17.27)$$

so

$$\alpha_x = \frac{\epsilon_2}{\epsilon_1}k_x\tan k_x d. \qquad (17.28)$$

It is straightforward to show that for the even symmetry E_z-field solution in the slab, (the cosine solution above)

$$\alpha_x = -\frac{\epsilon_2}{\epsilon_1}k_x\cot k_x d. \qquad (17.29)$$

To determine the propagation constant γ for the odd-symmetry modes we must solve the simultaneous Eqs. (17.24), (17.25), and (17.28). Elimination of γ from Eqs. (17.24) and (17.25) gives

$$\alpha_x = \sqrt{k_1^2 - k_2^2 - k_x^2} = \sqrt{\omega^2\mu_0(\epsilon_1 - \epsilon_2) - k_x^2} \qquad (17.30)$$

Fig. 17.5. Graphical solution of Eq. (17.31) for odd-symmetry
P-wave modes in a slab waveguide. Fig. (17.9) is similar.

We have assumed that $\mu_1 = \mu_2 = \mu_0$, as most optically transparent materials have relative permeability very close to unity.

From Eqs. (17.28) and (17.30):

$$\tan k_x d = \frac{\epsilon_1}{\epsilon_2} \sqrt{\frac{\omega^2 \mu_0 (\epsilon_1 - \epsilon_2)}{k_x^2} - 1} \qquad (17.31)$$

The solution to this equation can be illustrated graphically, as shown in Fig. (17.5). Whatever the value of ω there is always at least one solution to this equation. For small values of $k_x d$ Eq. (17.28) gives

$$\alpha_x = \frac{\epsilon_2}{\epsilon_1} k_x^2 d \qquad (17.32)$$

and from Eq. (17.30)

$$k_x^4 d^4 + (\frac{\epsilon_1}{\epsilon_2})^2 k_x^2 d^2 = (\frac{\epsilon_1}{\epsilon_2})^2 \omega^2 \mu_0 (\epsilon_1 - \epsilon_2) d^2 \qquad (17.33)$$

neglecting the term in $k_x^4 d^4$ gives

$$k_x \simeq \omega^2 \mu_0 (\epsilon_1 - \epsilon_2) = k_1 - k_2 \qquad (17.34)$$

and from Eq. (17.25), if $k_x \to 0$

$$\gamma \to \pm i k_1 \qquad (17.35)$$

Therefore as either $\epsilon_1 \to \epsilon_2$, or d becomes very small, α_x also decreases and energy extends far into medium 2. The propagation constant of the wave approaches the value it would have in medium 2 alone.

The solution of Eq. (17.30) with the smallest value of $k_x d$ is called the *fundamental* P-wave mode of the guide. This mode can propagate at any frequency – it has no cut-off frequency. However, this is not true for the even-symmetry E_z field solution. In this case, from Eqs. (17.29)

Fig. 17.6. Graphical Solution of Eq. (17.36) for even-symmetry
P-wave modes in a slab waveguide. Fig. (17.9) is similar.

and (17.30)

$$\cot k_x d = -\frac{\epsilon_1}{\epsilon_2}\sqrt{\frac{\omega^2\mu_0(\epsilon_1 - \epsilon_2)}{k_x^2} - 1} \qquad (17.36)$$

The graphical solution of this equation is given in Fig. (17.6). In order
for this equation to have a solution, $k_x d$ must be greater than $\pi/2$. This
cannot be so while at the same time $\omega \to 0$.

17.7 Dispersion Curves and Field Distribution

For a propagating mode to exist, from either Eq. (17.31) or (17.36)

$$\omega^2\mu_0(\epsilon_1 - \epsilon_2) \geq k_x^2. \qquad (17.37)$$

In this case Eq. (17.25) implies

$$\omega^2\mu_0(\epsilon_1 - \epsilon_2) \geq \gamma^2 + \omega^2\mu_0\epsilon_1, \qquad (17.38)$$

which writing $\gamma = \pm i\beta$ implies $\beta \geq \omega^2\mu_0\epsilon_2$. By solving Eq. (17.31) or
Eq. (17.36) for various values of ω we can plot the variation of β with ω
for the S and P wave modes. These curves are called dispersion curves.

From Eq. (17.25)

$$\beta^2 = k_1^2 - k_x^2 = \omega^2\mu_0\epsilon_1 - k_x^2 \qquad (17.39)$$

By solving for k_x from either Eq. (17.31) for the odd symmetry E_z
solution or Eq. (17.36) for the even symmetry solution, we can produce
the dispersion curves shown in Fig. (17.7). The curve labelled $m = 0$ is
the fundamental mode, the curves labelled $m = 1, 2, 3$ etc. are alternate
even, odd, even symmetry E_z field higher modes. Some of their E_z field
distributions within the slab are shown in Fig. (17.8). The E_x and H_y

Fig. 17.7.

Fig. 17.8. E_z field variation for P-wave odd and even-symmetry
modes in a slab waveguide. Fig. (17.10) is similar.

components of the P-wave solution are easily found from Eqs. (17.11)
and (17.12). For the fundamental mode,

$$E_x(\text{in medium 1}) = \frac{-\gamma}{\gamma^2 + k_1^2}\frac{\partial E_z}{\partial x} = -\frac{A\gamma}{k_x}\cos k_x x$$

$$E_x(\text{in medium 2}) = -\frac{\gamma}{\gamma^2 + k_2^2}\frac{\partial E_z}{\partial x} = -\frac{B\gamma}{\alpha_x}e^{-\alpha_x x} \tag{17.40}$$

$$H_y(\text{in medium 1}) = -\frac{i\omega\epsilon_1}{\gamma^2 + k_1^2}\frac{\partial E_z}{\partial z} = -\frac{iA\omega\epsilon_1}{k_x}\cos k_x x$$

$$H_y(\text{in medium 2}) = -\frac{i\omega\epsilon_2}{\gamma^2 + k_2^2}\frac{\partial E_z}{\partial x} = \frac{-iB\omega\epsilon_2}{\alpha_x}e^{-\alpha_x x} \tag{17.41}$$

The tangential component of magnetic field must be continuous at, say,
$x = d$. So from Eq. (17.41)

$$\frac{A}{B} = \frac{\epsilon_2 k_x e^{-\alpha_x d}}{\epsilon_1 \alpha_x \cos k_x d} \tag{17.42}$$

which from Eq. (17.28) gives

$$\frac{A}{B} = \frac{e^{-\alpha_x d}}{\sin k_x d} \tag{17.43}$$

The actual value of A (or B) is determined from the energy flux in the dielectric slab. The transverse intensity distribution in the guide is

$$I = E_x H_y,$$

which from Eqs. (17.40) and (17.41) gives

$$I_1 \text{ (in medium 1)} = \frac{i\gamma w \epsilon_1 A^2}{k_x^2} \cos^2 k_x x$$

$$I_2 \text{ (in medium 2)} = \frac{i\gamma w \epsilon_2 \beta^2}{\alpha_x^2} e^{-2\alpha_x x}$$

