

ENEE244-020x

Digital Logic Design

Lecture 4

Announcements

- HW 1 assigned (see course webpage). Due date: Sept. 11 @ 11:59pm. **Extended**
- HW 2 assigned (see course webpage). Due date: Sept. 22 @ 11:59pm.
- Makeup Quiz Policy
 - If you have an excused absence, you may email your TA ahead of time (and cc me) for permission to take the quiz in a different section
 - After Wednesday, no makeup quizzes allowed, even with an excused absence

Agenda

- Last time:
 - Binary Codes (1.7)
 - Error Detecting and Correcting Codes
- This time:
 - Boolean Algebra postulates and theorems (2.3-2.4)

Boolean Algebra

- 1854: George Boole develops Boolean algebra
- 1904: E. V. Huntington formulates the postulates of Boolean algebra
- 1938: Claude Shannon introduces switching algebra to model bistable electrical circuits

Definition of a Boolean Algebra

- A mathematical system consisting of:
 - A set of elements B [0/1 or T/F]
 - Two binary operators $(+)$ and $(\cdot)$ [OR/AND]
 - $=$ for equivalence, $()$ indicating order of operationsWhere the following axioms/postulates hold:
 - P1. Closure
For all $x, y \in B, x + y \in B, x \cdot y \in B$
 - P2. Identity
There exist identity elements in B , denoted 0,1 relative to $(+)$ and $(\cdot)$, respectively.
For all $x \in B, 0 + x = x + 0 = x, 1 \cdot x = x \cdot 1 = x$.

Definition of Boolean Algebra

– P3. Commutativity

The operations $(+)$, $(\cdot)$ are commutative

For all $x, y \in B$ $x + y = y + x$, $x \cdot y = y \cdot x$

– P4. Distributivity

***Each operation $(+)$, $(\cdot)$ is distributive over the other.

For all $x, y, z \in B$:

$$x + (y \cdot z) = (x + y) \cdot (x + z) \text{ [} x \text{ OR (} y \text{ AND } z \text{)]}$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \text{ [} x \text{ AND (} y \text{ OR } z \text{)]}$$

Definition of Boolean Algebra

- P5. Complement

For every element $x \in B$ there exists an element $x' \in B$ called the complement of x such that:

$(\overline{x})$

$$x + x' = 1$$

$$x \cdot x' = 0$$

- P6. Non-triviality

There exist at least two elements $x, y \in B$ such that $x \neq y$.

Principle of Duality

- (Except for P6), each postulate consists of two expressions s.t. one expression is transformed into the other by interchanging the operations (+) and ($\cdot$) as well as the identity elements 0 and 1.
- Such expressions are known as duals of each other.
- If some equivalence is proved, then its dual is also immediately true.
- E.g. If we prove: $(x \cdot x) + (x' \cdot x') = 1$, then we have by duality: $(x + x) \cdot (x' + x') = 0$

Theorems of Boolean Algebra

Theorem 1: The Idempotent law

For each element x in a Boolean algebra:

$$x + x = x$$

$$x \cdot x = x$$

Proof:

$$\begin{aligned} x + x &= 1 \cdot (x + x) \text{ [by P2]} \\ &= (x + x') \cdot (x + x) \text{ [by P5]} \\ &= x + (x \cdot x') \text{ [by P4]} \\ &= x + 0 \text{ [by P5]} \\ &= x. \text{ [by P2]} \end{aligned}$$

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 2: Null elements:

For each element x in a Boolean algebra:

$$x + 1 = 1$$

$$x \cdot 0 = 0.$$

$$\left. \begin{array}{l} x + 0 = x \\ x \cdot 1 = x \end{array} \right\} \text{identity}$$

$$\left. \begin{array}{l} x + x' = 1 \\ x \cdot x' = 0 \end{array} \right\} \text{comp}$$

Proof:

$$\begin{aligned} x + 1 &= 1 \cdot (x + 1) \text{ [by P2]} \\ &= (x + x') \cdot (x + 1) \text{ [by P5]} \\ &= x + (x' \cdot 1) \text{ [by P4]} \\ &= x + x' \text{ [by P2]} \\ &= 1. \text{ [by P5]} \end{aligned}$$

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 3: The Involution Law

$$x'' = x$$

For every x in a Boolean algebra, $(x')' = x$.

