ENEE244-010x
Digital Logic Design

Lecture 9

Announcements

- HW4 due today
- HW5 will go up on Wed, due on 10/21.
- Exam grading will take a bit longer. . .

Agenda

- Recap:
  - The simplification problem (4.1)
- This time:
  - Prime Implicants (4.2)
  - Prime Implicates (4.3)
  - Karnaugh Maps (4.4)

The Simplification Problem

- The determination of Boolean expressions that satisfy some criterion of minimality is the simplification or minimization problem.
- We will assume cost is determined by number of gate inputs.

Prime Implicants

- $f_1$implies f_2 $(f_1 \rightarrow f_2)$
  - There is no assignment of values to the n variables that makes f_1 equal to 1 and f_2 equal to 0.
  - Whenever f_1 equals 1, then f_2 must also equal 1.
  - Whenever f_2 equals 0, then f_1 must also equal 0.
- Concept can be applied to terms and formulas.

Examples

- Case of Disjunctive Normal Formula
  - Sum-of-products form
  - Each of the product terms implies the function being described by the formula
  - Whenever product term has value 1, function must also have value 1.
- Case of Conjunctive Normal Formula
  - Product-of-sums form
  - Each sum term is implied by the function
  - Whenever the sum term has value 0, the function must also have value 0.

Subsumes

- A term t_1 is said to subsume a term t_2 iff all the literals of the term t_2 are also literals of the term t_1.
- Example: $x\overline{y}\,\overline{z}, x\overline{z}$

  $x + \overline{y} + \overline{z}, x + \overline{z}$
- If a product term t_1 subsumes a product term t_2, then t_1 implies t_2.
  - Why?
- If a sum term t_3 subsumes a sum term t_4, then t_4 implies t_3.
  - Why?

Subsumes

- Theorem:
  - If one term subsumes another in an expression, then the subsuming term can always be deleted from the expression without changing the function being described.
- CNF: $(x + y)(x + y + z)$
- DNF: $xy + xyz$

Implicants and Prime Implicants

- A product term is said to be an implicant of a complete function if the product term implies the function.
- Each of the minterms in minterm canonical form is an implicant of the function.
- An implicant of a function is a prime implicant if the implicant does not subsume any other implicant with fewer literals.

Example

x	y	z	f
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

$\overline{x}\,\overline{y}z$ is also an implicant

$x\overline{y}z$ is an implicant. Is it a prime implicant?

$\overline{y}z$ is an implicant.
Is it a prime implicant?
Yes. $\overline{y}, z$ are not implicants

Prime Implicants

- Theorem:
  - When the cost for a minimal Boolean formula is such that decreasing the number of literals in the DNF formula decreases the cost of the formula, the minimal DNFs correspond to sums of prime implicants.
- Why?

Irredundant Disjunctive Normal Formulas

- Definition: An expression in sum-of-products form such that:
  - Every product term in the expression is a prime implicant
  - No product term may be eliminated from the expression without changing the function described by the expression.
- Theorem:
  - When the cost of a formula decreases when a literal is removed, the minimal DNFs correspond to irredundant disjunctive normal formulas.

Prime Implicates and Irredundant Conjunctive Expressions

- A sum term is said to be an implicate of a complete function if the function implies the sum term.
- Each of the maxterms in maxterm canonical form is an implicate of the function.
- An implicate of a function is a prime implicate if the implicate does not subsume any other implicate with fewer literals.

Example

x	y	z	f
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

$\overline{x} + y + z$ is an implicate. Is it a prime implicate?

$\overline{x} + \overline{y} + z$ is also an implicant

$\overline{x} + z$ is an implicate.
Is it a prime implicate?
Yes. $\overline{x}, z$ are not implicates

Prime Implicates

- Theorem:
  - When the cost for a minimal Boolean formula is such that decreasing the number of literals in the CNF formula decreases the cost of the formula, the minimal CNFs correspond to products of prime implicates.

Irredundant Conjunctive Normal Formulas

- Definition: An expression in product-of-sums form such that:
  - Every sum term in the expression is a prime implicate
  - No sum term may be eliminated from the expression without changing the function described by the expression.
- Theorem:
  - When the cost of a formula decreases when a literal is removed, the minimal CNFs correspond to irredundant conjunctive normal formulas.

Karnaugh Maps

- Method for graphically determining implicants and implicates of a Boolean function.
- Simplify Boolean functions and their logic gates implementation.
- Geometrical configuration of 2^ncells such that each of the n-tuples corresponding to the row of a truth table uniquely locates a cell on the map.
- Structure of Karnaugh map:
  - Two cells are physically adjacent within the configuration iff their respective n-tuples differ in exactly one element.
  - E.g. (0,1,1), (0,1,0)
  - E.g. (1,0,1), (1,1,0)

Three-Variable Maps

- Each cell is adjacent to 3 other cells.
- Imagine the map lying on the surface of a cylinder.

		yz			
		00	01	11	10
x	0	1	0	0	1
	1	1	1	0	0

Four-Variable Maps

- Each cell is adjacent to 4 other cells.
- Imagine the map lying on the surface of a torus.

yz

wx	00	01	11	10
00	1	1	0	1
01	1	1	0	0
11	0	0	0	0
10	1	0	0	1

Karnaugh Maps and Canonical Formulas

- Minterm Canonical Formula

x \ yz	00	01	11	10
0	1	0	0	1
1	1	1	0	0

$$f(x) = \overline{x}\,\overline{y}\,\overline{z} + \overline{x}y\overline{z} + x\overline{y}\,\overline{z} + x\overline{y}z$$
$$= \sum m(0,2,4,5)$$

Karnaugh Maps and Canonical Formulas

- Maxterm Canonical Formula

		yz			
		00	01	11	10
x	0	1	0	0	1
	1	1	1	0	0

$$f(x) = (x + y + \overline{z})(x + \overline{y} + \overline{z})(\overline{x}\,\overline{y}\,\overline{z})(\overline{x}\,\overline{y}z)$$
$$= \Pi M(1,3,6,7)$$