The total energy flux per unit width is

$$\phi = 2 \int_0^d I_1 dx + 2 \int_d^\infty I_2 dx$$

17.8 S-waves in the Slab Guide

For S-waves in the slab guide the electric field points only in the y direction $\frac{\partial}{\partial y} \equiv 0$, and there are H_x and H_z components of magnetic field. In this case we start from Eq. (17.17) and write

$$\frac{\partial^2 H_z}{\partial x^2} = -(\gamma^2 + k^2) H z \tag{17.44}$$

For the modes with odd symmetry of the H_z field

$$H_{z_1} = C \sin k_x x \qquad |x| \le d, \tag{17.45}$$

$$H_{x_1} = \frac{C\gamma k_x}{\gamma^2 + k_1^2} \cos k_x x = \frac{C\gamma}{k_x} \cos k_x x, \tag{17.46}$$

$$E_{y_1} = \frac{iC\omega\mu_1}{\gamma^2 + k_1^2} k_x \cos k_x x = \frac{iC\omega\mu_1}{k_x} \cos k_x x, \tag{17.47}$$

$$H_{z_2} = D e^{-\alpha_x x}, \qquad x \ge d, \tag{17.48}$$

$$H_{x_2} = -\frac{C\gamma\alpha_x}{\gamma^2 + k_2^2} e^{-\alpha_x x} = \frac{C\gamma}{\alpha_x} e^{-\alpha_x x}, \tag{17.49}$$

$$E_{y_2} = -\frac{iD\omega\mu_2\alpha_x}{\gamma^2 + k_2^2} e^{-\alpha_x x} = \frac{iD\omega\mu_2}{\alpha_x} e^{-\alpha_x x}. \tag{17.50}$$

It is straightforward to show, in a similar manner to before, that

$$\alpha_x = \frac{\mu_2}{\mu_1} k_x \tan k_x d \tag{17.51}$$

Fig. 17.9.

and

$$\tan k_{xd} = \frac{\mu_1}{\mu_2}\sqrt{\frac{\omega^2\mu_0(\epsilon_1 - \epsilon_2)}{k_x^2} - 1}. \tag{17.52}$$

Even as $\omega \to 0$ there is always at least one solution to the equation. This is the *fundamental*, $m = 0$, S-wave mode of the guide. The graphical solution of Eq. (17.52) in Fig. (17.9) illustrates this.

Modes which have even symmetry of the H_z field have field components

$$H_{z_1} = C\cos k_x x, \qquad | x | \le d, \tag{17.53}$$

$$H_{x_1} = -\frac{C\gamma}{k_x}\sin k_x x, \tag{17.54}$$

$$E_{y_1} = -\frac{iC\omega\mu_1}{k_x}\sin k_x x, \tag{17.55}$$

$$H_{z_2} = De^{-\alpha_x x}, \qquad x \ge d, \tag{17.56}$$

$$H_{x_2} = \frac{C\gamma}{\alpha_x}e^{-\alpha_x}, \tag{17.57}$$

$$E_{y_2} = \frac{iD\omega\mu_2}{\alpha_x}e^{-\alpha_x x}. \tag{17.58}$$

The defining equation for the modes is

$$\alpha_x = -\frac{\mu_2}{\mu_1}k_x\cot k_x d \tag{17.59}$$

Because most optical materials have $\mu \simeq \mu_0$ the cutoff frequencies of the S-wave modes above the fundamental are slightly lower, by a factor $\frac{\epsilon_2}{\epsilon_1}$, then the corresponding P-wave modes. Fig. (17.10) shows the schematic variation of some of the field components of the S-wave modes in a dielectric slab.

Fig. 17.10.

Fig. 17.11.

17.9 Practical Slab Guide Geometries

Slab guides that are infinite in one-dimension are convenient mathematically but do not exist in practice. Practical slab guides utilize index changes or discontinuities in both the x and y directions (for a guide intended to propagate waves in the z direction). Some examples are shown in Fig. (17.11). Such structures can be fabricated by methods similar to those used in integrated circuit manufacture. In each case the fields outside the guiding layer decay exponentially. The fields are broadly similar to those discussed previously for the infinite slab guide. However, the modes that exist are no longer pure P or S-wave modes, they possess all components of **E** and **H**, although they can still be predominantly P or S in character in some circumstances.

Slab guide geometries exist in most semiconductor diode lasers and in many LEDs. Consequently it is possible to integrate the light source with a slab waveguide and then incorporate active components such as modulators, couplers and switches into the structure. The resultant *in-*

tegrated optic package can serve as a complete transmitting module with full potential for amplitude or phase modulation, or multiplexing. We shall meet examples of these devices later, in Chapter 19. In order to realize a practical guided-wave optical communication link over long distances slab waveguides are not practical. The information-carrying light beam must be transmitted in an optical fiber – a cylindrical dielectric waveguide.

17.10 Cylindrical Dielectric Waveguides

The ray description of the slab waveguide is useful for describing the path of a meridional ray down a fiber. However, to obtain a more detailed understanding of the propagation characteristics of the fiber we again use Maxwell's equations. In cylindrical coordinates the curl equation (Eq. 17.9) is:

$$\text{curl } \mathbf{E} = [\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}]\hat{\mathbf{r}} + [\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}]\hat{\phi}$$

$$+ [\frac{1}{r}\frac{\partial(rE_\phi)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \phi}]\hat{\mathbf{z}} = i\omega\mu\mathbf{H} \qquad (17.60)$$

with a similar equation for curl $\mathbf{H}$. E_r, E_ϕ, E_z, respectively, are the radial, azimuthal, and axial components of the electric field, with corresponding unit vectors $\hat{\mathbf{r}}, \hat{\phi}$, and $\hat{\mathbf{z}}$. The Helmholtz equations (Eq. 17.5) become:

$$\frac{\partial^2 \mathbf{E}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{E}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathbf{E}}{\partial \phi^2} = -(\gamma^2 + k^2)\mathbf{E}$$

$$\frac{\partial^2 \mathbf{H}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{H}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathbf{H}}{\partial \phi^2} = -(\gamma^2 + k^2)\mathbf{H} \qquad (17.61)$$

where, as before, $k^2 = \omega^2\mu\epsilon$.

We start by examining the solution of Eq. (17.61) for the axial component of the field. We look for a separable solution of the form

$$E_z(r, \phi) = R(r)\Phi(\phi). \qquad (17.62)$$

The z dependence of E_z is $e^{i\omega t - \gamma z}$, as before, but this factor will be omitted for simplicity. Substituting from (17.62) into (17.61) gives

$$r^2\frac{R''}{R} + \frac{rR'}{R} + k_c^2 r^2 = -\frac{\Phi''}{\Phi} \qquad (17.63)$$

where primes denote differentiation and we have written $k_c^2 = \gamma^2 + k^2$.

