Introduction to Cryptology

Lecture 24
Announcements

• HW 10 due on 5/10
• Scholarly Paper EC due on 5/10
Agenda

• Last time:
  – Public Key Encryption (11.3)
  – El Gamal Encryption (11.4)
  – RSA Encryption and Weaknesses (11.5)

• This time:
  – Digital Signatures Definitions (12.2-12.3)
  – RSA Signatures (12.4)
  – Dlog-based signatures (12.5)
CONSTRUCTION 11.29
Let GenRSA be as before, and let \( \ell \) be a function with \( \ell(n) \leq 2n - 4 \) for all \( n \). Define a public-key encryption scheme as follows:

- **Gen:** on input \( 1^n \), run GenRSA\((1^n) \) to obtain \((N, e, d)\). Output the public key \( pk = (N, e) \), and the private key \( sk = (N, d) \).

- **Enc:** on input a public key \( pk = (N, e) \) and a message \( m \in \{0, 1\}^{\|N\| - \ell(n) - 2} \), choose a random string \( r \leftarrow \{0, 1\}^{\ell(n)} \) and interpret \( \hat{m} := 1\|r\|m \) as an element of \( \mathbb{Z}_N^* \). Output the ciphertext \( c := [\hat{m}^e \mod N] \).

- **Dec:** on input a private key \( sk = (N, d) \) and a ciphertext \( c \in \mathbb{Z}_N^* \), compute \( \hat{m} := [c^d \mod N] \), and output the \( \|N\| - \ell(n) - 2 \) least-significant bits of \( \hat{m} \).

The padded RSA encryption scheme.
Digital Signatures Definition

A digital signature scheme consists of three ppt algorithms $(Gen, Sign, Vrfy)$ such that:

1. The key-generation algorithm $Gen$ takes as input a security parameter $1^n$ and outputs a pair of keys $(pk, sk)$. We assume that $pk, sk$ each have length at least $n$, and that $n$ can be determined from $pk$ or $sk$.

2. The signing algorithm $Sign$ takes as input a private key $sk$ and a message $m$ from some message space (that may depend on $pk$). It outputs a signature $\sigma$, and we write this as $\sigma \leftarrow Sign_{sk}(m)$.

3. The deterministic verification algorithm $Vrfy$ takes as input a public key $pk$, a message $m$, and a signature $\sigma$. It outputs a bit $b$, with $b = 1$ meaning valid and $b = 0$ meaning invalid. We write this as $b := Vrfy_{pk}(m, \sigma)$.

Correctness: It is required that except with negligible probability over $(pk, sk)$ output by $Gen(1^n)$, it holds that $Vrfy_{pk}(m, Sign_{sk}(m)) = 1$ for every message $m$. 
Digital Signatures Definition: Security

Experiment $\text{SigForge}_{A,\Pi}(n)$:

1. $\text{Gen}(1^n)$ is run to obtain keys $(pk, sk)$.
2. Adversary $A$ is given $pk$ and access to an oracle $\text{Sign}_{sk}(\cdot)$. The adversary then outputs $(m, \sigma)$. Let $Q$ denote the set of all queries that $A$ asked to its oracle.
3. $A$ succeeds if and only if
   1. $\text{Vrfy}_{pk}(m, \sigma) = 1$
   2. $m \notin Q$.

In this case the output of the experiment is defined to be 1.

Definition: A signature scheme $\Pi = (\text{Gen}, \text{Sign}, \text{Vrfy})$ is existentially unforgeable under an adaptive chosen-message attack, if for all ppt adversaries $A$, there is a negligible function $\text{neg}$ such that:

$$\Pr[\text{SigForge}_{A,\Pi}(n) = 1] \leq \text{neg}(n).$$
RSA Signatures

CONSTRUCTION 12.5

Let GenRSA be as in the text. Define a signature scheme as follows:

- **Gen**: on input $1^n$ run GenRSA($1^n$) to obtain $(N, e, d)$. The public key is $(N, e)$ and the private key is $(N, d)$.

- **Sign**: on input a private key $sk = (N, d)$ and a message $m \in \mathbb{Z}_N^*$, compute the signature

\[ \sigma := [m^d \mod N]. \]

- **Vrfy**: on input a public key $pk = (N, e)$, a message $m \in \mathbb{Z}_N^*$, and a signature $\sigma \in \mathbb{Z}_N^*$, output 1 if and only if

\[ m \overset{?}{=} [\sigma^e \mod N]. \]

The plain RSA signature scheme.
Attacks

No message attack:

Choose $s \in \mathbb{Z}_N^*$, compute $s^e$.
Output $(m = s^e, \sigma = s)$ as the forgery.
For forging a signature on an arbitrary message:

To forge a signature on message \( m \), choose arbitrary \( m_1, m_2 \neq 1 \) such that \( m = m_1 \cdot m_2 \).
Query oracle for \((m_1, \sigma_1), (m_2, \sigma_2)\).
Output \((m, \sigma)\), where \( \sigma = \sigma_1 \cdot \sigma_2 \).
CONSTRUCTION 12.6

Let GenRSA be as in the previous sections, and construct a signature scheme as follows:

- **Gen**: on input $1^n$, run GenRSA($1^n$) to compute $(N, e, d)$. The public key is $\langle N, e \rangle$ and the private key is $\langle N, d \rangle$.
  
  As part of key generation, a function $H : \{0, 1\}^* \rightarrow \mathbb{Z}_N^*$ is specified, but we leave this implicit.

- **Sign**: on input a private key $\langle N, d \rangle$ and a message $m \in \{0, 1\}^*$, compute
  
  $$\sigma := [H(m)^d \mod N].$$

- **Vrfy**: on input a public key $\langle N, e \rangle$, a message $m$, and a signature $\sigma$, output 1 if and only if $\sigma^e \overset{?}{=} H(m) \mod N$.

The RSA-FDH signature scheme.
Random Oracles

• Assume certain hash functions behave exactly like a random oracle.
• The “oracle” is a box that takes a binary string as input and returns a binary string as output.
• The internal workings of the box are unknown.
• All parties (honest parties and adversary) have access to the box.
• The box is consistent.
• Oracle implements a random function by choosing values of $H(x)$ “on the fly.”
Principles of RO Model

1. If $x$ has not been queried to $H$, then the value of $H(x)$ is uniform.
2. If $A$ queries $x$ to $H$, the reduction can see this query and learn $x$.
3. The reduction can set the value of $H(x)$ to a value of its choice, as long as this value is correctly distributed, i.e., uniform.
Security of RSA-FDH

Theorem: If the RSA problem is hard relative to $\text{GenRSA}$ and $H$ is modeled as a random oracle, then the construction above is secure.
• Uses an instantiation of RSA-FDH for signing.
• SHA-1 should not be used “off-the-shelf” as an instantiation of $H$ because output length is too small and so practical short-message attacks apply.
• In PKCS #1 v2.1, $H$ is constructed via repeated application of an underlying cryptographic hash function.