Karnaugh Maps and Canonical Formulas

- Decimal Representation

		yz			
		00	01	11	10
x	0	0	1	3	2
	1	4	5	7	6

Karnaugh Maps and Canonical Formulas

- Decimal Representation

yz

wx

	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

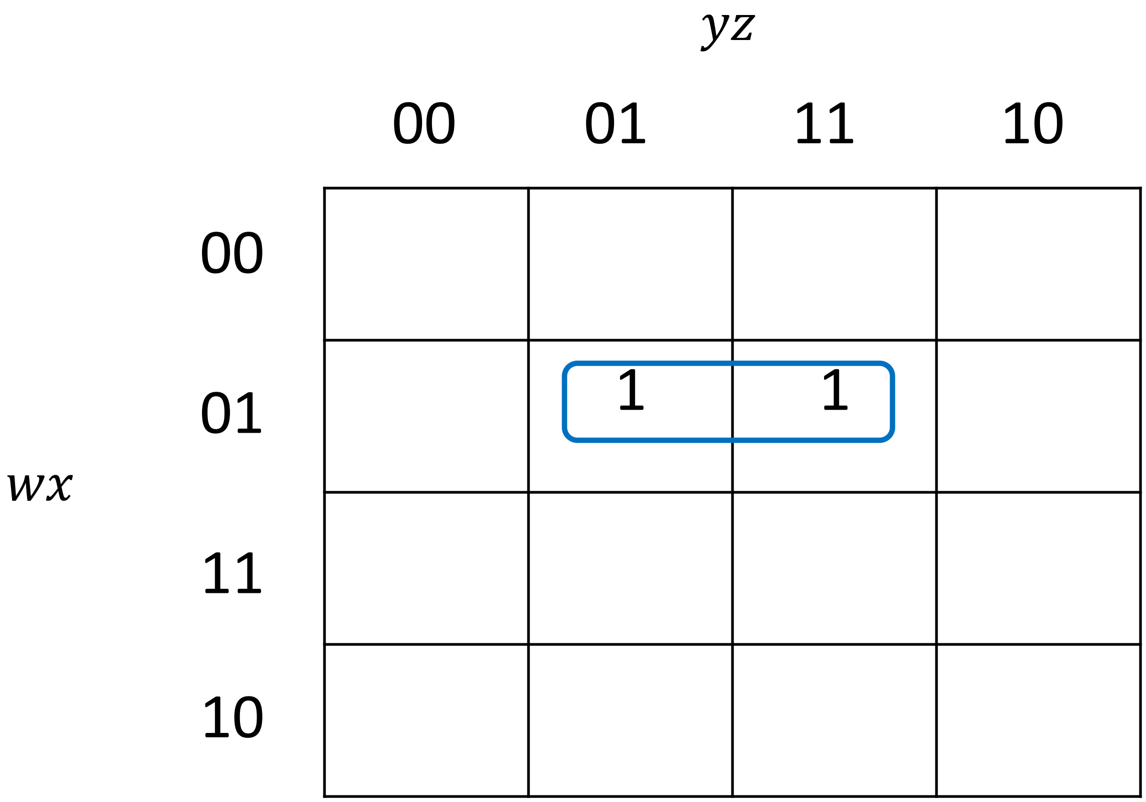
Product Term Representations on Karnaugh Maps

- Any set of 1-cells which form a $2^a \times 2^b$ rectangular grouping describes a product term with $n - a - b$ variables.
- Rectangular groupings are referred to as subcubes.
- The total number of cells in a subcube must be a power-of-two (2^{a+b}).
- Two adjacent 1-cells: $\overline{w}x\overline{y}z + \overline{w}xyz$

$$= \overline{w}xz(\overline{y} + y) = \overline{w}xz$$

Examples of Subcubes

Subcubes for elimination of one variable



yz

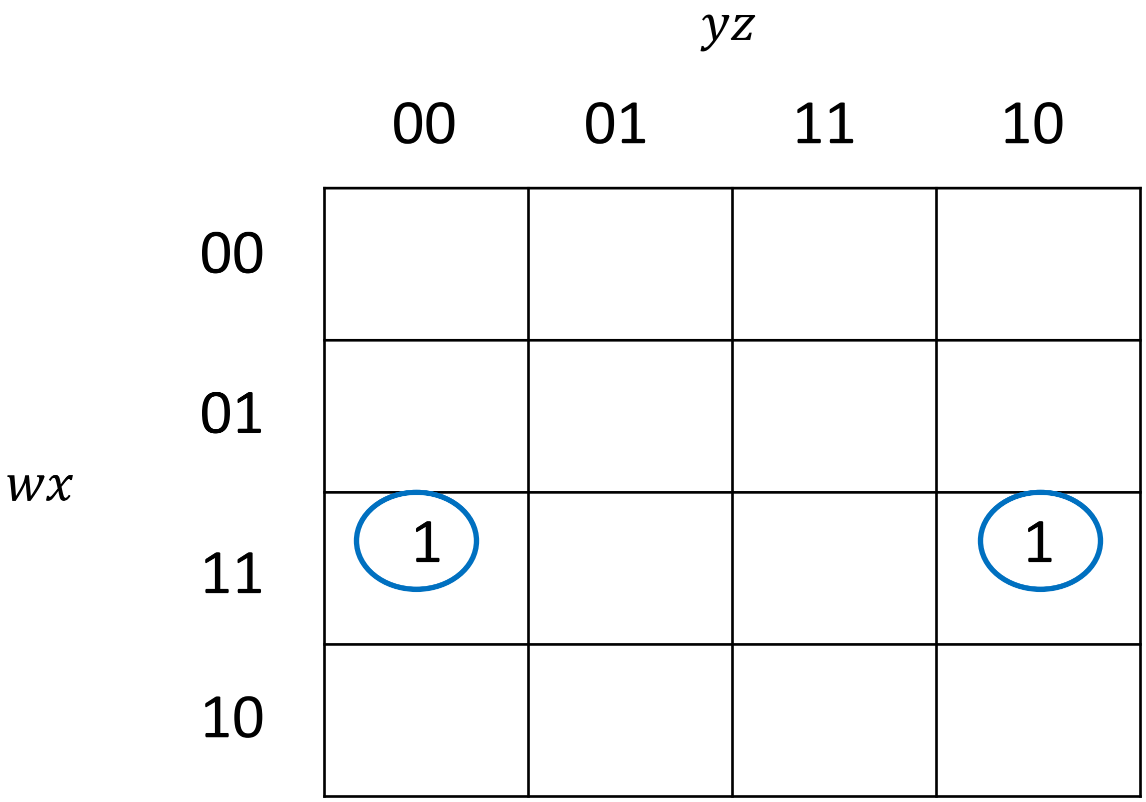
wx

	00	01	11	10
00				
01		1	1	
11				
10				

Variables in the product term are variables whose value is constant inside the subcube.