The only way for Eq. (17.63) to be satisfied, since the L.H.S. is only a function of r, and the R.H.S. is only a function of ϕ, is for each side

Fig. 17.12.

to be independently equal to a constant. Therefore,

$$r^2\frac{R''}{R} + \frac{rR'}{R} + k_c^2 r^2 = \nu^2 \tag{17.64}$$

which is usually written as

$$R'' + \frac{1}{r}\frac{\partial R}{\partial r} + (k_c^2 - \frac{\nu^2}{r^2})R = 0 \tag{17.65}$$

and

$$-\frac{\Phi''}{\Phi} = \nu^2 \tag{17.66}$$

The general solution of Eq. (17.66) is

$$\Phi(\phi) = A\sin\nu\phi + B\cos\nu\phi. \tag{17.67}$$

ν must be a positive or negative integer, or zero so that the function $\Phi(\phi)$ is single valued: that is $\Phi(\phi + 2\pi) = \Phi(\phi)$.

Eq. (17.65) is called Bessel's equation. Its solutions are Bessel functions, which are the cylindrical geometry equivalents of the real or complex exponential solutions we encountered in the slab guide.

If $k_c^2 > 0$ the general solution of Eq. (17.65) is

$$R(r) = CJ_\nu(k_c r) + DY_\nu(k_c r) \tag{17.68}$$

where $J_\nu(k_c r)$ is the Bessel function of the first kind and $Y_\nu(k_c v)$ is the Bessel function of the second kind.* These are oscillatory functions of r, as shown in Fig. (17.12).

If $k_c^2 < 0$ the general solution of Eq. (17.68) is

$$R(r) = CI_\nu(|k_c|r) + D'K_\nu(|k_c|r) \tag{17.69}$$

where $I(|k_c|r), K_\nu(|k_c|r)$ are the modified Bessel functions of the first

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* The Y_ν Bessel function of the second kind is also written as N_ν by some authors.

Fig. 17.13.

Fig. 17.14.

and second kinds, respectively. These are monotonically increasing and decreasing functions of r as shown in Fig. (17.13).

Before proceeding further we can make several physical observations to help simplify our analysis.

(i) We cannot use the equations above to solve for wave propagation in a graded index fiber - a fiber where the dielectric constant varies smoothly in the radial direction. In such fibers ϵ is a function of r (the quadratic index fiber is an example) and the wave equation analysis is much more complex. References XX and XXX can be consulted for details.

(ii) We can use these equations to find the field components in a step-index fiber, whose radial refractive index profile is as shown in Fig. (17.14). We find a solution for the core, where the dielectric constant $\epsilon = n_1^2$ and match this solution to an appropriate solution for the cladding, where $\epsilon_2 = n_2^2$.

(iii) In the core we expect an oscillatory form of the axial field to be ap-

propriate (as it was for the slab guide). Since the function $Y_\nu(k_c r)$ blows up as $r \to 0$ we reject it as physically unreasonable. Only the $J_\nu(k_c r)$ variation needs to be retained.

(iv) We are interested in fibers with $n_1 > n_2$ that guide energy. The fields in the cladding must decay as we go further away from the case. Since $I_\nu(k_c r)$ blows up as $r \to \infty$ we reject it as physically unreasonable. Only the $K_\nu(k_c r)$ variation needs to be retained.

(v)   In the core

$$k_{c_1}^2 > 0 \text{ implies } \gamma^2 + k_1^2 > 0$$

and for a propagating solution $\gamma \equiv i\beta$; therefore

$$k_1^2 > \beta^2$$

while in the cladding $k_{c_2}^2 < 0$, and therefore $\gamma^2 + k_2^2 < 0$ which implies $k_2^2 < \beta^2$. In summary

$$k_2 < \beta < k_1$$

The closer β approaches k_1, the more fields are confined to the core of the fiber, while for β approaching k_2, they penetrate far into the cladding. When β becomes equal to k_2 the fields are no longer guided by the core: they penetrate completely into the cladding.

(vi) If $k_c^2 > 0$, then the axial fields in the cladding would no longer be described by the K_v Bessel function, instead they would be described by the J_v Bessel function. These oscillatory fields no longer decay evanescently in the cladding and the wave is no longer guided by the core. We can take $k_c^2 = 0$ as a demarcation point between a wave being guided, or leaking from the core.

If we take the $\sin \nu\phi$ azimuthal variation for the axial electric field then in the core we have,

$$E_{z_1} = \; AJ_\nu(k_{c_1} r) \sin \nu\phi \quad (r \le a),$$

and in the cladding

$$E_{z_2} = \; CK_\nu(|\,k_{c_2}\,|\,r) \sin \nu\phi \quad (r \ge a), \tag{17.70}$$

where the subscripts 1 and 2 indicate core and cladding fields, respectively. These fields must be continuous at the boundary between core and cladding so

$$E_{z_1}(r = a) = E_{z_2}(r = a)$$

Far into the cladding the evanescent nature of the fields becomes apparent, since for large values of r

$$K_\nu(|k_{c_2}|r) \propto \frac{\exp(-|k_{c_2}|r)}{(|k_{c_2}|r)^{1/2}} \tag{17.71}$$

The corresponding magnetic fields are

$$H_z = BJ_\nu(k_{c_1})\cos\nu\phi \quad (r \leq a)$$

$$H_z = DK_\nu(k_{c_2})\cos\nu\phi \quad (r \geq a) \tag{17.72}$$

In this case, where we have assumed the $\sin(\nu\phi)$ for the electric field, we must have the $\cos(\nu\phi)$ variation for the H_z field to allow matching of the tangential fields (which include both z and ϕ components) at the core/cladding boundary. We must allow for the possibility of both axial electric and magnetic field components in the fiber: so-called *hybrid* modes. However, if $\nu = 0$ then Eq. (17.70) shows that $E_z = 0$: this is a TE mode. If we had chosen the $\cos(\nu\phi)$ variation for the axial electric field, then with $\nu = 0$ we would have a TM mode. Thus, TE and TM modes only exist for $\nu = 0$. To find the relationship between the field components orthogonal to the fiber axis and the E_z and H_z components we work from Eq. (17.60) and the corresponding equation for curl $\mathbf{H}$. Some tedious algebra will yield the results

$$E_r = -\frac{i}{k_c^2}\left(\frac{\gamma}{i}\frac{E_z}{\partial r} + \omega\mu\frac{1}{r}\frac{\partial H}{\partial\phi}\right) \tag{17.73}$$

$$E_\phi = -\frac{i}{k_c^2}\left(\frac{\gamma}{i}\frac{1}{r}\frac{\partial E_z}{\partial\phi} - \omega\mu\frac{\partial H_z}{\partial r}\right) \tag{17.74}$$

$$H_r = -\frac{i}{k_c^2}\left(\frac{\gamma}{i}\frac{\partial H_z}{\partial r} - \omega\epsilon\frac{1}{r}\frac{\partial E_z}{\partial\phi}\right) \tag{17.75}$$

$$H_\phi = -\frac{i}{k_c^2}\left(\frac{\gamma}{i}\frac{1}{r}\frac{\partial H_z}{\partial\phi} + \omega\epsilon\frac{\partial E_z}{\partial r}\right) \tag{17.76}$$

