ENEE244-010x
Digital Logic Design

Lecture 4

Announcements

- HW 2 up on course webpage, due on Monday, Sept. 21.

Agenda

- Last time:
  - Error Detecting and Correcting Codes (2.11, 2.12)

- This time:
  - Boolean Algebra axioms and theorems (3.1, 3.2)

Boolean Algebra

Definition of a Boolean Algebra

- A mathematical system consisting of:
  - A set of elements B [0/1 or T/F]
  - Two binary operators (+) and (·) [OR/AND]
  - = for equivalence, () indicating order of operations

  Where the following axioms/postulates hold:
  - P1. Closure

  For all $x, y \in B, x + y \in B, x \cdot y \in B$
  - P2. Identity

  There exist identity elements in B, denoted 0,1 relative to (+) and (·), respectively.

  For all $x \in B, 0 + x = x + 0 = x, 1 \cdot x = x \cdot 1 = x$.

Definition of Boolean Algebra

– P3. Commutativity

The operations (+), (·) are commutative

For all $x, y \in B$ $x + y = y + x,\ \ x \cdot y = y \cdot x$

– P4. Distributivity

***Each operation (+), (·) is distributive over the other.

For all $x, y, z \in B$:

$$x + (y \cdot z) = (x + y) \cdot (x + z)\ [x\ OR\ (y\ AND\ z)]$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)\ [x\ AND\ (y\ OR\ z)]$$

Definition of Boolean Algebra

– P5. Complement

For every element $x \in B$ there exists an element $\overline{x} \in B$ called the complement of x such that:

$$x + \overline{x} = 1$$

$$x \cdot \overline{x} = 0$$

– P6. Non-triviality

There exist at least two elements $x, y \in B$ such that $x \neq y$.

Principle of Duality

- (Except for P6), each postulate consists of two expressions s.t. one expression is transformed into the other by interchanging the operations (+) and (·) as well as the identity elements 0 and 1.
- Such expressions are known as duals of each other.
- If some equivalence is proved, then its dual is also immediately true.
- E.g. If we prove: $(x \cdot x) + (\overline{x} \cdot \overline{x}) = 1$, then we have by duality: $(x + x) \cdot (\overline{x} + \overline{x}) = 0$

Theorems of Boolean Algebra

Theorem 3.2: Null elements:

For each element x in a Boolean algebra:

$$x + 1 = 1$$

$$x \cdot 0 = 0.$$

Proof:

$$x + 1 = 1 \cdot (x + 1)$$

$$= (x + \overline{x}) \cdot (x + 1)$$

$$= x + (\overline{x} \cdot 1)$$

$$= x + \overline{x}$$

$$= 1.$$

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 3.4: The Idempotent law

For each element x in a Boolean algebra:

$$x + x = x$$

$$x \cdot x = x$$

Proof:

$$x + x = 1 \cdot (x + x)$$

$$= (x + \overline{x}) \cdot (x + x)$$

$$= x + (x \cdot \overline{x})$$

$$= x + 0$$

$$= x.$$

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 3.5: The Involution Law

For every x in a Boolean algebra, $\overline{(\overline{x})} = x$.

Proof. We will use the fact that the complement is unique (Theorem 3.1 in textbook).

$$x + \overline{x} = 1$$

$$\overline{x} + x = 1$$

$x = \overline{x}$ by uniqueness.

Moreover,

$$x \cdot \overline{x} = 0$$

$$\overline{x} \cdot x = 1$$

$x = \overline{x}$ by uniqueness.

Theorems of Boolean Algebra

Theorem 3.6: The absorption law

For each pair of elements x, y in a Boolean algebra:

$$x + x \cdot y = x$$

$$x \cdot (x + y) = x$$

Proof:

$$x + x \cdot y = (x \cdot 1) + (x \cdot y)$$

$$= x \cdot (1 + y)$$

$$= x \cdot 1$$

$$= x$$

Second part follows from principle of duality.

Theorems of Boolean Algebra

Theorem 3.7: DeMorgan's Law

For each pair of elements x, y in a Boolean algebra:

$$\overline{(x+y)} = \overline{x} \cdot \overline{y}$$

$$\overline{(x \cdot y)} = \overline{x} + \overline{y}$$

Proof:

$$(x+y) + \overline{x} \cdot \overline{y} = (x+y+\overline{x}) \cdot (x+y+\overline{y})$$

$$= (x+\overline{x}+y) \cdot (y+\overline{y}+x)$$

$$= (1+y) \cdot (1+x)$$

$$= 1 \cdot 1$$

$$= 1$$

Note: Using Associativity of each operation (+), (·). Theorem 3.8

Analogous for multiplicative case.

Second part follows from principle of duality.

In Class Activity