Product term: $\overline{w}xz$

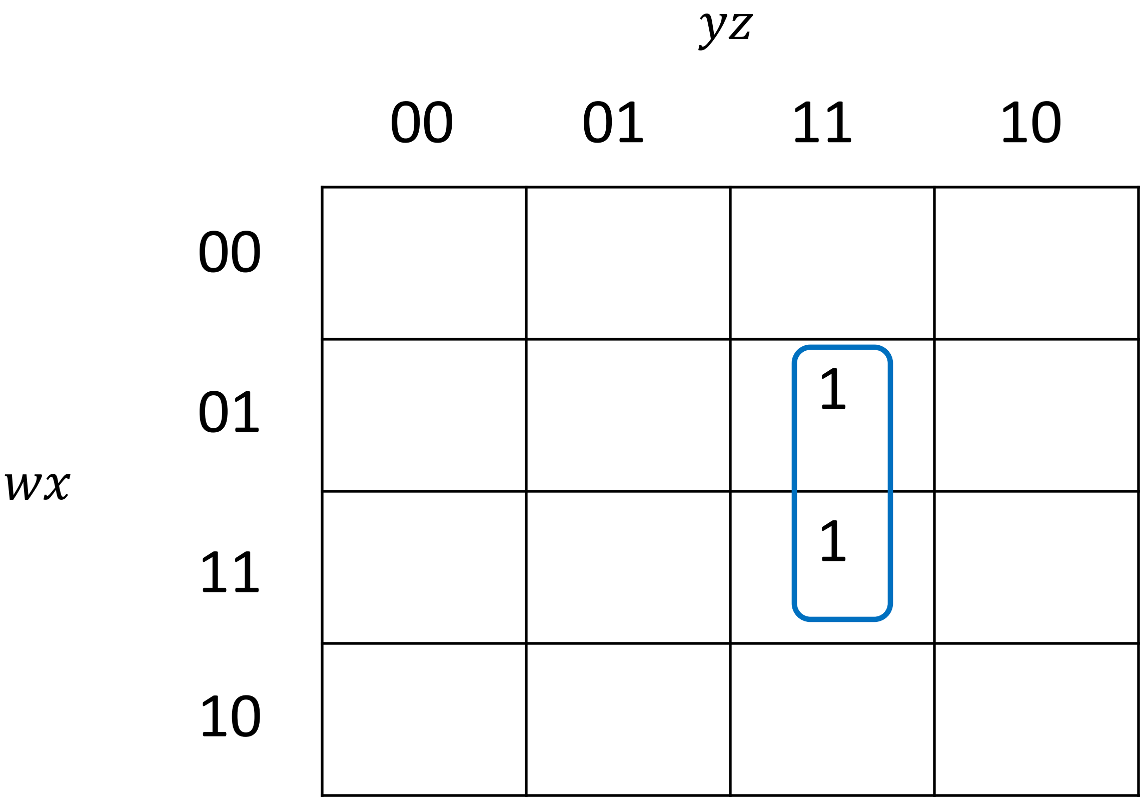
Subcubes for elimination of one variable



wx \ yz	00	01	11	10
00				
01				
11	1			1
10				

Product term: $wx\overline{z}$

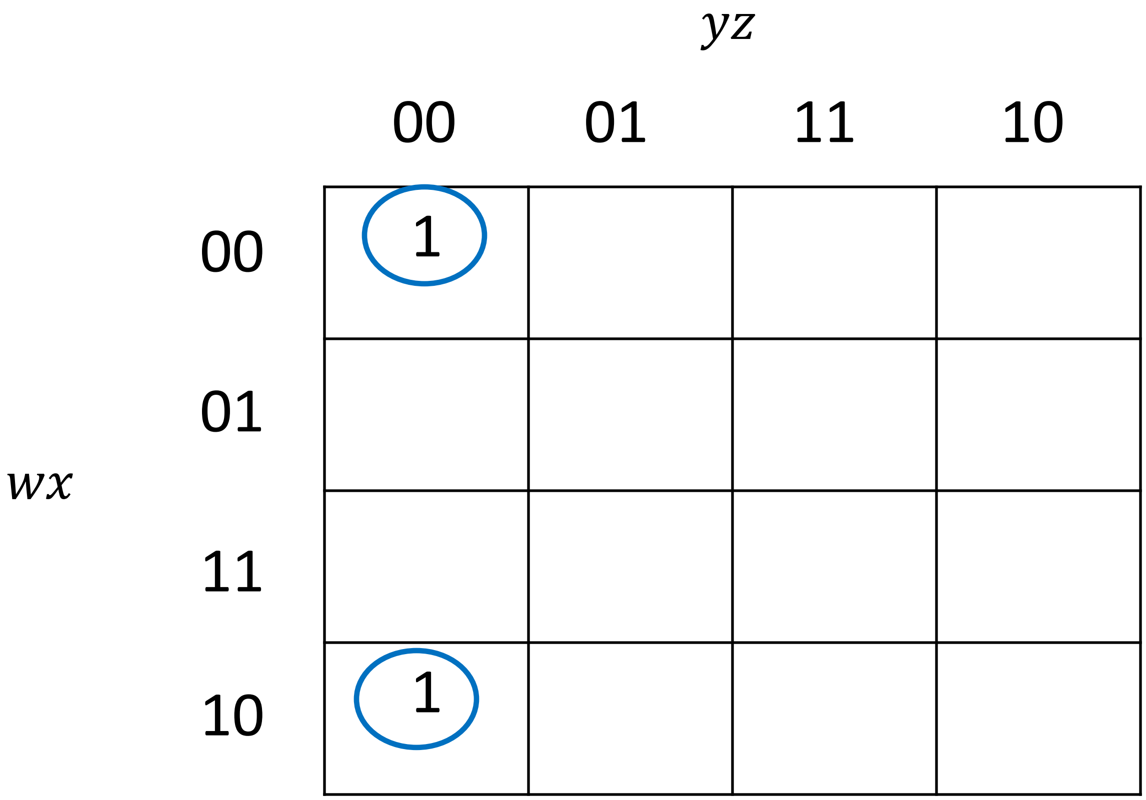
Subcubes for elimination of one variable