Note that for a TE mode $(E_z = 0), E_r \sim \partial H_z/\partial\phi$ so that if H_z has a variation $\sim \cos(\nu\phi)$, where ν is a constant, then E_r must vary as $\sin(\nu\phi)$. For a TM mode $(H_z = 0)$, $H_r \sim \partial E_z/\partial\phi$ with similar consequences. The full variation of the field in core and cladding is:

17.10.1 Core:

$$E_z = AJ_\nu(\frac{ur}{a})\sin\nu\phi \tag{17.77}$$

$$E_r = [-\frac{A\gamma}{(u/a)}J_\nu'(\frac{ur}{a}) + B\frac{i\omega\mu_0}{(u/a)^2}\frac{\nu}{r}J_\nu(\frac{ur}{a})]\sin\nu\phi \tag{17.78}$$

$$E_\phi = [-\frac{A\gamma}{(u/a)^2}\frac{\nu}{r}J_\nu(\frac{ur}{a}) + B\frac{i\omega\mu_0}{(u/a)}J_\nu'(\frac{ur}{a})]\cos\nu\phi \tag{17.79}$$

$$H_z = BJ_\nu(\frac{ur}{a})\cos\nu\phi \tag{17.80}$$

$$H_r = [A\frac{i\omega\epsilon_1}{(u/a)^2}\frac{\nu}{r}J_\nu(\frac{ur}{a}) - \frac{B\gamma}{(u/a)}J_\nu'(\frac{ur}{a})]\cos\nu\phi \tag{17.81}$$

$$H_\phi = [-A\frac{i\omega\epsilon_1}{(u/a)}J'_\nu(\frac{ur}{a}) + \frac{B\gamma}{(u/a)^2}\frac{\nu}{r}J_\nu(\frac{ur}{a})]\sin\nu\phi \qquad (17.82)$$

where $u = k_{c_1}a = (k_1^2 + \gamma^2)^{1/2}a$ is called the *normalized transverse phase constant* in the core region.

17.10.2 Cladding:

$$E_z = CK_\nu(\frac{wr}{a})\sin\nu\phi \qquad (17.83)$$

$$E_r = [\frac{C\gamma}{(w/a)}K'_\nu\left(\frac{wr}{a}\right) - D\frac{i\omega\mu_0}{(w/a)^2}\frac{\nu}{r}K_\nu(\frac{wr}{a})]\sin\nu\phi \qquad (17.84)$$

$$E_\phi = [\frac{C\gamma}{(w/a)^2}\frac{\nu}{r}K_\nu(\frac{wr}{a}) - D\frac{i\omega\mu_0}{(w/a)}K'_\nu(\frac{wr}{a})]\cos\nu\phi \qquad (17.85)$$

$$H_z = DK_\nu(\frac{wr}{a})\cos\nu\phi \qquad (17.86)$$

$$H_r = [-C\frac{i\omega\epsilon_2}{(w/a)^2}\frac{\nu}{r}K_\nu(\frac{wr}{a}) + \frac{D\gamma}{(w/a)}K'_\nu(\frac{wr}{a})]\cos\nu\phi \qquad (17.87)$$

$$H_\phi = [C\frac{i\omega\epsilon_2}{(w/a)}K'_\nu(\frac{wr}{a}) - \frac{D\gamma}{(w/a)^2}\frac{\nu}{r}K_\nu(\frac{wr}{a})]\sin\nu\phi \qquad (17.88)$$

where $w = | k_c | a = [-(k_2^2 + \gamma^2)]^{1/2}a$ is called the *normalized transverse attenuation coefficient in the cladding.* If $w = 0$, this determines the *cut-off* condition for the particular field distribution.

17.10.3 Boundary Conditions:

To determine the relationship between the constants A, B, C and D in Eqs. (17.77-17.88) we need to match appropriate tangential and radial components of the fields at the boundary. If we neglect magnetic behavior and put $\mu_1 = \mu_2 = \mu_0$, for the tangential fields:

$$E_{z_1} = E_{z_2}$$
$$E_{\phi_1} = E_{\phi_2}$$
$$H_{z_1} = H_{z_2}$$
$$H_{\phi_1} = H_{\phi_2} \qquad (17.89)$$

and for the radial fields

$$\epsilon_1 E_{r_1} = \epsilon_2 E_{r_2}$$
$$H_{r_1} = H_{r_2} \qquad (17.90)$$

where all fields are evaluated at $r = a$, and the subscripts 1 and 2 refer to core and cladding fields, respectively. It can be shown that use of these relationship in conjunction with Eqs. (17.77-17.88) leads to the

condition

$$\Big[\frac{J_\nu'(u)}{uJ_\nu(u)} + \frac{K_\nu'(w)}{wK_\nu(w)}\Big]\Big[\frac{\epsilon_1}{\epsilon_2}\frac{J_\nu'(u)}{uJ_\nu(u)} + \frac{K_\nu'(w)}{wK_\nu(w)}\Big]$$

$$= \nu^2\Big(\frac{1}{u^2} + \frac{1}{w^2}\Big)\Big(\frac{\epsilon_2}{\epsilon_2}\frac{1}{u^2} + \frac{1}{w^2}\Big) \qquad (17.91)$$

Furthermore,

$$u^2 + w^2 = (k_1^2 + \gamma^2)a^2 - (k_2^2 + \gamma^2)a^2 = (k_1^2 - k_2^2)a^2 \qquad (17.92)$$

which, since $k_1 = 2\pi n_1/\lambda_0, k_2 = 2\pi n_2/\lambda_0$, gives

$$u^2 + w^2 = 2k_1^2 a^2 \Delta. \qquad (17.93)$$

Δ is the relative refractive index difference between core and cladding

$$\Delta = \frac{(n_1^2 - n_2^2)}{2n_1^2} \simeq \frac{(n_1 - n_2)}{n_1}. \qquad (17.94)$$

A fiber is frequently characterized by its V number where

$$V = (u^2 + w^2)^{1/2}, \qquad (17.95)$$

which can be written as

$$V = \frac{2\pi a}{\lambda_0}(n_1^2 - n_2^2)^{1/2} = \frac{2\pi a}{\lambda_0}(N.A.)$$

$$= kn_1 a(2\Delta)^{1/2} \qquad (17.96)$$

V is also called the normalized frequency.

