ENEE244-010x
Digital Logic Design

Lecture 4
Announcements

• HW 2 up on course webpage, due on Monday, Sept. 21.
Agenda

• Last time:
  – Error Detecting and Correcting Codes (2.11, 2.12)

• This time:
  – Boolean Algebra axioms and theorems (3.1, 3.2)
Boolean Algebra
Definition of a Boolean Algebra

• A mathematical system consisting of:
  – A set of elements $B$ [0/1 or T/F]
  – Two binary operators (+) and (⋅) [OR/AND]
  – = for equivalence, () indicating order of operations
Where the following axioms/postulates hold:
  – P1. Closure
    For all $x, y \in B$, $x + y \in B$, $x \cdot y \in B$
  – P2. Identity
    There exist identity elements in $B$, denoted 0,1 relative to (+) and (⋅), respectively.
    For all $x \in B$, $0 + x = x + 0 = x$, $1 \cdot x = x \cdot 1 = x$. 
Definition of Boolean Algebra

– P3. Commutativity
The operations (+), (⋅) are commutative
For all $x, y \in B$ $x + y = y + x, \ x \cdot y = y \cdot x$

– P4. Distributivity
***Each operation (+), (⋅) is distributive over the other.
For all $x, y, z \in B$:

\[
x + (y \cdot z) = (x + y) \cdot (x + z) \ [x \ OR \ (y \ AND \ z)]
\]

\[
x \cdot (y + z) = (x \cdot y) + (x \cdot z) \ [x \ AND \ (y \ OR \ z)]
\]
Definition of Boolean Algebra

– P5. Complement
For every element $x \in B$ there exists an element $\bar{x} \in B$ called the complement of $x$ such that:

\[
x + \bar{x} = 1
\]
\[
x \cdot \bar{x} = 0
\]

– P6. Non-triviality
There exist at least two elements $x, y \in B$ such that $x \neq y$. 
Principle of Duality

• (Except for P6), each postulate consists of two expressions s.t. one expression is transformed into the other by interchanging the operations (+) and (⋅) as well as the identity elements 0 and 1.
• Such expressions are known as duals of each other.
• If some equivalence is proved, then its dual is also immediately true.
• E.g. If we prove: \((x \cdot x) + (\overline{x} \cdot \overline{x}) = 1\), then we have by duality: \((x + x) \cdot (\overline{x} + \overline{x}) = 0\)
Theorems of Boolean Algebra

Theorem 3.2: Null elements:
For each element $x$ in a Boolean algebra:
\[ x + 1 = 1 \]
\[ x \cdot 0 = 0. \]

Proof:
\[ x + 1 = 1 \cdot (x + 1) \]
\[ = (x + \overline{x}) \cdot (x + 1) \]
\[ = x + (\overline{x} \cdot 1) \]
\[ = x + \overline{x} \]
\[ = 1. \]

Second part follows from principle of duality.
Theorems of Boolean Algebra

Theorem 3.4: The Idempotent law
For each element $x$ in a Boolean algebra:

\[
x + x = x \\
x \cdot x = x
\]

Proof:

\[
x + x = 1 \cdot (x + x) \\
= (x + \overline{x}) \cdot (x + x) \\
= x + (x \cdot \overline{x}) \\
= x + 0 \\
= x.
\]

Second part follows from principle of duality.
Theorems of Boolean Algebra

Theorem 3.5: The Involution Law

For every $x$ in a Boolean algebra, $(\overline{x}) = x$.

Proof. We will use the fact that the complement is unique (Theorem 3.1 in textbook).

$$x + \overline{x} = 1$$

$$\overline{x} + x = 1$$

$x = \overline{x}$ by uniqueness.

Moreover,

$$x \cdot \overline{x} = 0$$

$$\overline{x} \cdot x = 1$$

$x = \overline{x}$ by uniqueness.
Theorems of Boolean Algebra

Theorem 3.6: The absorption law
For each pair of elements \( x, y \) in a Boolean algebra:
\[
\begin{align*}
x + x \cdot y &= x \\
x \cdot (x + y) &= x
\end{align*}
\]

Proof:
\[
\begin{align*}
x + x \cdot y &= (x \cdot 1) + (x \cdot y) \\
&= x \cdot (1 + y) \\
&= x \cdot 1 \\
&= x
\end{align*}
\]
Second part follows from principle of duality.
Theorems of Boolean Algebra

Theorem 3.7: DeMorgan’s Law
For each pair of elements $x, y$ in a Boolean algebra:

$$(x + y) = \overline{x} \cdot \overline{y}$$

$$(x \cdot y) = \overline{x} + \overline{y}$$

Proof:

$$(x + y) + \overline{x} \cdot \overline{y} = (x + y + \overline{x}) \cdot (x + y + \overline{y})$$

$$= (x + \overline{x} + y) \cdot (y + \overline{y} + x)$$

$$= (1 + y) \cdot (1 + x)$$

$$= 1 \cdot 1$$

$$= 1$$

Analogous for multiplicative case.
Second part follows from principle of duality.

Note: Using Associativity of each operation (+), (⋅). Theorem 3.8
In Class Activity