	00	01	11	10
00				
01			1	
11			1	
10				

Product term: xyz

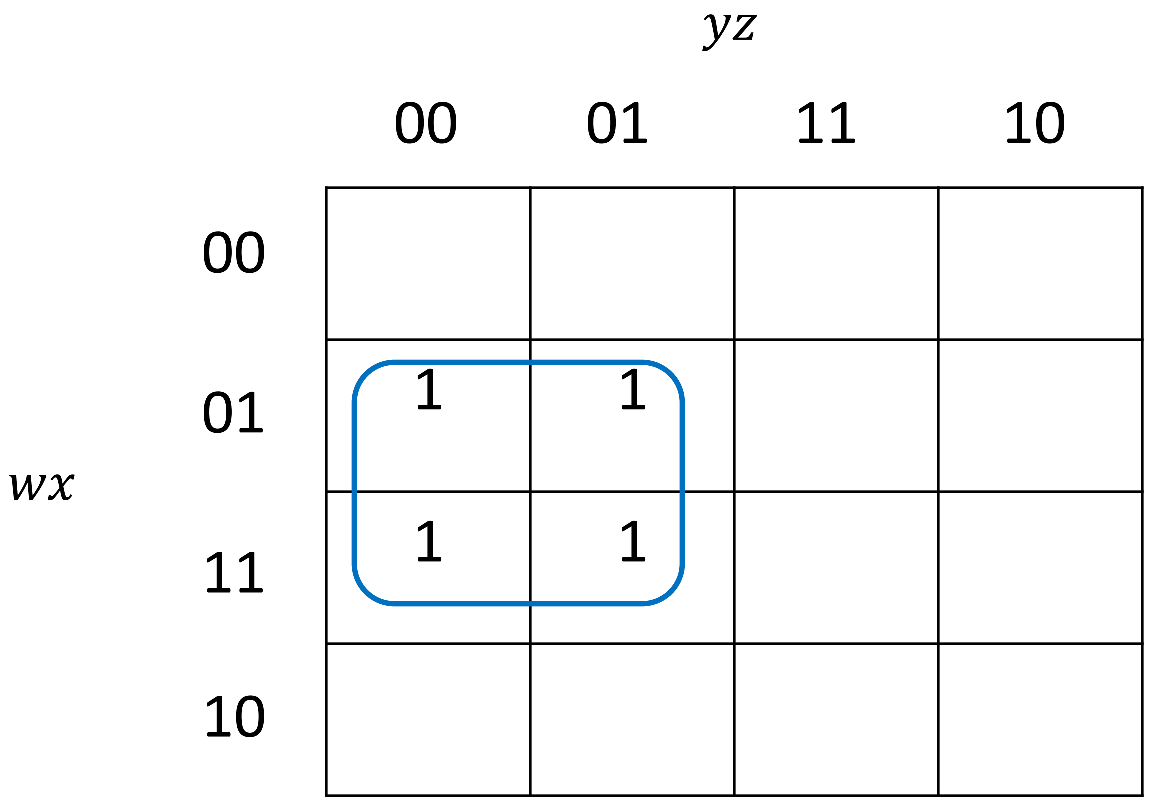
Subcubes for elimination of one variable



wx \ yz	00	01	11	10
00	1			
01				
11				
10	1			

Product term: $\bar{x}\,\bar{y}\,\bar{z}$

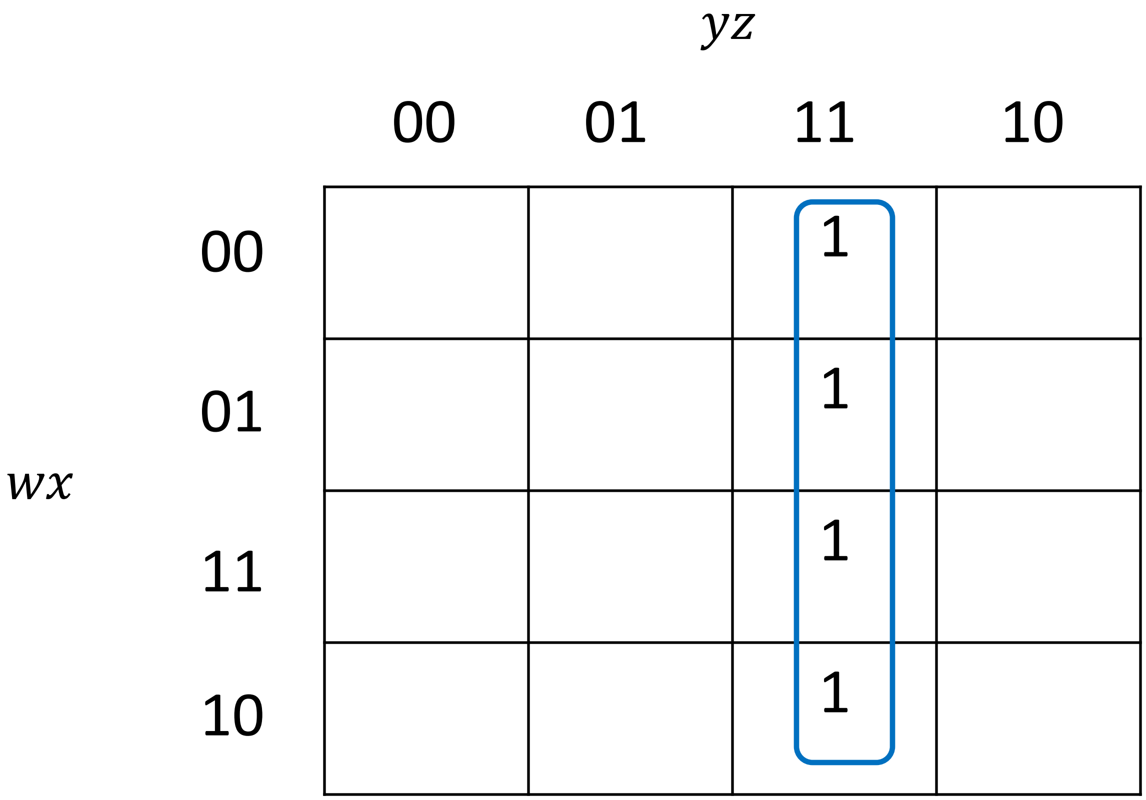
Subcubes for elimination of two variables