Eqs. (17.91) and (17.93) can be solved for u and w to determine the specific field variations in the fiber.

17.11 Modes and Field Patterns

The modes in a cylindrical dielectric waveguide are described as either TE, TM or hybrid. The hybrid modes fall into 2 categories, HE modes where the axial electric field E_z is significant compared to E_r or E_ϕ, and EH modes where the axial magnetic field H_z is significant compared to H_r or H_ϕ. These modes are further characterized by two integers ν and ℓ. A mode characterized as HE_{12}, for example, has azimuthal field variations like $\sin\phi$ or $\cos\phi$, and since $\ell = 2$, has 2 radial oscillations of the axial electric field within the core. For the TE and TM modes $\nu = 0$. For the E_z field variation given by Eq. (17.77)

$$E_z = AJ_\nu\Big(\frac{ur}{a}\Big)\sin\nu\phi \qquad (17.97)$$

the number of radial oscillations of the axial electric field is determined by how many times $J_\nu(x)$ has crossed the axis as x varies from 0 to ur/a. Each time the Bessel function oscillates from positive to negative,

or vice-versa, there is a new value of (ur/a) where the oscillatory function can merge smoothly with the radially decaying field distribution in the cladding.

The field distributions are easiest to write down for the $TE_{0\ell}$ and $TN_{0\ell}$ modes. Setting $\nu = 0$ in Eqs. (17.77)-(17.52) we get the fields of the $TE_{0\ell}$ mode, for example, in the core:

$$E_z = 0$$

$$E_r = 0$$

$$E_\phi = -B\frac{i\omega\mu_0}{(u/a)}J_1(\frac{ur}{a}) \qquad (17.98)$$

$$H_z = BJ_0(\frac{ur}{a})$$

$$H_r = \frac{B\gamma}{(u/a)}J_1(\frac{ur}{a})$$

We have made use of the relation $J_0'(x) = -J_1(x)$. The $TE_{0\ell}$ modes correspond to choosing the constants $A = C = 0$ in Eqs. (17.77) - (17.88).

We could have chosen the $\cos\nu\phi$ variation for the E_z field in which we would have got the fields of the $TM_{0\ell}$ mode by setting $\nu = 0$, which in this case implies $B = D = 0$.

The transverse fields of the mode give use to a characteristic Poynting vector or intensity distribution, which can be called the *mode* pattern. For the $TE_{0\ell}$ mode this is

$$S(r) = E_\phi H_r \propto J_1^2(\frac{ur}{a}) \qquad (17.99)$$

Because $J_1(0) = 0$, this is a donut-shaped intensity distribution as shown in Fig. (17.). The intensity distribution for the $TM_{0\ell}$ mode looks the same. We shall see shortly that these are *not* the *fundamental* modes of the step-index fiber. These modes cannot propagate at a given frequency for an arbitrarily small core diameter—they have a *cut-off* frequency. the only mode that has no cut-off frequency is the HE_{11} mode. To discuss this further, and in order to find a way of grouping modes with similar mode patterns together we need to work in the *weakly-guiding approximation*.

17.12 The Weakly-Guiding Approximation

In the weakly-guiding approximation we neglect the difference in refrac-

Fig. 17.15.

Fig. 17.16.

tive index between core and cladding and set $\epsilon_1 = \epsilon_2$. In this case the boundary condition relation, Eq. (17.91) becomes

$$\frac{J'_\nu(u)}{uJ_\nu(w)} + \frac{k'_\nu(w)}{wk_\nu(w)} = \pm\nu\left(\frac{1}{u^2} + \frac{1}{w^2}\right) \qquad (17.100)$$

For the $TE_{0\ell}$ and $TM_{0\ell}$ mode this gives

$$\frac{J_1(u)}{uJ_0(u)} + \frac{k_1(w)}{wK_0(w)} = 0 \qquad (17.101)$$

where we have made use of the relation $K'_0(w) = -K_1(w)$.

By the use of a series of relationships between Bessel functions, given in Appendix   , when the positive sign is taken on the right-hand side of Eq. (17.100), the following equation results

$$\frac{J_{\nu+1}(u)}{uJ_\nu(n)} + \frac{K_{\nu+1}(w)}{wK_\nu(w)} = 0 \qquad (17.102)$$

This equation describes the EH modes. When the negative sign is taken

in Eq. (17.100) the following equation for the HE modes is obtained

$$\frac{J_{\nu-1}(u)}{uJ_\nu(u)} - \frac{K_{\nu-1}(w)}{wK_\nu(w)} = 0 \qquad (17.103)$$

By the use of the Bessel function relations given in Appendix , Eq. (17.109) can be rearranged into the form

$$\frac{J_{\nu-1}(u)}{uJ_{\nu-2}(u)} + \frac{K_{\nu-1}(w)}{wK_{\nu-2}(w)} \qquad (17.104)$$

If we introduce a new integer m, which takes the values

$$m = \begin{cases} 1 & \text{for TE and TM modes,} \\ \nu+1 & \text{for EH modes,} \\ \nu-1 & \text{for HE modes,} \end{cases} \qquad (17.105)$$

then a unified boundary condition relation results that includes Eqs. (17.101), (17.102) and (17.104). It is

$$\frac{uJ_{m-1}(u)}{J_m(u)} = -\frac{wK_{m-1}(w)}{K_m(w)} \qquad (17.106)$$

If $m-1 < 0$, as would be the case for $HE_{\nu\ell}$ modes with $\nu = 1$, then the Bessel function relations $J_{-\nu} = (-1)^\nu J_\nu$ and $K_{-\nu} = K_\nu$ would be useful in evaluating Eq. (17.106).

If the V-number of the fiber is known, given by Eqs. (17.95) and (17.96), then the values of u and w that satisfy Eq. (17.106) can be determined. Once these parameters are known the fields in the fiber can be calculated from Eqs. (17.77) - (17.88).

17.13 Mode Patterns

In the weakly-guiding approximation the Poynting vector variation within the fiber is again calculated from

$$S = E_t H_t, \qquad (17.107)$$

where the transverse fields in the fiber are

$$\mathbf{E}_t = E_r \hat{\mathbf{r}} + E_\phi \hat{\phi}$$
$$\mathbf{H}_t = H_r \hat{\mathbf{r}} + H_\phi \hat{\phi}. \qquad (17.108)$$

The transverse intensity distribution has a characteristic mode pattern. Similarities between these mode patterns allow groups of TE, TM and hybrid modes to be grouped together as LP (linearly polarized) modes, as shown in Fig. (17.). Each LP mode is characterized by the two integers m and ℓ: ℓ describes the number of radial oscillations of the field that occur in the core; m describes the number of variations of E_x or E_y that occur as ϕ varies from 0 to 2π.

