Boolean Algebra

Axioms and Theorems:

P1. Closure

For all $x, y \in B, x + y \in B, x \cdot y \in B$

P2. Identity

There exist identity elements in B, denoted 0,1 relative to (+) and (·), respectively.

For all $x \in B, 0 + x = x + 0 = x, 1 \cdot x = x \cdot 1 = x$.

P3. Commutativity

The operations (+), (·) are commutative

For all $x, y \in B$ $x + y = y + x,\ x \cdot y = y \cdot x$

P4. Distributivity

Each operation (+), (·) is distributive over the other.

For all $x, y, z \in B$:

$x + (y \cdot z) = (x + y) \cdot (x + z)$ [$x\ OR\ (y\ AND\ z)$]

$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ [$x\ AND\ (y\ OR\ z)$]

P5. Complement

For every element $x \in B$ there exists an element $x' \in B$called the complement of x such that:

$x + x' = 1$

$x \cdot x' = 0$

P6. Non-triviality

There exist at least two elements $x, y \in B$ such that $x \neq y$.

Theorem 1: The Idempotent law

For each element x in a Boolean algebra:

$x + x = x$

$x \cdot x = x$

Theorem 2: Null elements:

For each element x in a Boolean algebra:

$x + 1 = 1$

$x \cdot 0 = 0.$

Theorem 3: The Involution Law

For every x in a Boolean algebra, $x'' = x$.

Theorem 4: The Associative Law

For every three elements x, y, z in a Boolean algebra:

$$(x + y) + z = x + (y + z)$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Theorem 5: DeMorgan's Law

For each pair of elements x, y in a Boolean algebra:

$$(x + y)' = x' \cdot y'$$

$$(x \cdot y)' = x' + y'$$

Theorem 6: The absorption law

For each pair of elements x, y in a Boolean algebra:

$$x + x \cdot y = x$$

$$x \cdot (x + y) = x$$

Class Exercise

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1. Prove $x + x'y = x + y$

2. Prove $x'y' + xy + x'y = x' + y$

3. Prove that no Boolean algebra has exactly three distinct elements.