	00	01	11	10
00				
01	1	1		
11	1	1		
10				

Product term: $x\overline{y}$

Subcubes for elimination of two variables



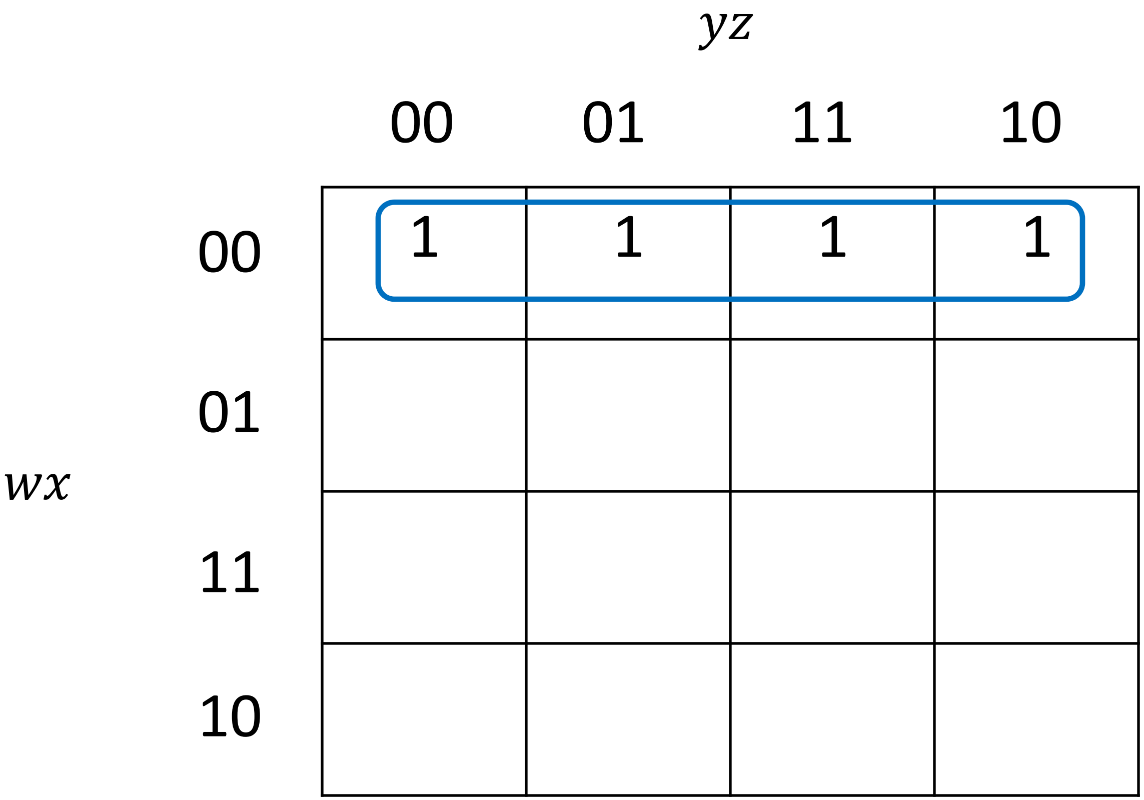
yz

wx

	00	01	11	10
00			1	
01			1	
11			1	
10			1	

Product term: yz

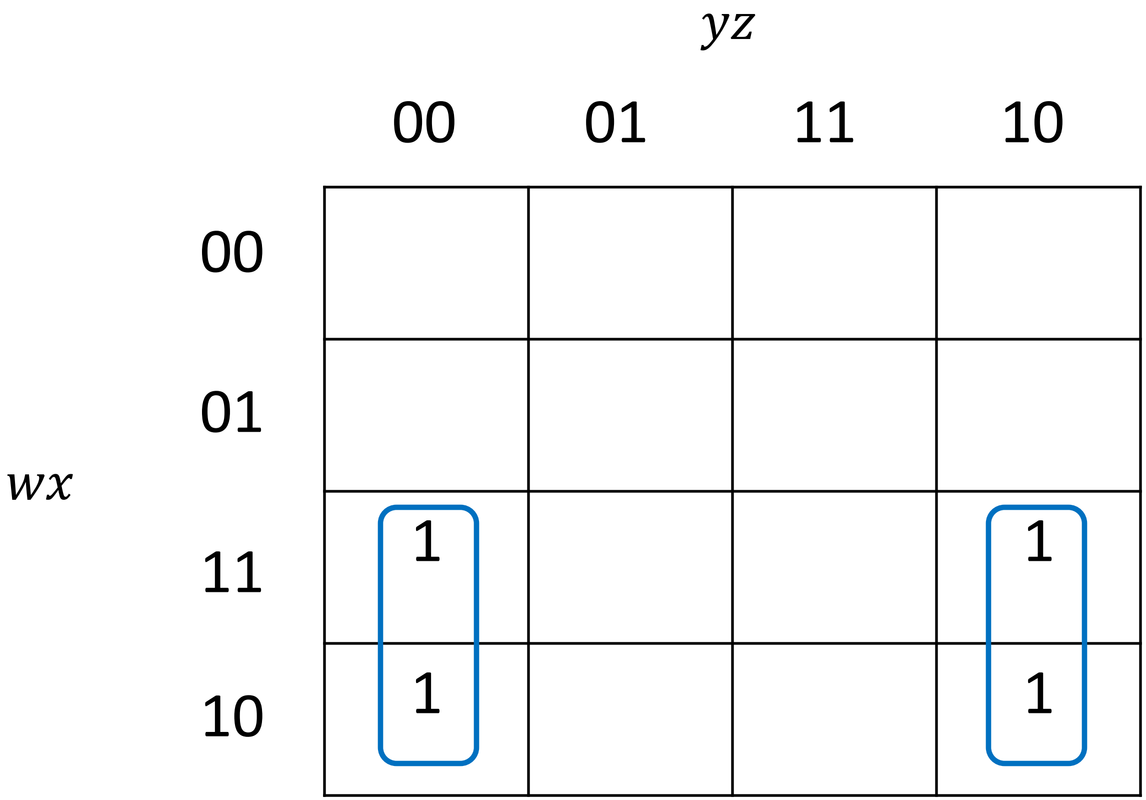
Subcubes for elimination of two variables