The schematic Poynting vector distributions for the LP modes shown in Fig. (17.) are the patterns appropriate to either $(E_x H_y)$ or $(E_y H_x)$. For example, for the $TE_{0\ell}$ mode discussed earlier

$$E_x = -E_\phi \sin \phi$$
$$E_y = E_\phi \cos \phi$$
$$H_x = H_r \cos \phi$$
$$H_y = H_r \sin \phi \qquad (17.109)$$

and the x-polarized intensity distribution can be written as

$$I_x \propto J_1^2 \left(\frac{ur}{a}\right) \sin^2 \phi \qquad (17.110)$$

For $LP_{m\ell}$ modes with $\ell \geq 1$ there are four constituent transverse or hybrid modes. This results because each hybrid mode consists of two degenerate modes, corresponding to the choice of the $\cos(\nu\phi)$ or $\sin(\nu\phi)$ azimuthal variation of the fields. The TM or TE modes (for which $\nu = 0$) are singly degenerate and axisymmetric.

17.14 Cutoff Frequencies

The field in the cladding decays with increasing distance from the core so long as $w > 0$. However, as $w \to 0$, the propagation constant β approaches the value k_2 appropriate to the cladding and the wave is no longer confined. The condition $w = 0$ determines the cutoff condition for the mode in question, which from Eq. (17.106) becomes

$$\frac{u J_{m-1}(u)}{J_m(u)} = 0 \qquad (17.111)$$

The unique property of the HE_{11} mode can be illustrated with this equation. This is the LP_{01} mode with $m = 0$.

Only the $J_0(u)$ Bessel function is non-zero at $u = 0$, so Eq. (17.111) has the solution $u = 0$ in this case. Therefore, the HE_{11} mode has no cutoff frequency—or viewed another way, it can propagate in a fiber of small core radius a. Since,

$$u = \left(k_1^2 + \gamma^2\right)^{1/2} a, \qquad (17.112)$$

and at cutoff

$$u = V = k_0 n_1 a (2\Delta)^{1/2} \qquad (17.113)$$

$u = 0$ implies ω (the wave frequency) $= 0$. All other modes in the fiber have a cutoff frequency, below which they cannot propagate. The cutoff frequencies are determined by the zeros $J_{m-1,\ell}$ of the Bessel function

Table 17. . *Low order zeros of Bessel functions.*

J_0	J_1	J_2	J_3
$J_{01} = 2.405$	$J_{11} = 3.832$	$J_{21} = 5.136$	$J_{31} = 6.380$
$J_{02} = 5.520$	$J_{12} = 7.106$	$J_{22} = 8.417$	$J_{32} = 9.761$
$J_{03} = 8.654$	$J_{13} = 10.173$	$J_{23} = 11.620$	$J_{33} = 13.015$

J_{m-1}, a few values of which are given in Table (17.). For the $TM_{0\ell}$ and $TE_{0\ell}$ modes the cutoff condition is

$$\frac{J_0(u)}{J_1(u)} = 0 \tag{17.114}$$

so $u = j_{0\ell}$ at cutoff.

For the $HE_{\nu\ell}$ hybrid mode with $\nu \geq 2$ the cutoff condition is

$$\frac{J_\nu(u)}{J_{\nu+1}(u)} = 0 \tag{17.115}$$

so $u = j_{\nu,\ell}$ at cutoff.

For the $EH_{\nu\ell}$ hybrid modes the cutoff condition is

$$\frac{J_{\nu-2}(u)}{J_{\nu-1}(u)} = 0, \tag{17.116}$$

so $u = j_{\nu-2,\ell}$ at cutoff.

The smallest $j_{m-1,\ell}$ is j_{01}, so the LP_{11} has the lowest non-zero cutoff frequency. The cutoff wavelength for this mode is found from

$$k_0 n_1 a (2\Delta)^{1/2} = 2.405 \tag{17.117}$$

so the cutoff wavelength is

$$\lambda_c = \frac{2\pi n_1 a (2\Delta)^{1/2}}{2.405} \tag{17.118}$$

If a given mode in the fiber is not below cutoff then its normalized phase constant u has to take values that are bounded by values at which the $J_{m-1}(u)$ and $J_m(u)$ Bessel functions in Eq. (17.106) are equal to zero. This is shown in Fig. (17.) [Ref.: D. Gloge, "Weakly Guiding Fibers," Appl. Opt., 10, 2252-2258, 1971.]

The value of u must be between the zeros of $J_{m-1}(u)$ and $J_m(u)$, that is for modes with $\ell = 0$, or 1.

$$J_{m-1,\ell} < u < J_{m,\ell}$$

If this condition is not met then it is not possible to match the J_2 Bessel function to the decaying K_ν Bessel function at the core boundary. At

the lower limit of the permitted values of u in Fig. (17.), the cutoff frequency, $V = u$, while at the upper limit $V \to \infty$.

17.14.1 Example:

For a fiber with $a = 3\mu$m, $n_1 = 1.5$ and $\Delta = 0.002$ the cutoff wavelength is $\lambda_c = 744nm$. This fiber would support only a single mode for $\lambda > \lambda_c$.

Alternatively, to design a fiber for single frequency operation at $\lambda_0 = 1.55\mu$m we want

$$a < \frac{2.405\lambda_0}{2\pi n_1 (2\Delta)^{1/2}} \tag{17.119}$$

For $n_1 = 1.5$ and $\Delta = 0.002$ this requires $a < 6.254\mu$m.

A fiber satisfying this condition would be a single-mode fiber for 1.55μ m, it would not necessarily be a single-mode fiber for shorter wavelengths. Single-mode fibers for use in optical communication applications at the popular wavelengths of 810nm, 1.3μm or 1.55μm all have small core diameters, although for mechanical strength and ease of handling their overall cladding diameter is typically 125μm. The fiber will have additional protective layers of polymer, fiber, and even metal armoring (for rugged applications such as underground or overhead communications links, or for laying under the ocean). Fig. (17.) shows the construction of a typical multi fiber cable of this kind.

17.15 Multimode Fibers

If the core of a step-index fiber has a larger radius relative to the operating wavelength than the value given by Eq. (17.119) then more than one mode can propagate in the fiber. The number of modes that can propagate in a given fiber can be determined from its V-number. For $V < 2.405$ the fiber will allow only the HE_{11} mode to propagate. For relatively small V-numbers the number of modes that are permitted to propagate can be determined by counting the number of Bessel function zeros $j_{\nu\ell} < V$. For large V-numbers a good estimate of the number M of permitted modes is $M = v^2/2$. [Ref. D. Gloge].

In a multimode fiber most of the modes will be well above cutoff and well-confined to the core. The fraction of the mode power that propagates in the cladding decreases markedly for these modes as is shown in Fig. (17.). [Fig. 5.11 from Gowar or Gloge Fig. 5.]

