

Boolean Algebra

Axioms and Theorems:

P1. Closure

For all $x, y \in B, x + y \in B, x \cdot y \in B$

P2. Identity

There exist identity elements in B , denoted 0,1 relative to (+) and ($\cdot$), respectively.

For all $x \in B, 0 + x = x + 0 = x, 1 \cdot x = x \cdot 1 = x$.

P3. Commutativity

The operations (+), ($\cdot$) are commutative

For all $x, y \in B, x + y = y + x, x \cdot y = y \cdot x$

P4. Distributivity

Each operation (+), ($\cdot$) is distributive over the other.

For all $x, y, z \in B$:

$$x + (y \cdot z) = (x + y) \cdot (x + z) \text{ [} x \text{ OR (} y \text{ AND } z \text{)]}$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \text{ [} x \text{ AND (} y \text{ OR } z \text{)]}$$

P5. Complement

For every element $x \in B$ there exists an element $x' \in B$ called the complement of x such that:

$$x + x' = 1$$

$$x \cdot x' = 0$$

P6. Non-triviality

There exist at least two elements $x, y \in B$ such that $x \neq y$.

Theorem 1: The Idempotent law

For each element x in a Boolean algebra:

$$x + x = x$$

$$x \cdot x = x$$

Theorem 2: Null elements:

For each element x in a Boolean algebra:

$$x + 1 = 1$$

$$x \cdot 0 = 0.$$

Theorem 3: The Involution Law

For every x in a Boolean algebra, $x'' = x$.

Theorem 4: The Associative Law

For every three elements x, y, z in a Boolean algebra:

$$(x + y) + z = x + (y + z)$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Theorem 5: DeMorgan's Law

For each pair of elements x, y in a Boolean algebra:

$$(x + y)' = x' \cdot y'$$

$$(x \cdot y)' = x' + y'$$

Theorem 6: The absorption law

For each pair of elements x, y in a Boolean algebra:

$$x + x \cdot y = x$$

$$x \cdot (x + y) = x$$

Class Exercise

9/10/26

1. Prove $x + x'y = x + y$

$$\begin{aligned} x + x'y &= (x + x') \cdot (x + y) & P4 \\ &= 1 \cdot (x + y) & P5 \\ &= x + y & P2 \end{aligned}$$

2. Prove $x'y' + xy + x'y = x' + y$

$$\begin{aligned} x'y' + xy + x'y &= x'y' + (x + x')y & P4 \\ &= x'y' + 1 \cdot y & P5 \\ &= x'y' + y & P2 \\ &= (x' + y)(y' + y) & P4 \\ &= (x' + y) \cdot 1 & P5 \\ &= x' + y & P2 \end{aligned}$$

3. Prove that no Boolean algebra has exactly three distinct elements.

Consider a set of 3 distinct elements:

$\{x, x', y\}$. y must have a complement.

Case 1: $y' = x$. Then by uniqueness of complement

$y = x'$. Then y is not distinct. Contradiction

Case 2: $y' = x'$. Then by uniqueness of complement

$y = x$. Then y is not distinct. Contradiction

Case 3: $y' = y$. Then by Def of Complement

$y + y = 1$ and $y + y = 0$. But by Th 1, $y + y = y = y \cdot y$. Then $y = 0$ and $y = 1$ contradiction