	00	01	11	10
00	1	1	1	1
01				
11				
10				

Product term: $\overline{w}\,\overline{x}$

Subcubes for elimination of two variables



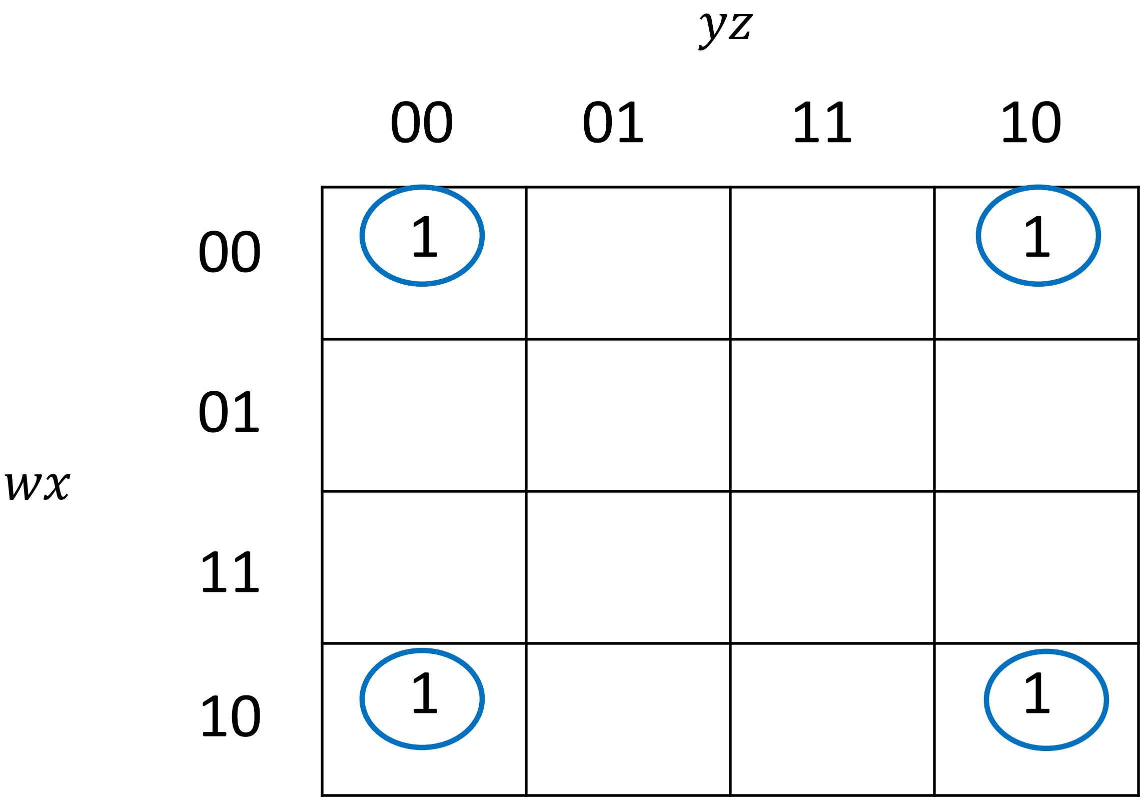
yz

wx

	00	01	11	10
00				
01				
11	1			1
10	1			1

Product term: $w\overline{z}$

Subcubes for elimination of two variables

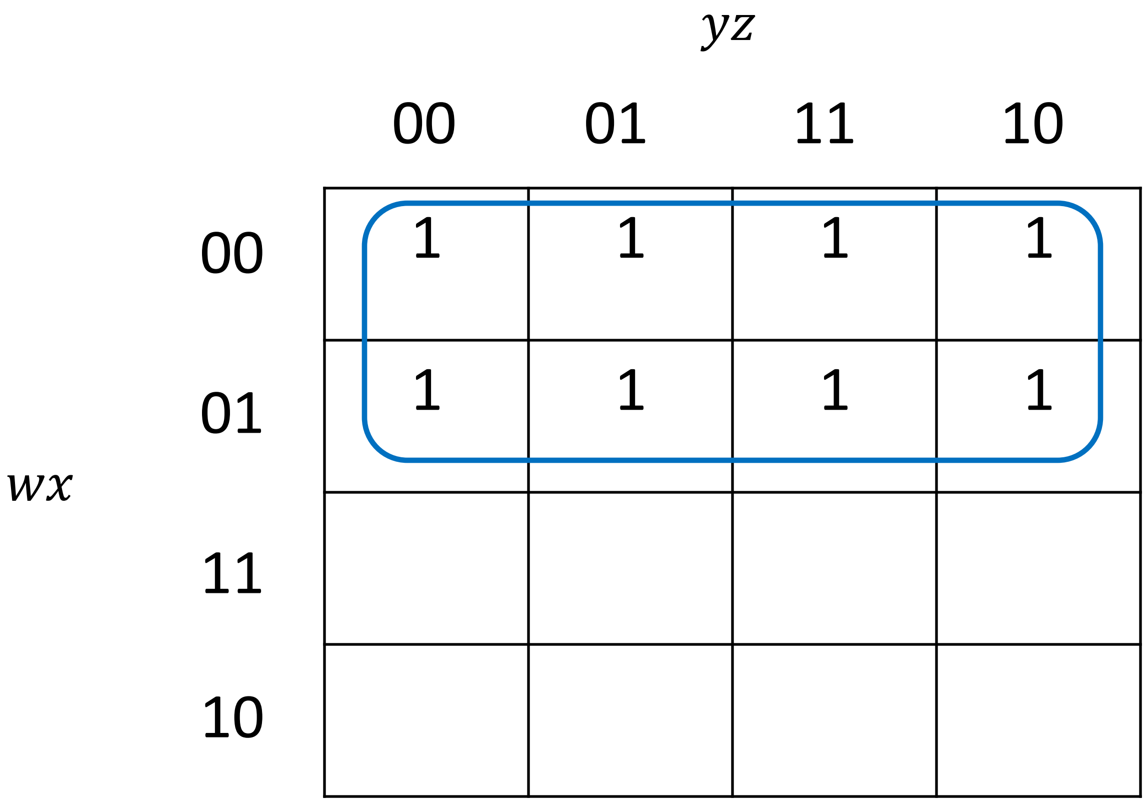


yz

wx	00	01	11	10
00	1			1
01				
11				
10	1			1

Product term: $\overline{x}\,\overline{z}$