17.16 Fabrication of Optical Fibers

There are two principal groups of fabrication techniques for the manufacture of low attenuation optical fibers: methods involving the direct drawing of fiber from molten glass in a crucible; and various vapor deposition methods in which a "preform" with appropriate graded index properties is produced and then drawn into a fiber. All these fabrication procedures rely on the use of highly pure starting material whether this be a borosilicate glass or silica, with added dopants. The brief discussion here can be supplemented with information from more specialized texts [P.K. Cheo, "Fiber Optics and Optoelectronics," Second Edition, Prentice Hall, Englewood Cliffs, New Jersey, 1990, J.E. Midwinter, "Optical Fibers for Transmission, Wiley, New York, 1979; S.E. Miller and A.G. Chynoweth, Eds., Optical Fiber Telecommunications, Academic, New York, 1979; P.C. Schultz, "Progress in optical waveguide process and materials," Appl. Opt., 18, 3684-3693, 1979.]

Fig. (17.) shows schematically the double crucible method for the continuous production of clad fibers. The starting material can be fed in in powder form, or as preformed glass rods. This technique cannot be used for high melting point glasses, such as fused silica. The core and cladding materials can be doped to give the core a higher index than the cladding, for example, by doping with tantalum and sodium, respectively. By varying the drawing temperature the degree of interdiffusion between core and cladding can be controlled to give a weakly-guiding "step-index" or "graded-index" fiber. The addition of thallium to the core, followed by its diffusion into the cladding during the drawing process also gives a graded index fiber. The double crucible method allows the continuous production of fiber with attenuation as low as 5dB/km.

The production of single-mode versus multimode fiber is determined by the size of the drawing aperture from the inner crucible, the temperature of the molten glass, and the rate at which the fiber is drawn.

For the production of the lowest attenuation fibers based on silica doped with germanium, fluorine or phosphorus, particularly for the manufacture of single-mode fiber, a doped preform is fabricated by some form of vapor deposition. The three principal vapor deposition processes are shown schematically in Fig. 17.   [D.B. Keck, "Single-mode fibers out perform multimode cables," IEEE Spectrum, March 1983, pp. 30-37].

In the outside chemical vapor deposition (OCVD) process materials such as $SiCl_4$, $GeCl_4$, $POCl_3$, BBr_3 and, BCl_3 are oxidized in a burner and deposited on a rotating silica mandrel as a "soot". By varying the vapor

to be decomposed, layers of soot particles can be built up, layer upon layer, to generate the final refractive index profile desired. The mandrel, with soot over layers is then melted into a preform in a furnace. The preform can then be heated and drawn into a fiber.

In the inside chemical vapor deposition (ICVD) process the chemical vapors are decomposed inside a silica tube by heating them externally or with an internal RF-generated plasma. The tube with its inner layers of soot can then be collapsed and inserted into a second silica (cladding) tube to serve as the core of the final drawn fiber.

The vapor-phase axial deposition (VAD) process is a variant on the OCVD process in which the core and cladding glasses are deposited simultaneously onto the end of the preform, which is continuously drawn up into an electric furnace. The perform is rotated as deposition proceeds and as it is drawn up into the furnace its diameter decreases. The final preform is then taken away and heated for fiber drawing [Ref. P.K. Cheo].

17.17 Dispersion in Optical Fibers

The use of optical fibers to transmit data at ever increasing rates, which in the laboratory have already exceeded 10 Gb/s over kilometer long fibers, has required careful consideration of the effects of dispersion in the fiber. In its simplest sense dispersion is the spreading in width of a narrow optical pulse as it travels along the fiber. This spreading is greatest in multimode fibers, where in a ray description each different mode corresponds to a different length ray trajectory inside the fiber. The different pulse transit times along these different ray paths leads to broadening or splitting, of a narrow optical pulse. This is called *modal* dispersion. For example, for a step-index fiber the transit time difference between a ray that propagates along the axis and one that enters just within the N.A. is easily shown to be

$$\Delta T = \frac{n_1^2}{n_2} \frac{\ell}{c_0} \Delta$$
$$\simeq \frac{n_1 \ell \Delta}{c_0} \tag{17.120}$$

For $\Delta = 0.002, n_1 = 1.5$, this modal dispersion is 10ns/km.

In single-mode fibers modal dispersion does not exist but dispersion still occurs. This residual dispersion comes from *material* dispersion $dn/d\lambda$, the variation of the refractive index of core and cladding with

wavelength (or frequency), and *waveguide* dispersion the variation of the waveguide propagation constant β with frequency. A detailed analysis of these effects has been given by Gowar, but a few general observations can be made.

17.17.1 Material Dispersion

The transit time of a pulse in a single-mode fiber depends, among other factors, on its group velocity,* defined as

$$v_g = \frac{d\omega}{d\beta} \tag{17.121}$$

In the simplest case of a pulse totally confined to the core

$$\beta = k_1 = \frac{\omega n_1}{c0}$$

It is easy to show that

$$v_g = \frac{c_0}{(n_1 - \lambda \frac{dn_1}{d\lambda})} \tag{17.121}$$

We can define a *group refractive index*, n_{g_1}, where

$$v_g = \frac{c_0}{n_{g_1}} \tag{17.122}$$

If a narrow optical pulse is travelling in the core, then its different spectral components will travel with different velocities. The spectral components covers a wavelength range $\Delta\lambda$ that depends on the monochromaticity of the source. For a light emitting diode (LED) $\Delta\lambda$ is typically $\simeq$ 40 nm at 800nm; for a semiconductor laser the equivalent spectral frequency width will be in the range 1-100MHz. The transit time of the pulse will be

$$T = \frac{\ell}{v_{g_1}} = \left(n_1 - \lambda \frac{dn_1}{d\lambda}\right)\frac{\ell}{c_0}, \tag{17.123}$$

and a narrow pulse (impulse) will speed to a width

$$\Delta T = -\frac{\ell}{c_0}\lambda^2 \frac{d^2 n}{d\lambda^2}\left(\frac{\Delta\lambda}{\lambda}\right). \tag{17.124}$$

This pulse broadening is minimized, for a source with specific spectral purity $\Delta\lambda/\lambda$ when the quantity

$$|Y_m| = |\lambda^2 \frac{d^2 n}{d\lambda^2}| \tag{17.125}$$

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* The group velocity is a measure of the velocity with which the centroid of a pulse, which itself contains a range of frequencies, will travel. It is therefore, an important parameter in characterizing the performance of an optical communication system that transmits information as a train of light pulses.

is minimized. For doped silica fibers $Y_m = 0$ for wavelengths in the range $1.22\mu m < \lambda < 1.37\mu m$, the minimum being at $\lambda_{md} = 1.276\mu$m for a pure silica fiber. The spreading of the pulse will not actually be zero, even at this operating wavelength, because of the spectral width of the source. The residual pulse broadening that occurs for a source with $\lambda_0 = \lambda_{md}$ can be shown to be

$$\frac{\Delta T}{\ell} = (-)\frac{\lambda_0^3}{8c_0}\left(\frac{d^3n}{d\lambda^3}\right)_{\lambda_0}\left(\frac{\Delta\lambda}{\lambda}\right)^2$$

For pure silica at $\lambda_0 = 1.276\mu$m, $\Delta T/\ell = 2\times 10^{-11}(\Delta\lambda/\lambda)^2 s/m$.