Proof. We will use the fact that the complement is unique (proof in later slide).

$$x + x' = 1 \text{ [by P5]}$$

$$x' + x = 1 \text{ [by P3]}$$

$$x = x'' \text{ by uniqueness.}$$

Moreover,

$$x \cdot x' = 0 \text{ [by P5]}$$

$$x' \cdot x = 0 \text{ [by P3]}$$

$$x = x'' \text{ by uniqueness.}$$

$$x \quad y \quad z$$

$$y = x'$$

$$z = x'$$

$$\downarrow \\ y = z$$

Theorems of Boolean Algebra

Theorem 6: The absorption law

For each pair of elements x, y in a Boolean algebra:

$$\begin{aligned}x + x \cdot y &= x \\x \cdot (x + y) &= x\end{aligned}$$

Proof:

$$\begin{aligned}x + (x \cdot y) &= (x \cdot 1) + (x \cdot y) \text{ [by P2]} \\&= x \cdot (1 + y) \text{ [by P4]} \\&= x \cdot 1 \text{ [by Th 2]} \\&= x \text{ [by P2]}\end{aligned}$$

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 4: The Associative Law

$$(x + y) + z = x + (y + z)$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Proof. We will prove that

$$(x + y) + z = [x + (y + z)] \cdot [(x + y) + z] = x + (y + z)$$

For the left hand side: $(x + y) + z = [x + (y + z)] \cdot [(x + y) + z]$:

$$\begin{aligned} [x + (y + z)] \cdot [(x + y) + z] &= [x + (y + z)] \cdot (x + y) + [x + (y + z)] \cdot z \quad [\text{by P4}] \\ &= ([x + (y + z)] \cdot x + [x + (y + z)] \cdot y) + [x + (y + z)] \cdot z \quad [\text{by P4}] \\ &= ([x \cdot x + (y + z) \cdot x] + [x \cdot y + (y + z) \cdot y]) + [x \cdot z + (y + z) \cdot z] \quad [\text{by P4}] \\ &= ([x + (y + z) \cdot x] + [x \cdot y + (y + z) \cdot y]) + [x \cdot z + (y + z) \cdot z] \quad [\text{by Th1}] \\ &= (x + [x \cdot y + y]) + [x \cdot z + z] \quad [\text{by Th 6}] \\ &= (x + y) + z \quad [\text{by Th 6}] \end{aligned}$$

The right hand side: $[x + (y + z)] \cdot [(x + y) + z] = x + (y + z)$ is analogous

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 5: DeMorgan's Law

For each pair of elements x, y in a Boolean algebra:

$$(x + y)' = x' \cdot y'$$
$$(x \cdot y)' = x' + y'$$

Note: Using
Associativity
of each
operation
(+), ($\cdot$).
Th 4

Proof: 

$$\begin{aligned}(x + y) + (x' \cdot y') &= (x + y + x') \cdot (x + y + y') \text{ [by P4]} \\ &= (x + x' + y) \cdot (y + y' + x) \text{ [by P3]} \\ &= (1 + y) \cdot (1 + x) \text{ [by P5]} \\ &= 1 \cdot 1 \text{ [by Th 2]} \\ &= 1 \text{ [by P2]}\end{aligned}$$

$$\begin{array}{r}(x + y) + \\ (x' \cdot y') = 1 \\ \hline (x + y) \cdot (x' \cdot y') = 0\end{array}$$

Analogous for multiplicative case.

Second part follows from principle of duality.

Operator Precedence

first perform $\cdot$
then perform $+$ } same as regular arithmetic.

Uniqueness of Complement

Theorem: If $x + a = 1$ and $x \cdot a = 0$ then $a = x'$.

Proof: We will prove that $a = ax' = x'$

For the left hand side: $a = ax'$:

$$\begin{aligned} a &= a \cdot 1 \quad [\text{by P2}] \\ &= a \cdot (x + x') \quad [\text{by P5}] \\ &= a \cdot x + a \cdot x' \quad [\text{by P4}] \\ &= 0 + a \cdot x' \quad [\text{by assumption on } a] \\ &= a \cdot x' \quad [\text{by P2}] \end{aligned}$$

The right hand side: $ax' = x'$ is analogous.

In Class Activity