Subcubes for elimination of three variables



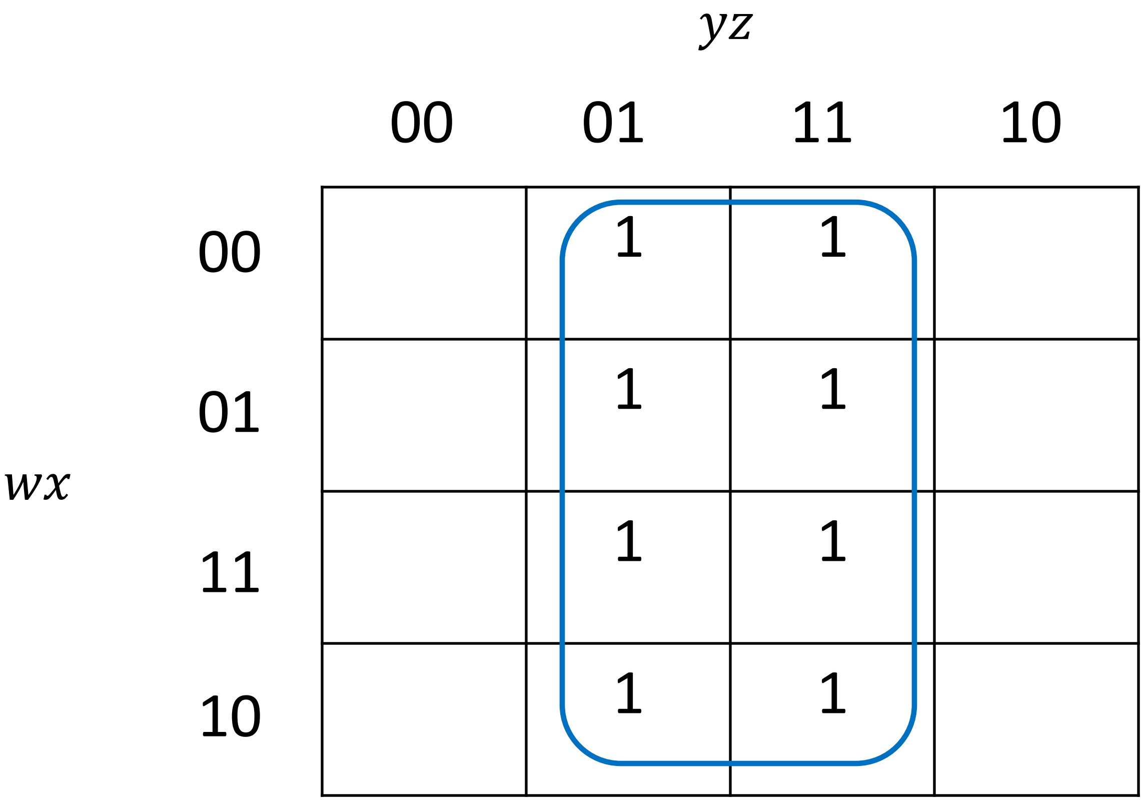
yz

wx

	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11				
10				

Product term: $\overline{w}$

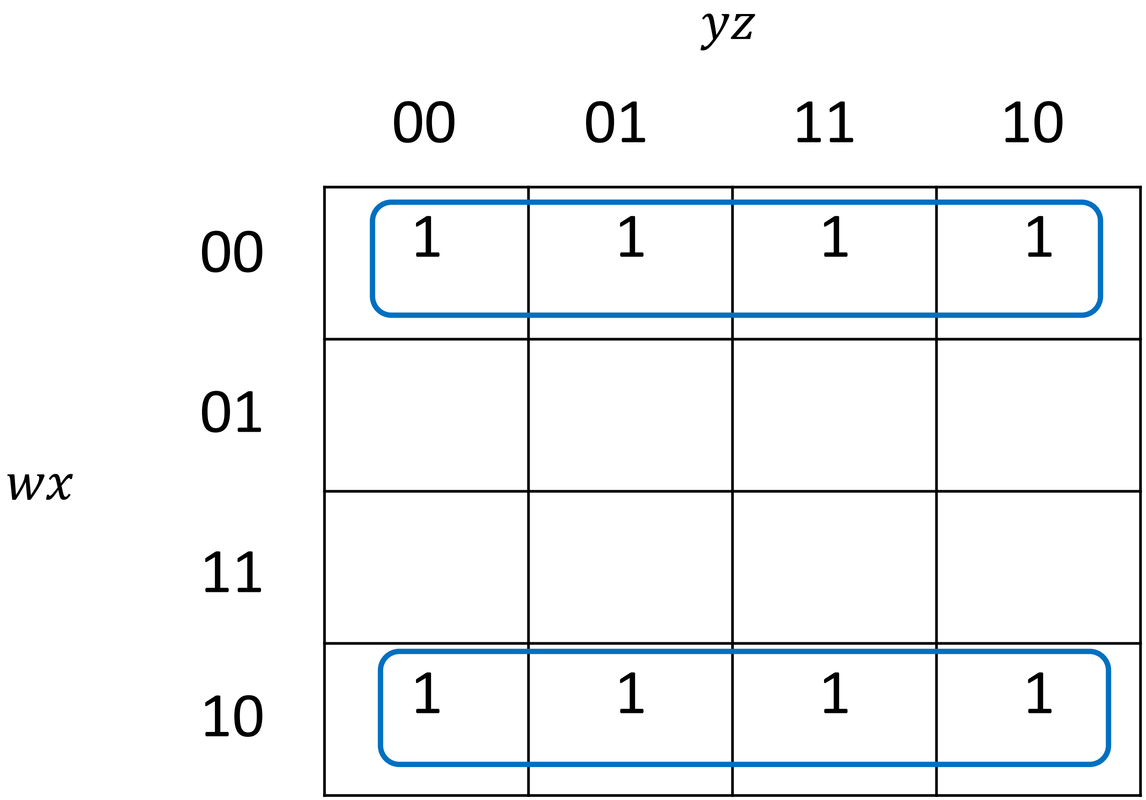
Subcubes for elimination of three variables



	00	01	11	10
00		1	1	
01		1	1	
11		1	1	
10		1	1	

Product term: $\overline{w}$

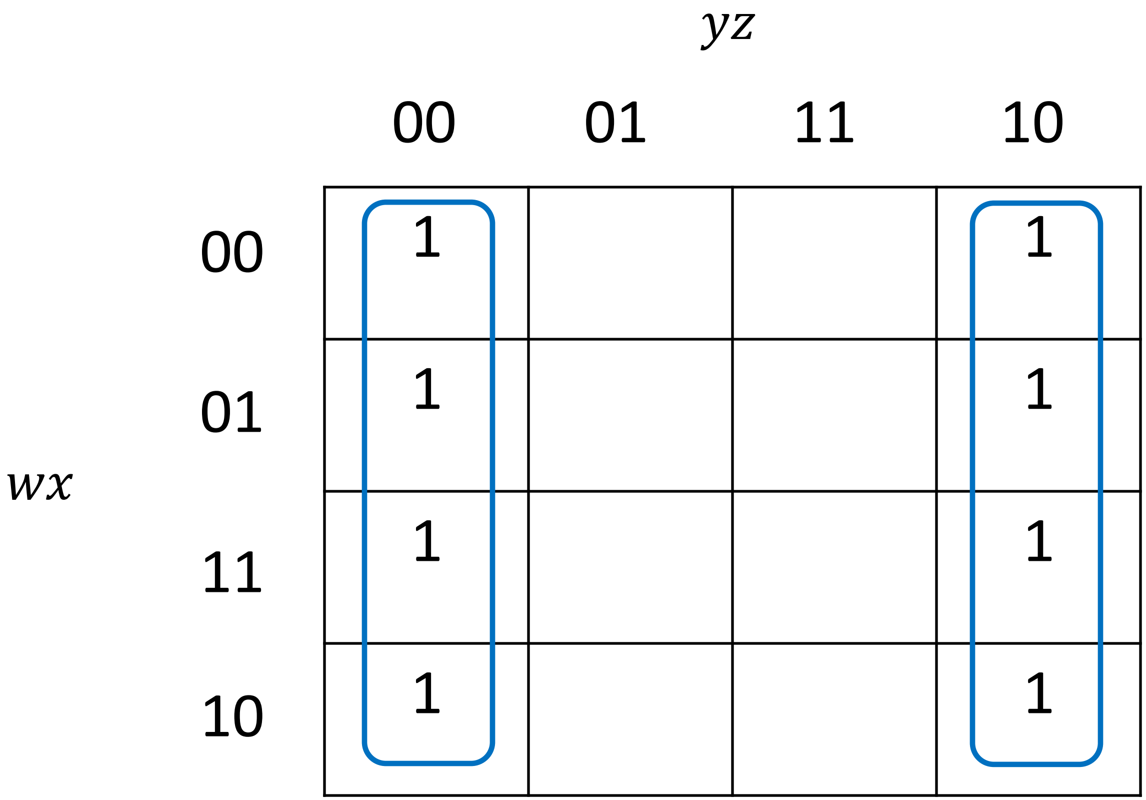
Subcubes for elimination of three variables