For a high data rate optical communication link using a single mode semiconductor laser running at 10 Gbit/s the effective laser linewidth is broadened to $\simeq$ 20 GHz, so $\Delta\lambda/\lambda = 0.00009$ and $\Delta T/\ell \simeq 1.5 \times 10^{-16} s/km$.

Operation at a wavelength of 1.55μm, where the attenuation of silica is a minimum, but $Y_m = -0.01$, leads to more broadening. Even so, this broadening is extremely small. For the optical link described above running at 1.55μm, $\Delta T/\ell \simeq 3.5ps/km$.

17.17.2 Waveguide Dispersion

When a mode propagates, its propagation constant is $\gamma = i\beta$ where β simultaneously satisfies both the equations

$$\beta = [k_1^2 - (u/a)^2]^{1/2}$$

and

$$\beta = [(w/a)^2 + k_2^2]^{1/2}$$

At cutoff $w = 0$, $\beta = k_2$ and the wave propagates completely in the cladding. As the frequency ω of the wave rises far above the cut-off value $\beta \to k_1$. The variation of β with ω is called a waveguide dispersion curve. At any point on a dispersion curve the *phase* velocity of the mode is ω/β, while the *group* velocity is $d\omega/d\beta$. The dispersion curve for a particular mode is bounded by the β values appropriate for core and cladding as shown in Fig. (17.). The so called *light lines* have ω/β slopes c_1 and c_2, where c_1 and c_2 are the velocity of light in core and cladding, respectively.

17.18 Coupling Optical Sources and Detectors to Fibers

In order to couple light from a source into a fiber efficiently the light

Fig. 17.17.

from the source must, in the first instance, be injected within the numerical apeature. In the case of multimode fibers this is generally sufficient to ensure efficient collection and efficient guiding of the injected source light. However, when step-index single-mode fibers are comenced the injected light must additionally be matched as closely as possible to the lateral intensity profile of the mode that will propagate in the fiber. This cannot be done exactly because the injected light will typically have a Gaussian radial intensity profile, whereas the single mode lateral intensity profile is a zero-order Bessel function with exponentially decaying wings. For optimum coupling a circularly symmetric Gaussian laser beam should be focused so that its beam waist is on, and parallel to, the cleaned end face of the fiber, as shown in fig. (17.17).

Optimum coupling of the Gaussian beam into the HE_{11} mode occurs when $w_0 = 0.61a$ and the v-number of the fiber is $V = 3.8$,[1] in otherwords close to the value where the next highest mode HE_{12} will propagate. In this situation, if Fresnel losses at the fiber entrance face are ignored, the maximum coupling efficiency is 85%. The coupling efficiency from a Gaussian beam to a single mode or multimode fiber is somewhat complex to calculate in the general case[1]. To optimize coupling the input Gaussian must be matched to the equivalent Gaussian profile of the propagating mode. A useful approximation that can be used for a step-node fiber of core radius a is

$$w_{\text{mode}} = \frac{a}{sqrt2 \ln v}$$

If the end of the fiber is illuminated with too large a focused spot significant energy enters higher order mode, which leak out of the cladding. This is really demonstrated in the laboratory. If a visible laser is focused into a fiber the first few centimeters of the fiber will glow brightly

as non-propagating modes are excited and then energy leaks from the cladding.

In the arrangement shown in Fig. (17.) the lens can very conveniently be replaced by a GRIN lens, which allow butt coupling of source to GRIN. The parameters of the GRIN lens must be chosen comnensurate with the input laser spot size to give an appropriate beam waist at the output face of the GRIN.

If the laser source to be coupled to the fiber is not generating a circularly symmetric Gaussian beam, then anamorphic focussing optics must be used to render the beam circular and focus it onto the end face of the fiber. A typical semiconductor laser may generate an *elliptical* Gaussian beam, one in which the beam is characterized by different spot sizes in two orthogonal directions. In this case a combination of a cylindrical and spherical lens can be used to give improved focusing. If the emitting facet of the semiconductor laser is sufficiently small then a microfabricated spherical lens can be used to sample source and fiber as we have seen in Chapter 13.

The problem of coupling an optical detector to a fiber efficiently is an easy one. Most optical detectors are of larger area than the model spot leaving the fiber so direct butt coupling from fiber to detector is straightforward.

17.18.1 Fiber Connectors

If two identical fibers are to be joined together this can be done with either mechanical coaxial connectors or through a *fusion* spheric. In either case the flat, orthogonally cleared end of the two fibers must be brought together so that their two faces are in very close proximity, parallel, and the fibers are coaxial, as shown in Fib. (17.18). Any tilt or offset of the fibers will reduce the coupling efficency.

Precision, mechanical connectors are able to couple two fibers with typical losses of 0.1-0.2dB. Generally, the connector will contain a small quantity of index-matching fluid that is squared in the small interface gap between the 2 fibers.

In a fusion sphere the two fiber ends in Fig. (17. a) are brought into contact and external heat quickly applied to melt and fuse the two fibers together. Such spheres will have an insertion loss of 0.1dB.

17.19 Problems for Chapter 17

Fig. 17.18. Alignment of fibers for connection.

(1) The Goos-Hänchen shift can be regarded as equivalent to the penetration of the totally internally reflected ray into the cladding so that the "effective" thickness of the slabs is $2d_{\text{eff}}$, as shown below. Derive an expression for d_{eff} using the parameters in Eq. (17.h).

(2) Prove that for an optical fiber being operated at a center wavelength λ_0 with a source of spectral width $\Delta\lambda$ the minimum pulse broadening as a function of distance caused by material dispersion is

$$\frac{\Delta T}{\ell} = (-)\frac{\lambda_0^3}{8c_0}\left(\frac{d^3n}{d\lambda^3}\right)_{\lambda_0}\left(\frac{\Delta\lambda}{\lambda}\right)^2$$

(3) A fiber with a parabolic index profile has

$$n(r) = n_0 - \frac{1}{2}n_2 r^2$$

with $n_0 = 1.5$ and $n_2 = 10^9 m^{-2}$. Design a single lens focusing arrangement that will optimally couple a laser with $\lambda_0 = 180nm$ and $w_0 = 2mm$ into the fiber. How would you modify the focusing arrangement if the laser emitted a Gaussian beam with $w_0 = 5\mu m$ from the laser facet?

(4) Repeat question (3) but design a focusing system using a parabolic index GRIN lens that is 20mm long. What parabolic profile parameter n_2 for the GRIN lens is needed?

REFERENCES

Snyder and Love.