	00	01	11	10
00	1	1	1	1
01				
11				
10	1	1	1	1

Product term: $\overline{x}$

Subcubes for elimination of three variables



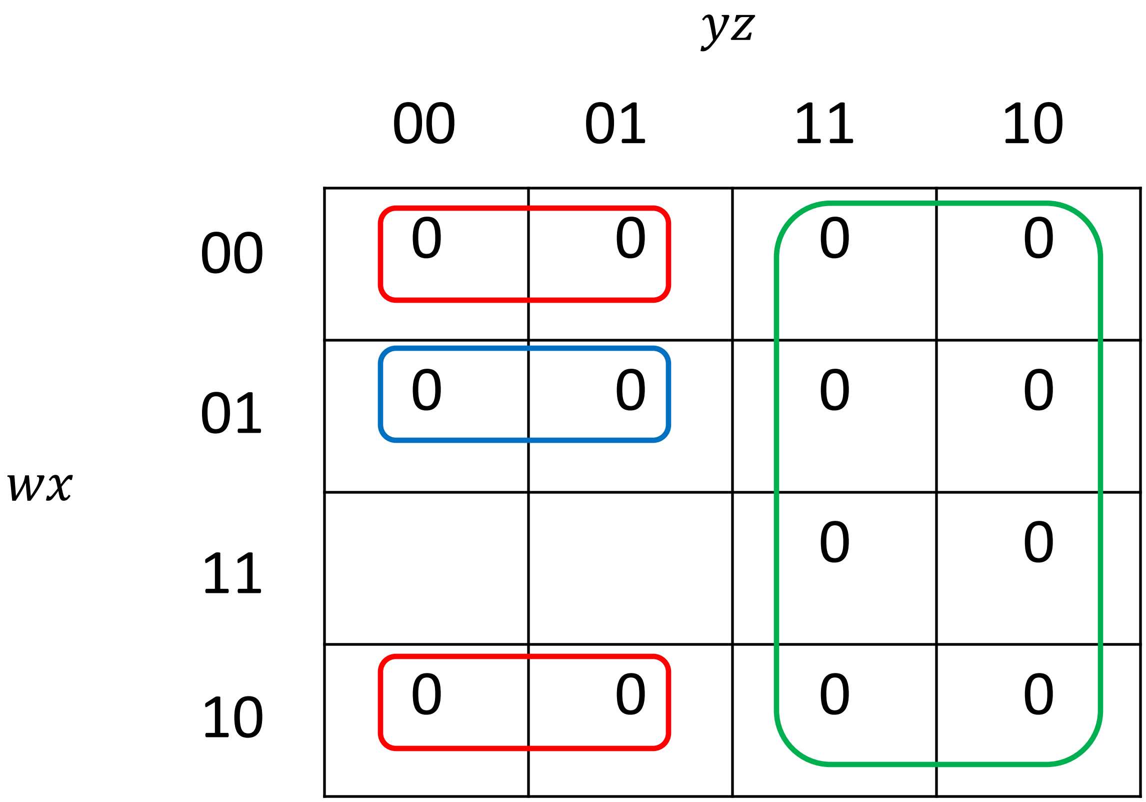
yz

wx

	00	01	11	10
00	1			1
01	1			1
11	1			1
10	1			1

Product term: $\overline{z}$

Subcubes for sum terms



wx \ yz	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11			0	0
10	0	0	0	0

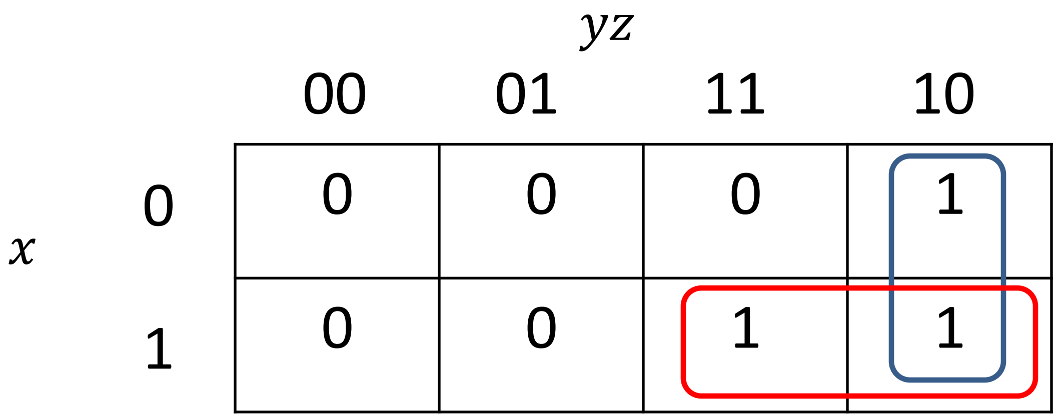
Sum terms: $w + \overline{x} + y$

$x + y$

$\overline{y}$

Using K-Maps to Obtain Minimal Boolean Expressions

Example



$$\begin{array}{c|cccc} & \multicolumn{4}{c}{yz} \\ x & 00 & 01 & 11 & 10 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array}$$

$$\left.\begin{array}{l} y\bar{z} \\ xy \end{array}\right\}$$ Both are prime implicants

How do we know this is minimal?

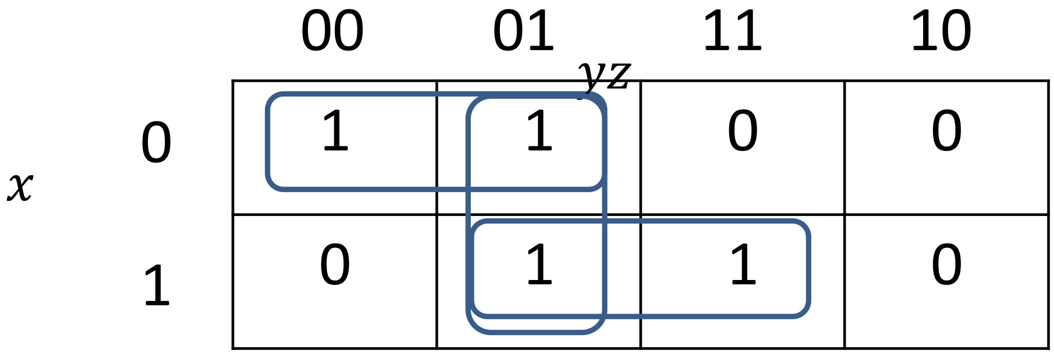
$$f = y\bar{z} + xy$$

Finding the set of all prime implicants in an n-variable map:

- If all 2^n entries are 1, then function is equal to 1.
- For i = 1, 2. . . n
  - Search for all subcubes of dimensions $2^a \times 2^b = 2^{n-i}$ that are not totally contained within a single previously obtained subcube.
  - Each of these subcubes represents an i variable product term which implies the function.
  - Each product term is a prime implicant.

Essential Prime Implicants

- Some 1-cells appear in only one prime implicant subcube, others appear in more than one.



	00	01	11	10
0	1	1	0	0
1	0	1	1	0

- A 1-cell that can be in only one prime implicant is called an essential prime implicant.

Essential Prime Implicants

- Every essential prime implicant must appear in all the irredundant disjunctive normal formulas of the function.
- Hence must also appear in a minimal sum.
  - Why?

General Approach for Finding Minimal Sums

- Find all prime implicants using K-map
- Find all essential prime implicants using K-map
- **If all 1-cells are not yet covered, determine optimal choice of remaining prime implicants using K-map.